BUCHSBAUM CRITERION OF SEGRE PRODUCTS OF BUCHSBAUM MODULES

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The purpose of this paper is to give a survey of a study on the Buchsbaum property of Segre product of Buchsbaum vector bundles on multiprojective spaces based on my talk at the Commutative Algebra Conference 2015, Japan. A base field k is always an algebraically closed field throughout this paper. Let us begin with Grothendieck's theorem.

Theorem 1. *Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^1_k$ of rank r. Then $\mathcal{E}$ is isomorphic to $\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1_k}(a_i)$ for some $a_i \in \mathbb{Z}$.*

Sketch of Proof. Let us take an integer $\ell \in \mathbb{Z}$ such that $\Gamma(\mathcal{E}(\ell)) \neq 0$ and $\Gamma(\mathcal{E}(\ell - 1)) = 0$. Then we have an inclusion $\mathcal{O}_{\mathbb{P}^n_k} \to \mathcal{E}(\ell)$, which gives a direct summand. $\square$

Horrocks [5] has given a generalization of Theorem 1.

Theorem 2. *Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^n_k$ of rank r. Assume that $\mathcal{E}$ is ACM, that is, $\mathrm{H}^i_*(\mathcal{E}) = \oplus_{\ell\in\mathbb{Z}} \mathrm{H}^i(\mathbb{P}^n_k, \mathcal{E}(\ell)) = 0$ for $1 \leq i \leq n-1$. Then $\mathcal{E}$ is isomorphic to $\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n_k}(a_i)$ for some $a_i \in \mathbb{Z}$.*

Sketch of Proof. We will prove by induction on n. Let us take $\mathcal{F} = \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n_k}(a_i)$ by taking integers a_i from an isomorphism $\mathcal{E}|_H \cong \oplus_{i=1}^r \mathcal{O}_H(a_i)$. Then we have only to take a section of $\Gamma(\mathcal{F}^\vee \otimes \mathcal{E})$ by using the hypothesis of induction, which gives an isomorphism $\mathcal{F} \cong \mathcal{E}$. $\square$

On the other hand we have another way to prove a splitting criterion of ACM vector bundles on $\mathbb{P}^n_k$ by the Auslander-Buchsbaum theorem.

Theorem 3. *Let A be a noetherian local ring. Let M be a finitely generated A-module with* $\operatorname{proj\,dim}_A M < \infty$. *Then* $\operatorname{proj\,dim}_A M + \operatorname{depth}_A M = \operatorname{depth} A$.

Sketch of another proof of Theorem 2. Let $\mathcal{E}$ a vector bundle on $\mathbb{P}^n_k = \operatorname{Proj} S$, where $S = k[x_0, \cdots, x_n]$. Then we can take $\mathcal{E} = \widetilde{M}$, where M is a Cohen-Macaulay graded S-module. Hilbert Syzygy Theorem implies $\operatorname{proj\,dim}_S M < \infty$. By Auslander-Buchsbaum theorem (graded case), $\operatorname{proj\,dim}_S M = 0$, that is, M is graded free. $\square$

This observation arises the following question.

Question 4. *Find splitting criteria of vector bundles on $\mathbb{P}^m_k \times \mathbb{P}^n_k$.*

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Facts 5. *Let $\mathcal{E}$ be a vector bundle on $X = \mathbb{P}^m_k \times \mathbb{P}^n_k$. Then there are cohomological criteria for:*

(1) *$\mathcal{E}$ is a direct sum of $\mathcal{O}_X$, $\mathcal{O}_X(0,1)$ and $\mathcal{O}_X(1,0)$ twisted by line bundles $\mathcal{O}_X(\ell,\ell)$, that is, $\mathcal{E} \cong (\oplus_\ell \mathcal{O}_X(\ell,\ell)) \oplus (\oplus_{\ell'} \mathcal{O}_X(\ell', 1+\ell')) \oplus (\oplus_{\ell''} \mathcal{O}_X(1+\ell'', \ell''))$, see* [1].

(2) *$\mathcal{E}$ is a direct sum of line bundles of the form $\mathcal{O}_X(\ell_1,\ell_2)$ with $-r_1 \le \ell_1 - \ell_2 \le r_2$, see* [8].

(3) *$\mathcal{E}$ is a direct sum of line bundles of $\mathcal{O}_X$, $\mathcal{O}_X(0,1)$, $\mathcal{O}_X(1,0)$, $p_1^*\mathcal{O}_{\mathbb{P}^m} \otimes p_2^*\Omega^a_{\mathbb{P}^n}(a+1)$ and $p_1^*\Omega^a_{\mathbb{P}^m}(a+1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^m}$, where $0 \le a \le n-1$ twisted by line bundles of the form $\mathcal{O}_X(\ell,\ell)$, see* [1].

If $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^n_k}(\ell)$, $\mathcal{E}$ is ACM. If $\mathcal{E} \cong \Omega^i_{\mathbb{P}^n_k}(\ell)$, $\mathcal{E}$ is Buchsbaum. Now we are going to study the Buchsbaum case for splitting criteria by giving the definition and basic properties of Buchsbaum modules, see [10].

Definition 6. Let $(R, \mathfrak{m})$ be a Noetherian local ring. A finitely generated R-module M is a Buchsbaum module if $\mathit{length}_R(M/\mathfrak{q}M) - e(\mathfrak{q}; M)$ is independent of the choice of a parameter ideal $\mathfrak{q}$ for M.

Definition 7. Let $R = k[x_0, \cdots, x_n]$ be a polynomial ring over k with standard grading and $\mathfrak{m} = R_+$. A finitely generated graded R-module M is called a graded Buchsbaum module if $M_\mathfrak{m}$ is a Buchsbaum $R_\mathfrak{m}$-module.

Proposition 8. *Let $R = k[x_0, \cdots, x_n]$ be a polynomial ring over k with standard grading and $\mathfrak{m} = R_+$. Let M be a finitely generated graded R-module of $\dim M = d+1$. The following conditions are equivalent:*

(1) *M is a graded Buchsbaum R-module.*

(2) *For any homogeneous (linear) system of parameters $z_0, \cdots, z_d$ of R-module M, $[(z_0, \cdots, z_{i-1})M : z_i] = [(z_0, \cdots, z_{i-1})M : \mathfrak{m}]$ holds for $i = 0, \cdots, d$.*

(3) *For any homogeneous (linear) system of parameters $z_0, \cdots, z_d$ of R-module M, $\mathfrak{m}\mathrm{H}^i_\mathfrak{m}(M/(z_j, \cdots, z_d)M) = 0$ holds for $j = 0, \cdots, d+1$ and $0 \le i \le j-1$.*

(4) *Canonical maps $\mathrm{Ext}^i_R(k, M) \to \mathrm{H}^i_\mathfrak{m}(M)$ are surjective for $0 \le i \le d$.*

For vector bundles we use the following definition.

Definition 9. Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^n_k$.

(1) $\mathcal{E}$ is called ACM if $\mathrm{H}^i_*(\mathcal{E}) = 0$, $1 \le i \le n-1$.

(2) $\mathcal{E}$ is called Buchsbaum if for any r-plane with $1 \le r \le n$ we have $\mathfrak{m}\mathrm{H}^i_*(\mathcal{E}|_L) = 0$ for $i = 1, \cdots, r-1$.

(3) $\mathcal{E}$ is called quasi-Buchsbaum if $\mathfrak{m}\mathrm{H}^i_*(\mathcal{E}) = 0$, $1 \le i \le n-1$.

Let us state surprising results on the structure of Buchsbaum bundles, which has been proved by Goto [3] and Chang [2] in different ways.

Theorem 10. *Let $\mathcal{E}$ be a Buchsbaum vector bundle on $\mathbb{P}^n_k$. Then $\mathcal{E}$ is isomorphic to a direct sum of vector bundles of the form $\Omega^i_{\mathbb{P}^n_k}(\ell)$.*

In order to generalize to vector bundles on multiprojective space, we will study the following questions.

Question 11. *Let $\mathcal{E}$ be a vector bundle on $X = \mathbb{P}_k^m \times \mathbb{P}_k^n$. When is $\mathcal{E}$ isomorphic to a direct sum of vector bundles of the form $p_1^*\Omega_{\mathbb{P}^m}^i(a) \otimes p_2^*\Omega_{\mathbb{P}^n}^j(b)$?*

Question 12. *Let $\mathcal{E} = p_1^*\Omega_{\mathbb{P}^m}^i(a) \otimes p_2^*\Omega_{\mathbb{P}^n}^j(b)$ on $X = \mathbb{P}_k^m \times \mathbb{P}_k^n$.*

(1) *When is $\mathcal{E}$ ACM?*
(2) *When is $\mathcal{E}$ Buchsbaum?*
(3) *When is $\mathcal{E}$ quasi-Buchsbaum?*

The ACM and quasi-Buchsbaum property is described in terms of cohomologies of $\mathcal{E}$. It is an easy calculation by the Künneth formula. From now on we will study the Buchsbaum property of $\mathcal{E}$. The following is our main result coming from Theorem 18. The rest of the paper gives an outline of the proof. The details will be written in [9].

Theorem 13. *On the Segre product $\mathbb{P}_k^m \times \mathbb{P}_k^n$ let us take a vector bundle $\mathcal{E} = p_1^*\Omega_{\mathbb{P}^m}^i(a) \otimes p_2^*\Omega_{\mathbb{P}^n}^j(b)$ for $1 \le i \le m$ and $1 \le j \le n$, where p_1 and p_2 are projections.*

(1) *In case either $(i,j) = (m,n)$, or $i < m$ and $j < n$, $\mathcal{E}$ is Buchsbaum if and only if $-n+j-i-1 \le b-a \le m+j-i+1$.*
(2) *In case $i < m$ and $j = n$, $\mathcal{E}$ is Buchsbaum if and only if $-i \le b-a \le m-i+n+1$.*

Remark 14. Let R and S be the polynomial rings over a field k. Let M be a Buchsbaum R-module with $\operatorname{depth} M \ge 2$. Let N be a Buchsbaum S-module with $\operatorname{depth} N \ge 2$. Cohomological data $\dim_k[\mathrm{H}_{\mathfrak{m}}^i(M)]_\ell$ and $\dim_k[\mathrm{H}_{\mathfrak{m}}^i(N)]_\ell$ give structure of direct sums of syzygies over appropriate polynomial subrings $R' \subseteq R$ and $S' \subseteq S$. Then the Cohen-Macaulay and Buchsbaum property of the Segre product $M\#N$ of M and N are described in terms of the data by the Theorem 13.

Now let us describe spectral sequence theory for Buchsbaum modules according to [6, 7].

Let $R = k[x_0, \cdots, x_n]$ be the polynomial ring over $\bar{k} = k$. Let M be a finitely generated graded R-module of $\operatorname{depth} M \ge 2$. Let $0 \to M \to I^\bullet$ be the minimal injective resolution of M in the category of graded R-modules, constructed as in [4]. We set $I^i = {}'I^i \oplus {}''I^i$, where $\operatorname{Ass}_R({}'I^i) = \{\mathfrak{m}\}$ and $\mathfrak{m} \notin \operatorname{Ass}_R({}''I^i)$. Let us put $\bar{I}^\bullet = (0 \to M \xrightarrow{\varepsilon} {}''I^\bullet[-1])$. Then $\bar{I}^\bullet \cong \mathbb{R}\Gamma_{\mathfrak{m}}(M)$. Let $K_\bullet$ be a Koszul complex $K_\bullet((x_0, \cdots, x_n); R)$. Let us consider a double complex $B^{\bullet\bullet} = \operatorname{Hom}_R(K_\bullet, \bar{I}^\bullet)$, that is, $B^{p,q} = \operatorname{Hom}_R(K_p, \bar{I}^q)$. Take filtrations ${}'F_t(B^{\bullet\bullet}) = \sum_{p \ge t} B^{p,q}$ and ${}''F_t(B^{\bullet\bullet}) = \sum_{q \ge t} B^{p,q}$. The filtrations ${}'F_t$ and ${}''F_t$ give spectral sequences $\{{}'F_r^{p,q}\}$ and $\{{}''F_r^{p,q}\}$ respectively:

$$\left\{\begin{array}{ll} {}'F_1^{p,q} = \operatorname{Ker} d''^{p,q}/\operatorname{Im} d''^{p,q-1} \quad \Rightarrow & \\ & H^{p+q}(B^{\bullet\bullet}). \\ {}''F_1^{p,q} = \operatorname{Ker} d'^{p,q}/\operatorname{Im} d'^{p-1,q} \quad \Rightarrow & \end{array}\right.$$

Note that ${}'\mathrm{F}_1^{p,q} \cong \mathrm{Hom}_R(K_p, \mathrm{H}^q_{\mathfrak{m}}(M))$ and ${}''\mathrm{F}_1^{p,0} \cong \mathrm{Ext}^p_R(k, M)$ and ${}''\mathrm{F}_1^{p,q} = 0$ for $q \neq 0$.

Proposition 15. *Under the above conditions the following conditions are equivalent:*

(1) *M is a graded Buchsbaum R-module.*
(2) *$d_r^{p,q} : \mathrm{E}_r^{p,q} \to \mathrm{E}_r^{p+r,q-r+1}$ is a zero map for all p, q and r with $q \leq d$ and $r \geq 1$.*
(3) *$d_r^{0,q} : \mathrm{E}_r^{0,q} \to \mathrm{E}_r^{r,q-r+1}$ is a zero map for all q and r with $q \leq d$ and $r \geq 1$.*

Now we will describe the behaviour of the syzygy modules in the spectral sequence. Syzygy modules are typical Buchsbaum modules.

Let R be the polynomial ring $k[x_0, \cdots, x_n]$. Let L_i be a graded free R-module $R(-i)^{\oplus e_i}$ of rank $e_i = {}_{n+1}\mathrm{C}_i$, $i = 0, \cdots, n+1$. Let us consider the Koszul resolution of a graded R-module k: $0 \to L_{n+1} \to L_n \to \cdots \to L_1 \to L_0 \to k \to 0$. Let us take $E_i = \mathrm{Coker}\,(L_{i+1} \to L_i)$, $i = 1, \cdots, n$ and $E_{n+1} = L_{n+1}$. Then the i-th syzygy moduleE_i is a Buchsbaum module such as $\mathrm{H}^i_{\mathfrak{m}}(E_i) \cong k$ for $i = 1, \cdots, n$, $\mathrm{H}^q_{\mathfrak{m}}(E_i) = 0$ for $q \neq i, n+1$ and E_i is generated by elements of degree i.

Lemma 16. *Under the above notation we have*

(1) $\mathrm{Soc}\,\mathrm{H}^{n+1}_{\mathfrak{m}}(E_i)(= [0 : \mathfrak{m}]_{\mathrm{H}^{n+1}_{\mathfrak{m}}(E_i)}) \cong k(n-i+2)^{\oplus e_i}$ *for* $1 \leq i \leq n$
(2) $\mathrm{Soc}\,\mathrm{H}^{n+1}_{\mathfrak{m}}(E_{n+1}) \cong k$
(3) $\mathrm{Ext}^{i+j}_R(k, E_i) \cong k(j)^{\oplus e_j}$ *for* $1 \leq i \leq n+1$ *and* $0 \leq i+j \leq n+1$, *where* $k(j)^{\oplus e_j} = 0$ *for* $j < 0$.

Let us take $M = E_i$. Note $\mathrm{depth}_R E_i = i \geq 2$ and $\widetilde{M} = \Omega^{i-1}_{\mathbb{P}^n_k}$ on $\mathbb{P}^n_k = \mathrm{Proj}\,R$. A complex $\bar{I}^\bullet = (0 \to M \xrightarrow{\varepsilon} {}''I^\bullet[-1])$ gives $\bar{I}^\bullet \cong \mathbb{R}\Gamma_{\mathfrak{m}}(M)$. Then we have a spectral sequence $\{\mathrm{E}_r^{p,q}\}$ with

$$\mathrm{E}_1^{p,q} = \mathrm{Hom}_R(K_p, \mathrm{H}^q_{\mathfrak{m}}(M)) \Rightarrow \mathrm{H}^{p+q}(B^{\bullet\bullet}) = \mathrm{Ext}^{p+q}_R(k, M).$$

Lemma 17. *Let R be the polynomial ring $k[x_0, \cdots, x_n]$ over a field k. Let M be the i-th syzygy module E_i for $2 \leq i \leq n$, that is, $\Omega^{i-1}_{\mathbb{P}^n_k} = \widetilde{M}$ on $\mathbb{P}^n_k = \mathrm{Proj}\,R$. In the spectral sequence $\{\mathrm{E}_r^{p,q}\}$ with $\mathrm{E}_1^{p,q} = \mathrm{Hom}_R(K_p, \mathrm{H}^q_{\mathfrak{m}}(M))$ $\Rightarrow \mathrm{H}^{p+q}(B^{\bullet\bullet}) \cong \mathrm{Ext}^{p+q}_R(k, M)$,*

- $\varphi_i : \mathrm{H}^i(B^{\bullet\bullet}) = \mathrm{Ext}^i_R(k, M) \to \mathrm{E}_\infty^{0,i} = \mathrm{E}_1^{0,i} = \mathrm{H}^i_{\mathfrak{m}}(M)$
- $\psi_i : \mathrm{E}^{0,n+1}_{n-i+2} = [0 : \mathfrak{m}]_{\mathrm{H}^{n+1}_{\mathfrak{m}}(M)} \to \mathrm{E}^{n-i+2,i}_{n-i+2} = \mathrm{H}^i_{\mathfrak{m}}(M)(n-i+2)^{\oplus e_i}$

are isomorphisms.

Let $R = k[x_0, \cdots, x_m]$ and $S = k[y_0, \cdots, y_n]$ be the polynomial rings. A Segre product of R and S is defined as $R\#S = \oplus_{\ell \in \mathbb{Z}}(R_\ell \otimes_k S_\ell)$, which is a graded ring. $\mathrm{Proj}\,R\#S \cong \mathrm{Proj}\,R \times \mathrm{Proj}\,S$. For a graded R-module M and a graded S-module N, a Segre product of M and N is defined as $M\#N = \oplus_{\ell \in \mathbb{Z}}(M_\ell \otimes_k N_\ell)$, which is a graded $(R\#S)$-module. Let $I^\bullet$ be the minimal injective resolution of a graded R-module M. Let $J^\bullet$ be the minimal injective resolution of a graded S-module N. We take $I^i = {}'I^i \oplus {}''I^i$, where $\mathrm{Ass}_R({}'I^i) = \{\mathfrak{m}\}$, $\mathfrak{m} \notin \mathrm{Ass}_R({}''I^i)$ and $J^i = {}'J^i \oplus {}''J^i$, where $\mathrm{Ass}_R({}'I^i) = \{\mathfrak{m}\}$, $\mathfrak{m} \notin \mathrm{Ass}_R({}''I^i)$. Then let us put $\bar{I}^\bullet = (0 \to M \xrightarrow{\varepsilon} {}''I^\bullet[-1])$ and $\bar{J}^\bullet = (0 \to N \xrightarrow{\varepsilon} {}''J^\bullet[-1])$. Now we put $T = R\#S$ and $\mathfrak{m} = T_+$. Let us take a complex $W^\bullet = (0 \to M\#N \to ({}''I^\bullet \# {}''J^\bullet)[-1])$, that is, $W^\ell = \oplus_{i+j=\ell-1}({}''I^i \# {}''J^j)$ for $\ell \geq 1$. Then we have $W^\bullet \cong \mathbb{R}\Gamma_{\mathfrak{m}}(M\#N)$.

Thanks to the construction of a resolution $W^\bullet$, we will describe explicitly cycle elements of the cohomology modules of the Segre product, and we will obtain the following result.

Theorem 18. *Let $R = k[x_0, \cdots, x_m]$ and $S = k[y_0, \cdots, y_n]$ be the polynomial rings over a field k. Let E_i be an i-th syzygy R-module for $2 \leq i \leq m+1$ and F_j be a j-th syzygy S-module for $2 \leq j \leq n+1$. Put $M = E_i(a)$ and $N = F_j(b)$ for $a, b \in \mathbb{Z}$.*

(1) *In case $(i,j) = (m+1, n+1)$, $M\#N$ is Buchsbaum if and only if $\mathfrak{m}(M\#\mathrm{H}^{n+1}_{\mathfrak{m}}(N)) = 0$ and $\mathfrak{m}(\mathrm{H}^{m+1}_{\mathfrak{m}}(M)\#N) = 0$.*
(2) *In case $(i,j) \neq (m+1, n+1)$, $M\#N$ is Buchsbaum if and only if $M\#\mathrm{H}^{n+1}_{\mathfrak{m}}(N) = 0$ and $\mathrm{H}^{m+1}_{\mathfrak{m}}(M)\#N = 0$.*

Finally we end with posing the following problem.

Problem 19. *Let $\mathcal{E}$ be a vector bundle on $X = \mathbb{P}^m_k \times \mathbb{P}^n_k$. Give cohomological criteria whether $\mathcal{E}$ is a direct sum of line bundles of $p_1^*\Omega^i_{\mathbb{P}^m}(a) \otimes p_2^*\Omega^j_{\mathbb{P}^n}(b)$ with some restriction for i, j, a and b.*

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