CASTELNUOVO-MUMFORD REGULARITY FOR PROJECTIVE VARIETIES

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1. Castelnuovo-Mumford Regularity Basics

Let k be an algebraically closed field. Let $S = k[X_0, \cdots, X_N]$ be the polynomial ring over k. Let $\mathfrak{m} = S_+ = (X_0, \cdots, X_n)$ be the homogeneous maximal ideal of S. Let $\mathbb{P}_k^N = \operatorname{Proj} S$ be the projective N-space.

Definition 1.1 ([21])**.** Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}_k^N$. Let m be an integer. The coherent sheaf $\mathcal{F}$ is said to be m-regular if

$$\mathrm{H}^i(\mathbb{P}_k^N, \mathcal{F}(m-i)) = 0$$

for $i \geq 1$. This condition is equivalent to saying that

$$\mathrm{H}^i(\mathbb{P}_k^N, \mathcal{F}(j)) = 0$$

for all i and j with $i \geq 1$ and $i + j \geq m$

Proposition 1.2 ([21])**.** *If $\mathcal{F}$ is m-regular, then $\mathcal{F}(m)$ is generated by global sections.*

Remark 1.3. Let $(X, \mathcal{L})$ be a polarized variety such that $\mathcal{L}$ is generated by global sections. A coherent sheaf $\mathcal{F}$ on X is said to be m-regular if

$$\mathrm{H}^i(\mathbb{P}_k^N, \mathcal{F} \otimes \mathcal{L}^{m-i}) = 0$$

for $i \geq 1$. This condition is equivalent to saying that

$$\mathrm{H}^i(\mathbb{P}_k^N, \mathcal{F} \otimes \mathcal{L}^j) = 0$$

for all i and j with $i \geq 1$ and $i+j \geq m$. If $\mathcal{F}$ is m-regular, then $\mathcal{F} \otimes \mathcal{L}^m$ is generated by global sections.

Definition 1.4. For a coherent sheaf $\mathcal{F}$, $\operatorname{reg} \mathcal{F}$ is defined as the least integer m such that $\mathcal{F}$ is m-regular. We call $\operatorname{reg} \mathcal{F}$ as the Castelnuovo-Mumford regularity of $\mathcal{F}$. For a projective scheme $X \subseteq \mathbb{P}_k^N$, $\operatorname{reg} X$ is defined as $\operatorname{reg} \mathcal{I}_X$, where $\mathcal{I}_X$ is the ideal sheaf of X, and is called as the Castelnuovo-Mumford regularity of X.

Let $I_X = \Gamma_* \mathcal{I}_X = \oplus_{\ell \in \mathbb{Z}} \Gamma(\mathbb{P}_k^N, \mathcal{I}_X(\ell))$ be the defining ideal of X. Let $R = S/I_X$ be the coordinate ring of X. Then we have the minimal free resolution of I_X as graded S-module

$$0 \to F_s \to \cdots \to F_1 \to F_0 \to I_X,$$

where $F_i = \oplus_j S(-\alpha_{ij})$.

Proposition 1.5 ([4, 8])**.** *Under the above condition, we have*

$$\operatorname{reg} X = \max_{i,j}\{\alpha_{ij} - i\}.$$

Proof. "$\leq$" is an easy consequence of the free resolution of cohomologies. "$\geq$" follows from (1.2). □

The Castelnuovo-Mumford regularity measures a complexity of the defining ideal of projective scheme. The purpose of our study is to describe the Castelnuovo-Mumford regularity in terms of the basic invariants of projective scheme.

Remark 1.6. We always have $\operatorname{reg} X \geq 1$. If X is nondegenerate, that is, I_X is generated by elements of degree ≥ 2, then $\operatorname{reg} X \geq 2$.

Conjecture 1.7 (Regularity Conjecture [8])**.** *Let $X \subseteq \mathbb{P}^N_k$ be a nondegenerate projective variety. Then we have*

$$\operatorname{reg} X \leq \deg X - \operatorname{codim} X + 1.$$

Remark 1.8. The conjecture can be extended for a nondegenerate reduced scheme which is connected in codimension 1. However, the hypotheses "irreducible" and "reduced" are indispensable. In fact, a nondegenerate double line in $\mathbb{P}^3_k$ is irreducible, but the r.h.s. of the inequality is 1. Moreover, a skew line in $\mathbb{P}^3_k$ is nondegenerate and reduced, but the r.h.s. is also 1. If you prefer a version of polarized variety, the conjecture is described as

$$\operatorname{reg}(X, \mathcal{L}) \leq \Delta(X, \mathcal{L}) + 2$$

for a nondegenerate polarized variety $(X, \mathcal{L})$ such that $\mathcal{L}$ is generated by global sections.

The Regularity Conjecture is proved for $\dim X = 1$ by Gruson-Lazarsfeld-Peskine [10], and is proved if X is a smooth surface and $\operatorname{char} k = 0$ by Lazarsfeld [15]. For higher dimensional case, an weaker bound is proved under the assumption that X is smooth and $k = \mathbb{C}$. For $\dim X = 3$, $\operatorname{reg} X \leq \deg X - \operatorname{codim} X + 2$ is proved by Kwak [14]. For $\dim X = n \leq 14$, $\operatorname{reg} X \leq \deg X - \operatorname{codim} X + (n-2)(n-1)/2$ is proved by Chiantini-Chiarli-Greco [5].

2. Gruson-Lazarsfeld-Peskine Theorem

First of all, we state the Gruson-Lazarsfeld-Peskine Theorem, (2.1) and (2.2) for projective curves.

Theorem 2.1. *Let $C \subseteq \mathbb{P}^N_k$ be a nondegenerate projective curve of degree d.*

$$\operatorname{reg} C \leq d + 2 - N$$

Theorem 2.2. *Let $C \subseteq \mathbb{P}^N_k$ be a nondegenerate projective curve of degree d. If $g = p_g(C) \geq 1$, then $\operatorname{reg} C \leq d + 1 - N$ unless C is a smooth elliptic normal curve.*

Remark 2.3. If $\operatorname{reg} C \leq n$, then C has no $(n+1)$-secant lines by Bezout theorem.

Theorem 2.1 follows immediately from (2.4) and (2.5). In this section, we will describe a sketch of the proof of (2.4).

Lemma 2.4. *Let $p : \tilde{C} \to C \subseteq \mathbb{P}^N_k$ be the normalization of C. Let $\mathcal{M} = p^*\Omega_{\mathbb{P}^N_k}(1)$. Assume $H^1(\tilde{C}, \wedge^2\mathcal{M} \otimes \mathcal{A}) = 0$ for some $\mathcal{A} \in \operatorname{Pic} \tilde{C}$. Then $\operatorname{reg} C \leq \mathrm{h}^0(\mathcal{A})$.*

Lemma 2.5. *Let* $p : \tilde{C} \to C \subseteq \mathbb{P}_k^N$. *Let* $d = \deg p^*\mathcal{O}_{\mathbb{P}_k^N}(1)$. *Then there exists an ample line bundle* $\mathcal{A}$ *such that* $\mathrm{h}^0(\mathcal{A}) = d + 2 - N$ *and* $\mathrm{h}^1(\wedge^2\mathcal{M} \otimes \mathcal{A}) = 0$.

Sketch of the proof of Lemma 2.4. Let $\mathcal{O}_{\tilde{C}}(1) = p^*\mathcal{O}_{\mathbb{P}_k^N}(1)$ and $V = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}_k^N}(1)) \subseteq \mathrm{H}^0(\mathcal{O}_{\tilde{C}}(1))$. Let $\pi : \tilde{C} \times \mathbb{P}_k^N \to \tilde{C}$ be the first projection, and let $f : \tilde{C} \times \mathbb{P}_k^N \to \mathbb{P}_k^N$ be the second projection. Let Γ be the graph of $p : \tilde{C} \to \mathbb{P}_k^N$. By using the exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \pi^*\mathcal{M} & \to & V \otimes \mathcal{O}_{\tilde{C}\times\mathbb{P}_k^N} & \to & \pi^*\mathcal{O}_{\tilde{C}}(1) & \to & 0 \\ & & & & \| & & & & \\ 0 & \to & f^*\Omega_{\mathbb{P}_k^N}(1) & \to & V \otimes \mathcal{O}_{\tilde{C}\times\mathbb{P}_k^N} & \to & f^*\mathcal{O}_{\tilde{C}}(1) & \to & 0, \end{array}$$

the graph $\Gamma(\subseteq \tilde{C} \times \mathbb{P}_k^N)$ is defined by a composite map $\pi^*\mathcal{M} \to f^*\mathcal{O}_C(1)$. Then we have the exact sequence

$$\pi^*\mathcal{M} \otimes f^*\mathcal{O}_{\mathbb{P}_k^N}(-1) \to \mathcal{O}_{\tilde{C}\times\mathbb{P}_k^N} \to \mathcal{O}_\Gamma \to 0.$$

After tensoring with $\pi^*\mathcal{A}$, we take the Koszul resolution

$$\pi^*(\wedge^2\mathcal{M}\otimes\mathcal{A})\otimes f^*\mathcal{O}_{\mathbb{P}_k^N}(-2) \to \pi^*(\mathcal{M}\otimes\mathcal{A})\otimes f^*\mathcal{O}_{\mathbb{P}_k^N}(-1) \to \pi^*\mathcal{A} \to \mathcal{O}_\Gamma\otimes\pi^*\mathcal{A} \to 0,$$

which gives the exact sequences

$$\pi^*(\wedge^2\mathcal{M} \otimes \mathcal{A}) \otimes f^*\mathcal{O}_{\mathbb{P}_k^N}(-2) \to \mathcal{F}_1 \to 0, \tag{1}$$

$$0 \to \mathcal{F}_1 \to \pi^*(\mathcal{M} \otimes \mathcal{A}) \otimes f^*\mathcal{O}_{\mathbb{P}_k^N}(-1) \to \mathcal{F}_0 \to 0 \tag{2}$$

and

$$0 \to \mathcal{F}_0 \to \pi^*\mathcal{A} \to \mathcal{O}_\Gamma \otimes \pi^*\mathcal{A} \to 0. \tag{3}$$

Note that $R^jf_* = 0$ for $j \geq 2$ and $R^jf_*((\pi^* \wedge^i \mathcal{M} \otimes \mathcal{A}) \otimes f^*\mathcal{O}_{\mathbb{P}_k^N}(-i)) = \mathrm{H}^j(\tilde{C}, \wedge^i\mathcal{M} \otimes \mathcal{A}) \otimes \mathcal{O}_{\mathbb{P}_k^N}(-i)$ by projection formula. The sequence (1) gives

$$\mathrm{H}^1(\wedge^2\mathcal{M} \otimes \mathcal{A}) \otimes \mathcal{O}_{\mathbb{P}_k^N}(-2) \to R^1f_*\mathcal{F}_1 \to 0,$$

and we have $R^1f_*\mathcal{F}_1 = 0$ by the assumption. Then the sequence (2) gives an exact sequence

$$0 \to \mathcal{F}_1 \to \mathrm{H}^0(\mathcal{M} \otimes \mathcal{A}) \otimes \mathcal{O}_{\mathbb{P}_k^N}(-1) \to f_*\mathcal{F}_0 \to 0 \tag{4}$$

and an isomorphism

$$\mathrm{H}^1(\mathcal{M} \otimes \mathcal{A}) \otimes \mathcal{O}_{\mathbb{P}_k^N}(-1) \cong R^1f_*\mathcal{F}_0,$$

which implies that $R^1f_*\mathcal{F}_0$ is locally free. Furthermore, the sequence (3) gives an exact sequence

$$0 \to f_*\mathcal{F}_0 \to \mathrm{H}^0(\mathcal{A}) \otimes \mathcal{O}_{\mathbb{P}_k^N} \to p_*\mathcal{A} \to R^1f_*\mathcal{F}_0 \to \mathrm{H}^1(\mathcal{A}) \otimes \mathcal{O}_{\mathbb{P}_k^N} \to 0.$$

Since a morphism from a torsion sheaf $p_*\mathcal{A}$ to a locally free sheaf $R^1f_*\mathcal{F}_0$ is zero, we have a short exact sequence

$$0 \to f_*\mathcal{F}_0 \to \mathrm{H}^0(\mathcal{A}) \otimes \mathcal{O}_{\mathbb{P}^N_k} \to p_*\mathcal{A} \to 0. \tag{5}$$

By (4) and (5), we have a exact sequence

$$\mathrm{H}^0(\mathcal{M} \otimes \mathcal{A}) \otimes \mathcal{O}_{\mathbb{P}^N_k}(-1) \to \mathrm{H}^0(\mathcal{A}) \otimes \mathcal{O}_{\mathbb{P}^N_k} \to p_*\mathcal{A} \to 0.$$

Let $\mathcal{J}(\subseteq \mathcal{O}_{\mathbb{P}^N_k})$ be the zeroth Fitting ideal of $p_*\mathcal{A}$, explicitly, $\mathcal{J}$ is the image of $\wedge^{n_0}u$, where $u : \mathrm{H}^0(\mathcal{M} \otimes \mathcal{A}) \otimes \mathcal{O}_{\mathbb{P}^N_k}(-1) \to \mathrm{H}^0(\mathcal{A}) \otimes \mathcal{O}_{\mathbb{P}^N_k}$ and $n_0 = \mathrm{h}^0(\mathcal{A})$, see, e.g., [6] for the definition of Fitting ideals. Since $\operatorname{Supp} p_*\mathcal{A} = C$, we see $\mathcal{J} \subseteq \mathcal{I}_C$. On the other hand, $\operatorname{Supp} \mathcal{I}_C/\mathcal{J}$ is finite. Hence we have only to show that $\mathcal{J}$ is n_0-regular. By taking the Eagon-Northcott complex of u, see (2.6), we have a complex

$$\cdots \to \mathcal{O}_{\mathbb{P}^N_k}(-n_0-2)^{\oplus} \to \mathcal{O}_{\mathbb{P}^N_k}(-n_0-1)^{\oplus} \to \mathcal{O}_{\mathbb{P}^N_k}(-n_0)^{\oplus} \xrightarrow{\varepsilon} \mathcal{J} \to 0$$

such that ε is surjective and the complex is exact away from C, which gives $\mathcal{J}$ is n_0-regular.

□

Proposition 2.6. *Let $\mathcal{E}$ and $\mathcal{F}$ be locally free sheaves of* $\operatorname{rank} \mathcal{E} = e$ *and* $\operatorname{rank} \mathcal{F} = f$ *on a scheme X. Let $u : \mathcal{E} \to \mathcal{F}$. Then there is a complex*

$$0 \to \wedge^e \mathcal{E} \otimes S^{e-f}(\mathcal{F}^*) \to \cdots \to \wedge^{f+1}\mathcal{E} \otimes S^1(\mathcal{F}^*) \to \wedge^f \mathcal{E} \to \wedge^f \mathcal{F} \to 0,$$

which is called as the Eagon-Northcott complex. If $u : \mathcal{E} \to \mathcal{F}$ is surjective, then the complex is exact.

3. Generic Projection and Regularity Conjecture

In this section, we describe the higher dimensional case for the regularity conjecture. The following theorem extends the result of Kwak for 3-fold [14].

Theorem 3.1. *([5]) Let X be a nondegenerate smooth projective variety of $\mathbb{P}^N_{\mathbb{C}}$. If $n = \dim X \leq 14$, then* $\operatorname{reg} X \leq \deg X - \operatorname{codim} X + 1 + (n-2)(n-1)/2$.

We will describe an idea of the proof of (3.1). Let $p : X(\subseteq \mathbb{P}^N_{\mathbb{C}}) \to \mathbb{P}^{n+1}_{\mathbb{C}}$ be a generic projection. The proof consists of (3.2), (3.4) and (3.5).

Lemma 3.2. *Let $\mathcal{F} = \mathcal{G} \oplus \mathcal{O}_{\mathbb{P}^{n+1}_{\mathbb{C}}}(-3) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{n+1}_{\mathbb{C}}}(-n)$. If there is a surjective morphism $\mathcal{F} \to p_*\mathcal{O}_X$, then* $\operatorname{reg} X \leq d - N + n + 1 + (n-1)(n-2)/2$.

The proof of (3.2) proceeds as in Lazarsfeld [15].

Definition 3.3. Let $p : X(\subseteq \mathbb{P}^N_{\mathbb{C}}) \to \mathbb{P}^{n+1}_{\mathbb{C}}$ be a projection. Let $S_j = \{z \in \mathbb{P}^{n+1}_{\mathbb{C}} | \deg p^{-1}(z) = j\}$. The projection p is said to be good if $\dim S_j \leq \max\{-1, n-j+1\}$ for all j.

Lemma 3.4. *([5, (2.4)]) If $p : X(\subseteq \mathbb{P}^N_{\mathbb{C}}) \to \mathbb{P}^{n+1}_{\mathbb{C}}$ is good, there exists a surjective morphism $\mathcal{F} \to p_*\mathcal{O}_X$.*

The result of Kwak is extended to the higher dimensional case thanks to (3.4).

Lemma 3.5. *(Mather's theory [16]) If $n \leq 14$, then p is good.*

4. Uniform Position Principle, Socle Lemma, and Castelnuovo-Mumford Regularity

Let C be a nondegenerate projective curve of $\mathbb{P}_k^{N+1}$. Let H be a generic hyplerplane and $X = C \cap H \subseteq H \cong \mathbb{P}_k^N$. In this section, we will study a bound $\operatorname{reg} X \leq \lceil (\deg X - 1)/N \rceil + 1$.

Definition 4.1. Let $X (\subseteq \mathbb{P}_k^N)$ be a reduced zero-dimensional scheme such that X spans $\mathbb{P}_k^N$. The zero-dimensional scheme X is said to be in uniform position if the Hilbert function of Z is described as $\mathrm{H}_Z(t) = \min\{\mathrm{H}_X(t), \deg Z\}$ for any subscheme Z of X. This condition is equivalent to saying that for any subschemes Z_1 and Z_2 with $\deg Z_1 = \deg Z_2$, $\mathrm{h}^0(\mathcal{I}_{Z_1}(\ell)) = \mathrm{h}^0(\mathcal{I}_{Z_2}(\ell))$ for all $\ell \in \mathbb{Z}$. The zero-dimensional scheme X is said to be in linear general position if any $N+1$ points of X span $\mathbb{P}_k^N$. The zero-dimensional scheme X is said to be in linear semi-uniform position if there are integers $v(i, X)$, simply written as $v(i)$, $0 \leq i \leq N$ such that every i-plane L in $\mathbb{P}_k^N$ spanned by linearly independent $i+1$ points of X contains exactly $v(i)$ points of X.

Remark 4.2. Under the condition, we note that "uniform position" implies "linear general position", see [13, (4.3)], and "linear general position" implies "linear semi-uniform position".

Remark 4.3. A generic hyperplane section of a nondegenerate projective curve is in linear semi-uniform position, see [2], and in uniform position if $\operatorname{char} k = 0$, see [1].

Definition 4.4. Let R be the coordinate ring of a zero-dimensional scheme $X \subseteq \mathbb{P}_k^N$. Let $\underline{h} = \underline{h}(X) = (h_0, \cdots, h_s)$ be the h-vector of $X \subseteq \mathbb{P}_k^N$, where $h_i = \dim_k [R]_i - \dim_k [R]_{i-1}$ and s is the largest integer such that $h_s \neq 0$.

Remark 4.5. Under the above condition, we have $h_0 = 1$, $h_1 = N$, and $h_0 + \cdots + h_s = \deg X$. Let $t = \min\{t | \Gamma(\mathbb{P}_k^N, \mathcal{O}_{\mathbb{P}_k^N}(t)) \to \Gamma(X, \mathcal{O}_X(t)) \,\text{is surjective}\}$. Then we have $\operatorname{reg} X = t + 1 = s + 1$

Proposition 4.6. *Let C be a nondegenerate projective curve of $\mathbb{P}_k^{N+1}$ over an algebraically closed field k. Let H be a generic hyplerplane and $X = C \cap H \subseteq H \cong \mathbb{P}_k^N$. Let $\underline{h} = \underline{h}(X) = (h_0, \cdots, h_s)$ be the h-vector of $X \subseteq \mathbb{P}_k^N$.*
(i) If $\operatorname{char} k = 0$, then $h_i \geq h_1$ for $i = 1, \cdots, s-1$.
(ii) If $\operatorname{char} k > 0$, then $h_1 + \cdots + h_i \geq i h_1$ for $i = 1, \cdots, s-1$.

(i) is an easy consequence of Uniform Position Lemma, see, e.g. [1]. Also, [13, Section 4] is a good reference. (ii) follows from [2].

Proposition 4.7. *Let X be a generic hyperplane section of a nondegenerate projective curve. Then*

$$\operatorname{reg} X \leq \left\lceil \frac{\deg X - 1}{\operatorname{codim} X} \right\rceil + 1.$$

Now we will give two proofs of (4.7). The first one uses the classical Castelnuovo method, which works only for the case $\operatorname{char}(k) = 0$. The second one uses (4.6) for any characteristic case.

Proof. For $\operatorname{char} k = 0$, we need to show that $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}_k^N}(\ell)) \to \mathrm{H}^0(\mathcal{O}_X(\ell))$ is surjective, where $\ell = \lceil (d-1)/N \rceil$. Let P be a closed point of X. Then we put $X \backslash \{P\} = \{P_{1,1}, \cdots, P_{1,N}, P_{2,1}, \cdots, P_{2,N}, \cdots, P_{\ell-1,1}, \cdots, P_{\ell-1,N}, P_{\ell,1}, \cdots, P_{\ell,m}\}$, where $m = d-1-N(\ell-1)$. Since X is in linear general position, we can take ℓ hyperplanes $H_1, \cdots, H_\ell$ of $\mathbb{P}_k^N$ such that $X \cap H_i = \{P_{i1}, \cdots, P_{iN}\}$ for $i = 1, \cdots, \ell-1$ and $X \cap H_\ell = \{P_{\ell 1}, \cdots, P_{\ell m}\}$. Thus we have $X \cap (H_1 \cup \cdots \cup H_\ell) = X \backslash \{P\}$, which implies the assertion.

For any characteristic case, let R be the coordinate ring of a zero-dimensional scheme $X \subseteq \mathbb{P}_k^N$. By (4.6), we have $h_1 + \cdots + h_i \geq i h_i$ for all $i = 1, \cdots, s-1$, that is, $\mathrm{H}_X(t) \geq \min\{\deg(X), tN+1\}$ Since $\deg(X) = h_0 + \cdots + h_s$ and $\mathrm{codim}(X) = h_1 = N$, we obtain $\lceil (\deg(X)-1)/\mathrm{codim}\,(X) \rceil = \lceil (h_1 + \cdots + h_s)/h_1 \rceil \geq s$. Hence the assertion is proved.
□

Now we will study Castelnuovo-type bounds on the regularity for higher dimensional case. Let $X \subseteq \mathbb{P}_k^N$ be a nondegenerate projective variety of $\dim X = n$. Let H be a generic hyperplane.

Remark 4.8. Under the above condition, we have $\mathrm{reg}\,(X \cap H) \geq \mathrm{reg}\, X$. If X is ACM, i.e., the coordinate ring R is Cohen-Macaulay, then $\mathrm{reg}\,(X \cap H) = \mathrm{reg}\, X$. More generally, if X is arithmetically Buchabaum, i.e., R is Buchsbaum, then $\mathrm{reg}\,(X \cap H) = \mathrm{reg}\, X$, see [23].

Proposition 4.9 ([11, 23]). *For a nondegenerate progective variety $X \subseteq \mathbb{P}_k^N$, if X is arithmetically Buchsbaum, then*

$$\mathrm{reg}\, X \leq \left\lceil \frac{\deg X - 1}{\mathrm{codim}\, X} \right\rceil + 1.$$

We will introduce an invariant evaluating the intermediate colomologies of the projective varieties. Let $X \subseteq \mathbb{P}_k^N$ be a projective scheme. A graded S-module $\mathrm{M}^i(X) = \oplus_{\ell \in \mathbb{Z}} \mathrm{H}^i(\mathbb{P}_K^N, \mathcal{I}_X(\ell))$, is called the deficiency module of X, which is a generalization of the Hartshorne-Rao module for the curve case. Then we define $k(X)$ as the minimal nonnegative integer v such that $\mathfrak{m}^v \mathrm{M}^i(X) = 0$ for $1 \leq i \leq \dim(X)$, see [17], if there exists. If not, we put $k(X) = \infty$. It is known that the numbers $k(X)$ are invariant in a liaison class, see [17].

Further, we define $\bar{k}(X)$ as the maxmal number $k(X \cap V)$ for any complete intersection V of $\mathbb{P}_k^N$ with $\mathrm{codim}\,(X \cap V) = \mathrm{codim}\,(X) + \mathrm{codim}\,(V)$, possibly $V = \mathbb{P}_k^N$.

Remark 4.10. In general, $k(X) \leq \bar{k}(X)$. X is locally Cohen-Macaulay and equidimensional if and only if $\bar{k}(X) < \infty$. X is ACM if and only if $k(X) = 0$, equivalently, $\bar{k}(X) = 0$.

Conjecture 4.11 ([19]). *Let X be a nondegenerate irreducible reduced projective variety in $\mathbb{P}_K^N$ over an algebraically closed field k. Then we have*

$$\mathrm{reg}\,(X) \leq \left\lceil \frac{\deg(X) - 1}{\mathrm{codim}\,(X)} \right\rceil + \max\{\bar{k}(X), 1\}.$$

Furthermore, assume that $\deg(X)$ *is large enough. Then the equality holds only if X is a divisor on a variety of minimal degree.*

Theorem 4.12 ([18]). *Let X be a nondegenerate irreducible reduced projective variety in $\mathbb{P}_K^N$ over an algebraically closed field k. Assume that X is not ACM.*

Then we have

$$\operatorname{reg}(X) \le \left\lceil \frac{\deg(X)-1}{\operatorname{codim}(X)} \right\rceil + (\bar{k}(X)-1)\dim X + 1.$$

Furthermore, assume that $\deg(X)$ *is large enough. Then the equality holds only if* X *is a divisor on a rational ruled surface.*

Theorem 4.13. *Let* $C \subseteq \mathbb{P}^N_k$ *be a nondegenerate projective curve over an algebraically closed field of* $\operatorname{char} k = 0$. *Assume that* C *is not ACM. Then*

$$\operatorname{reg} C \le \left\lceil \frac{\deg C - 1}{\operatorname{codim} C} \right\rceil + k(C).$$

Assume that $\deg C \ge (\operatorname{codim} C)^2 + 2\operatorname{codim} C + 2$. *If the equality holds, then* C *lies on a rational ruled surface.*

Proof. Let $X = C \cap H$ be a generic hyperplane section. Let $m = \operatorname{reg} X$. Let $k = k(C)$. From the exact sequence

$$\mathrm{H}^1_*(\mathcal{I}_C)(-1) \xrightarrow{\cdot h} \mathrm{H}^1_*(\mathcal{I}_C) \to \mathrm{H}^1_*(\mathcal{I}_X) \to \mathrm{H}^2_*(\mathcal{I}_C)(-1) \xrightarrow{\cdot h} \mathrm{H}^2_*(\mathcal{I}_C)(-1),$$

where h is a defining equation of H, we have $\mathrm{h}^2(\mathcal{I}_{\mathcal{C}}(m-2)) \le \mathrm{h}^2(\mathcal{I}_{\mathcal{C}}(m-1)) \le \cdots \le 0$ and $\mathrm{H}^1(\mathcal{I}_C(m+k-1)) = h \cdot \mathrm{H}^1(\mathcal{I}_C(m+k-2)) = \cdots = h^k \cdot \mathrm{H}^1(\mathcal{I}_C(m+k-1)) = 0$. Hence we have

$$\operatorname{reg} C \le \operatorname{reg} X + k - 1 \le \left\lceil \frac{\deg C - 1}{\operatorname{codim} C} \right\rceil + k(C).$$

For the second part, we will use (4.14), which is a consequence of the theory of 1-generic matrices [7]. Let $(h_0, \cdots, h_s)$ be the h-vector of the one-dimensional graded ring R. In other words, we write $h_i = \dim_K(R_i) - \dim_K(R_{i-1})$ for all nonnegative integers i, and s for the maximal integer such that $h_s \neq 0$. Note that $h_0 = 1$, $h_1 = N$, $s = a(R) + 1$ and $\deg(X) = h_0 + \cdots + h_s$. Suppose that X does not lie on a rational normal curve. By (4.14), we have that $h_i \ge h_1 + 1$ for all $2 \le i \le s-2$, and $h_{s-1} \ge h_1$. Thus we have

$$\begin{aligned} \frac{\deg(X)-1}{N} &= \frac{h_1 + \cdots + h_s}{h_1} \\ &\ge 1 + \overbrace{\frac{N+1}{N} + \cdots + \frac{N+1}{N}}^{s-3} + 1 + \frac{h_s}{N} \\ &= a(R) + \frac{a(R) - 2 + h_s}{N} \\ &\ge a(R) + \frac{a(R)-1}{N}. \end{aligned}$$

Since $a(R) + 1 \ge (\deg(X) - 1)/N$, we see that $a(R) \le N + 1$. Hence we have

$$\deg(X) - 1 \le N(a(R) + 1) \le N(N+2),$$

which contradicts the hypothesis. Now let C be a nondegenerate projective curve.

Let $X = C \cap H$ be a generic hyperplane section. Since X is contained in a rational normal curve Z in $H (\cong \mathbb{P}^N_k)$. We have only to show there exists a surface Y containing C such that $Y \cap H = Z$. There is an isomorphism $\Gamma(\mathcal{I}_{Z/\mathbb{P}^N_k}(2)) \cong \Gamma(\mathcal{I}_{X/H}(2))$. Indeed, If there exists a hyperquadric Q such that $X \subseteq Q$ and $Z \not\subseteq$

Q, then $X \subseteq Z \cap Q$ by Bezout Theorem. On the other hand, $\Gamma(\mathcal{I}_{C/\mathbb{P}^N_k}(2)) \to \Gamma(\mathcal{I}_{X/H}(2))$ is surjective. Indeed, let K be the kernel of $\mathrm{H}^1_*\mathcal{I}_C(-1) \xrightarrow{\cdot h} \mathrm{H}^1_*\mathcal{I}_C$. From the exact sequence

$$\Gamma_*\mathcal{I}_C \to \Gamma_*\mathcal{I}_X \to \mathrm{H}^1_*\mathcal{I}_C(-1) \xrightarrow{\cdot h} \mathrm{H}^1_*\mathcal{I}_C \to \mathrm{H}^1_*\mathcal{I}_X,$$

we need to prove that $[K]_2 = 0$. By Socle Lemma (4.15), $a_-(K) > a_-(\mathrm{H}^1_*\mathcal{I}_X) \geq 2$.

Thus we see that Z is the intersection of the hyperquadrics containing X and that Y' is the intersection of the hyperquadrics of C. Since $Y' \cap H = Z$, there is an irreducible components of Y' such that $Y \cap H = Z$. □

Lemma 4.14. *(*[24, (2.3)]*) Assume that X is in uniform position. If X does not lie on a rational normal curve, then $h_i \geq h_1 + 1$ for $2 \leq i \leq s-2$.*

Example 1. There is a counterexample in case Let X a complete intersection of type $(2,2,4)$ in $\mathbb{P}^3_k$. In this case, $\operatorname{reg} X = 6$ and $\deg X = 16$, so $\operatorname{reg} X = \lceil(\deg X - 1)/\operatorname{codim} X\rceil + 1$. However, X does not lie on a rational normal curve. So we really need the condition on the degree $\deg(X) \geq N^2 + 2N + 2$.

Example 2. Let C be a smooth non-hyperelliptic curve of genus $g = rmp_g(C) \geq 5$. Let $C \subseteq \mathbb{P}^{g-1}_k$ be the canonical embedding. Then $\operatorname{reg} C = \lceil(\deg C - 1)/(g-2)\rceil + 1 = 4$. In this case, C is contained in a surface of minimal degree if and only if C is either trigonal or plane quintic.

Lemma 4.15 (Socle Lemma [12])**.** *Let $S = k[X_0, \cdots, X_N]$ be the polynomial ring over a field k of charateristic 0. For a graded S-module N, we define $a_-(N) = \min\{i | [N]_i \neq 0\}$. Let $M (\neq 0)$ be a finitely generated graded S-module. For a exact sequence of graded S-modules*

$$0 \to K \to M(-1) \xrightarrow{\cdot h} M \to C \to 0,$$

where $h \in S_1$ is a generic element. If $K \neq 0$, then $a_-(K) > a_-([0 : \mathfrak{m}]_C)$.

Corollary 4.16. *Let $C \subseteq \mathbb{P}^3_k$ be a space curve with maximal regularity. Assume that $\operatorname{char}(k) = 0$, $\deg C > 10$ and C is not ACM. Then C is a divisor of either type $(a, a+2)$ or $(a, a+3)$ on a smooth quadric surface.*

Proof. The assertion follows immediately from (4.13) □

5. Generic Hyperplane Section of Projective Curve in Positive Characteristic

In this section, we study the regularity bound of Castelnuovo-type for positive characteristic case. In Section 4, we show that if a generic hyperplane section of projective curve with its degree large enough has a maximal regularity, then the zero-dimensional scheme lie on a rational normal curve in characteristic zero case. We will describe how to extend this result to the positive characteristic case by the classical method of Castelnuovo. There is a relationship between the monodromy group of the projective curve and the configuration of the generic hyperplane section of the curve, as following Rathmann [22]. Let $C \subseteq \mathbb{P}^{N+1}_k$ and $X \subseteq \mathbb{P}^N_k$ be again a nondegenerate projective curve and its generic hyperplane section respectively. Let $M \subseteq C \times (\mathbb{P}^{N+1}_k)^*$ be the incidence correspondence parametrizing the pairs $(x, H) \in M$, that is, a point x of C and a hyperplane H of $\mathbb{P}^{N+1}_k$ such that x is contained in H. Since M is a $\mathbb{P}^N_k$-bundle over C via the first projection, M is irreducible and reduced. By Bertini's theorem, M is generically étale finite over $P = (\mathbb{P}_k^{N+1})^*$ via the second projection. Thus the function field $K(M)$ of M is separable finite over $K(P)$, in particular, $K(M)$ is a simple extension of $K(P)$. So we fix a splitting field Q for this simple extension. Let G_C be the Galois group $\mathrm{Gal}(Q/K(P))$. Then G_C is a subgroup of the full symmetric group S_d and is called the monodromy group of $C \subseteq \mathbb{P}_k^N$, where $d = \deg(C)$. The following is a basic result on the monodromy group of projective curve.

Proposition 5.1. *Let $X \subseteq \mathbb{P}_k^N$ be a generic hyperplane section of nondegenerate projective curve $C \subseteq \mathbb{P}_k^{N+1}$.*
(i) (See [1]). *If $\mathrm{char}(k) = 0$, then $G_C = S_d$.*
(ii) (See [22, (1.8)]). *If either $G_C = S_d$ or $G_C = A_d$, then X is in uniform position.*
(iii) (See [22, (1.6)]). *Let $1 \le t \le N+1$. G_C is t-transitive if and only if any t points of X is linearly independent.*

Proposition 5.2. (See [22, (2.5)]). *Let X be a generic hyperplane section in $\mathbb{P}^N$ of a nondegenerate projective curve C of $\mathbb{P}^{N+1}$ for $N \ge 3$. Let G_C be the monodromy group of C. If X is not in uniform position, then either of the following holds:*
(a) *$v(1) = 3$, and G_C is exactly 2-transitive.*
(b) *$v(1) = 2$, $v(2) \ge 4$, and G_C is exactly 3-transitive.*
(c) *$\deg(C) = 11, 12, 23$ or 24, and G_C is the Mathieu group M_{11}, M_{12}, M_{23}, M_{24} respectively. Moreover M_{11} and M_{23} are exactly 4-transitive and M_{12} and M_{24} are exactly 5-transitive.*

Theorem 5.3 ([3, 20])**.** *Let $X \subseteq \mathbb{P}_k^N$ be a generic hyperplane section of a nondegenerate projective curve for $N \ge 3$. Assume that X is not in uniform position and $\deg(X) \ge N^2 + 2N + 2$. Then we have $\mathrm{reg}\,(X) \le \lceil(\deg(X) - 1)/N\rceil$.*

What we have to prove is that $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}_k^N}(t)) \to \mathrm{H}^0(\mathcal{O}_X(t))$ is surjective, that is, $\mathrm{H}^1(\mathcal{I}_X(t)) = 0$, where $t = \lceil(\deg(X) - 1)/N\rceil - 1$.

Lemma 5.4. *Under the condition of (5.3), let $t = \lceil(\deg(X) - 1)/N\rceil - 1$. For any fixed point $P \in X$, there exists a (possibly reducible) hypersurface F of degree t in $\mathbb{P}_k^N$ such that $X \cap F = X \backslash \{P\}$.*

Proof. So we will prove for the case $N = 3$ by the classical method. Since $v(1) = 3$, we have $v(2) \ge 7$ and put $v = v(2)$. For a point P of X, we fix 2 points Q_1 and Q_2 in $X \backslash \{P\}$. Then we take different 2-planes $L_1, \cdots, L_a$ containing the line $\ell = \ell(Q_1, Q_2)$ spanned by Q_1 and Q_2 such that the union $\cup_{j=1}^a L_j$ covers X. We remark that $a \ge 3$. Since each 2-plane contains exactly v points of X and the line ℓ contains exactly 3 points of X, we see $d = a(v - 3) + 3$. We may assume that P is contained in L_a. Let $b = \lceil(v - 3)/2\rceil$. Since $(X \cap L_a) \backslash \{P, Q_1, Q_2\}$ consists of exactly $v - 3$ points, there are 2-planes $L_1', \cdots, L_b'$ such that $P \notin L_i'$ for $i = 1, \cdots, b$ and the union $\cup_{j=1}^b L_j'$ of 2-planes covers $(X \cap L_a) \backslash \{P, Q_1, Q_2\}$. By taking $F = (\cup_{i=1}^{a-1} L_i) \cup (\cup_{j=1}^b L_j')$, we have $(X \cap F) = X \backslash \{P\}$ and the degree of the union F of 2-planes is $a+b-1$. Thus we have only to show that $(a-1) + \lceil(v-3)/2\rceil \le \lceil((av - 3a + 3) - 1)/3\rceil - 1$. The inequality $(a-1) + (v-3)/2 \le (av - 3a + 2)/3 - 1$ is equivalent to saying that $(2a - 3)(v - 6) \ge 5$, which is easily shown for $v \ge 7$ and $a \ge 4$. Moreover, the case $v = 7$ and $a = 3$ satisfies $(a - 1) + \lceil(v - 3)/2\rceil = \lceil(av - 3a + 2)/3\rceil - 1$. Hence the assertion is proved. □

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References

[1] E. Arbarello, M. Cornalba, P.A. Griffiths and J. Harris, Geometry of algebraic curves I, Grundlehren der math. Wissenschaften 167, Springer, 1985.

[2] E. Ballico, On singular curves in positive characteristic, Math. Nachr. 141 (1989), 267 – 273.

[3] E. Ballico and C. Miyazaki, Generic hyperplane section of curves and an application to regularity bounds in positive characteristic, J. Pure Appl. Algebra 155 (2001), 93 – 103.

[4] D. Bayer and D. Mumford, What can be computed in algebraic geometry?, Computational algebraic geometry and commutative algebra (ed. D. Eisenbud and L. Robbiano), pp. 1 – 48, Cambridge University Press 1993.

[5] Chiantini, Chiarli and Greco, Bounding Castelnuovo-Mumford regularity for varieties with good general projections, J. Pure Appl. Algebra 152(2000), 57–64.

[6] D. Eisenbud, Commutative algebra, With a view toward algebraic geometry, GTM 150, Springer 1995.

[7] D. Eisenbud, The geometry of syzygies, A second course in commutative algebra and algebraic geometry, GTM 229, Springer, 2005

[8] D. Eisenbud and S. Goto, Linear free resolutions and minimal multiplicity, J. Algebra 88 (1984), 89 – 133.

[9] T. Fujita, Classification theories of polarized varieties, London Math. Soc. Lecture Note Series 155, Cambridge University Press, 1990.

[10] L. Gruson, C. Peskine and R. Lazarsfeld, On a theorem of Castelnuovo, and the equations defining space curves, Invent. Math. 72(1983), 491–506.

[11] L. T. Hoa and C. Miyazaki, Bounds on Castelnuovo-Mumford regularity for generalized Cohen-Macaulay graded rings, Math. Ann. 301 (1995), 587 – 598.

[12] C. Huneke and B. Ulrich, General hyperplane sections of algebraic varieties, J. Algebraic Geometry, 2 (1993), 487 – 505.

[13] M. Kreuzer, On the canonical module of a 0-dimensional scheme, Can. J. Math. 46 (1994), 357-379.

[14] S. Kwak Castelnuovo regularity for smooth subvarieties of dimensions 3 and 4. J. Algebraic Geom. 7 (1998), 195–206.

[15] R. Lazarsfeld A sharp Castelnuovo bound for smooth surfaces, Duke Math. J. 55(1987), 423–429.

[16] J. Mather, Generic projections. Ann. of Math. 98(1973), 226–245.

[17] J. Migliore, Introduction to liaison theory and deficiency modules, Progress in Math. 165, Birkhäuser, 1998.

[18] C. Miyazaki and W. Vogel, Bounds on cohomology and Castelnuovo-Mumford regularity, J. Algebra, 185 (1996), 626 – 642.

[19] C. Miyazaki, Sharp bounds on Castelnuovo-Mumford regularity, Trans. Amer. Math. Soc. 352 (2000), 1675 – 1686.

[20] C. Miyazaki, Castelnuovo-Mumford regularity and classical method of Castelnuovo, to appear in Kodai Math. J.

[21] D. Mumford, Lectures on curves on an algebraic surface, Annals of Math. Studies 59 (1966), Princeton UP.

[22] J. Rathmann, The uniform position principle for curves in characteristic p, Math. Ann. 276 (1987), 565 – 579.

[23] J. Stückrad and W. Vogel, Castelnuovo's regularity and cohomological properties of sets of points in $\mathbf{P^n}$. Math. Ann. 284 (1989), 487–501.

[24] K. Yanagawa, Castelnuovo's Lemma and h-vectors of Cohen-Macaulay homogeneous domains, J. Pure Appl. Algebra 105 (1995), 107 – 116.

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