BOUNDS ON THE CASTELNUOVO-MUMFORD REGULARITY OF PROJECTIVE VARIETIES FROM A VIEWPOINT OF COMMUTATIVE ALGEBRA

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Abstract. This paper investigates the Castelnuovo-Mumford regularity of generic hyperplane section of projective curve. The classical Castelnuovo method plays an important role in order to study the extremal examples for the bounds for the Castelnuovo-Mumford regularity.

1. Introduction

This paper investigates the Castelnuovo-Mumford regularity of a generic hyperplane section of projective curve. Let $T = k[y_0, \cdots, y_{N+1}]$ be the polynomial ring over an algebraically closed field k. Then we put $\mathbb{P}_k^{N+1} = \mathrm{Proj}(T)$. Let C be an irreducible reduced nondegenerate projective curve in $\mathbb{P}_k^{N+1}$, that is, the defining ideal I_C is generated by elements of degree ≥ 2 in T and T/I_C is an integral domain of dimension 2. Let X be a generic hyperplane section of C, that is, $X = C \cap H$, where H is a generic hyperplane of $\mathbb{P}_k^{N+1}$. So X is a zero-dimensional subscheme of $\mathbb{P}_k^N = \mathrm{Proj}(S)$, where S is the polynomial ring $k[x_0, \cdots, x_N]$. Let I be the defining ideal of X and R be the coordinate ring of X, that is, $R = S/I$. For a coherent sheaf $\mathcal{F}$ on $\mathbb{P}_k^N$ and an integer $m \in \mathbb{Z}$, $\mathcal{F}$ is said to be m-regular if $\mathrm{H}^i(\mathbb{P}_k^N, \mathcal{F}(m-i)) = 0$ for all $i \geq 1$. For a projective scheme $Y \subseteq \mathbb{P}_k^N$, Y is said to be m-regular if the ideal sheaf $\mathcal{I}_Y$ is m-regular. So, in this case, X is m-regular if and only if $\mathrm{H}^1(\mathbb{P}_k^N, \mathcal{I}_X(m-1)) = 0$, where $\mathcal{I}_X$ is the ideal sheaf of X. The Castelnuovo-Mumford regularity of $X \subseteq \mathbb{P}_k^N$ is the least such integer m and is denoted by $\mathrm{reg}(X)$. Note that $\mathrm{reg}(X) = a(R) + 2$, where $a(R)$ is the a-invariant of the coordinate ring R. Here, for a graded ring R over a field k with the irrelevant ideal $\mathfrak{m}$, the a-invariant $a(R)$ is defined as the maximal integer ℓ with $[\mathrm{H}_{\mathfrak{m}}^{\dim(R)}(R)]_\ell \neq 0$. The interest in this concept stems partly from the well-known fact that X is m-regular if and only if for every $p \geq 0$ the minimal generators of the pth syzygy module of the defining ideal I of $X \subseteq \mathbb{P}_k^N$ occur in degree $\leq m+p$. In this sense, it is important to study upper bounds on the Castelnuovo-Mumford regularity for projective schemes in order to describe the minimal free resolutions of the defining ideals. The following result is a starting point of our research on the Castelnuovo-Mumford regularity for generic hyperplane sections of projective curves. Throughout this paper, for a rational number $n \in \mathbb{Q}$, $\lceil n \rceil$ denotes the smallest integer which is not less than n.

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Proposition 1.1. (See [1, 2]). *Let $X \subseteq \mathbb{P}_k^N$ be a generic hyperplane section of nondegenerate projective curve. Then we have* $\mathrm{reg}(X) \leq \lceil (\deg(X)-1)/\mathrm{codim}(X) \rceil + 1$.

Before describing a sketch of the proof of (1.1), we define "uniform position", "linear general position" and "linear semi-uniform position" for zero-dimensional schemes. Let $X \subset \mathbb{P}_k^N$ a reduced zero-dimensional scheme such that X spans $\mathbb{P}_k^N$ as k-vector space. Then X is said to be in uniform position if $\mathrm{H}_Z(t) = \max\{\deg(Z), \mathrm{H}_X(t)\}$ for all t, for any subscheme Z of X, where H_Z and H_X denote the Hilbert function of Z and X respectively. This condition is equivalent to saying that, for any subschemes Z_1 and Z_2 of X with $\deg(Z_1) = \deg(Z_2)$, $\mathrm{h}^0(\mathbb{P}_k^N, \mathcal{I}_{Z_1}(\ell)) = \mathrm{h}^0(\mathbb{P}_k^N, \mathcal{I}_{Z_2}(\ell))$ for all integers $\ell \in \mathbb{Z}$. A reduced zero-dimensional scheme X is said to be in linear semi-uniform position if there are integers $v(i, X)$, simply written as $v(i)$, $0 \leq i \leq N$ such that every i-plane L in $\mathbb{P}_k^N$ spanned by linearly independent $i+1$ points of X contains exactly $v(i)$ points of X. A generic hyperplane section of a nondegenerate projective curve is in linear semi-uniform position, see [2]. We say X is in linear general position if $v(i) = i+1$ for all $i \geq 1$. Further, we note that "uniform position" implies "linear semi-uniform position". The property of h-vectors for 0-dimensional scheme in linear semi-unniform position yields the proof of Proposition 1.1. Now we describe a sketch of the proof for the readers' convenience.

Sketch of the proof of Proposition 1.1. Let $\underline{h} = (h_0, \cdots, h_s)$ be the h-vector of the zero-dimensional scheme $X \subseteq \mathbb{P}_k^N$, where s is the smallest integer such that $h_s \neq 0$. Note that $s = \mathrm{reg}(X) - 1 = a(R) + 1$. Since X is in linear semi-uniform position, we have $h_1 + \cdots + h_i \geq i h_i$ for all $i = 1, \cdots, s-1$, that is, $\mathrm{H}_X(t) \geq \min\{\deg(X), tN+1\}$ by [2]. Since $\deg(X) = h_0 + \cdots + h_s$ and $\mathrm{codim}(X) = h_1 = N$, we obtain $\lceil (\deg(X) - 1)/\mathrm{codim}(X) \rceil = \lceil (h_1 + \cdots + h_s)/h_1 \rceil \geq s$. Hence the assertion is proved. □

Let us classify extremal cases for regularity bounds in Proposition 1.1. Our main theorem extends the results of [4, (2.4)].

Theorem 1.2. *Let $X \subseteq \mathbb{P}_k^N$ be a generic hyperplane section of nondegenerate projective curve C. Assume that* $\deg(X) \geq N^2 + 2N + 2$. *If the equality* $\mathrm{reg}(X) = \lceil (\deg(X)-1)/\mathrm{codim}(X) \rceil + 1$ *holds, then X is contained in a rational normal curve in $\mathbb{P}_k^N$.*

First let us study when the extremal case in (1.1) happens for the case $N = 1$, that is, a generic hyperplane section of plane curve. Such curve is defined by one equation of degree $\deg C = d$, and we easily have $\mathrm{reg}(X) = d$. Thus we have $\mathrm{reg}(X) = \lceil (\deg(X) - 1)/\mathrm{codim}(X) \rceil + 1$ for the case $N = 1$.

Before studying the case $N \geq 2$, we will describe a relationship between the monodromy group of the projective curve and the configuration of the generic hyperplane section of the curve, as following Rathmann [13]. Let $C \subseteq \mathbb{P}_k^{N+1}$ and $X \subseteq \mathbb{P}_k^N$ be again a nondegenerate projective curve and its generic hyperplane section respectively. Let $M \subseteq C \times (\mathbb{P}_k^{N+1})^*$ be the incidence correspondence parametrizing the pairs $(x, H) \in M$, that is, a point x of C and a hyperplane H of $\mathbb{P}_k^{N+1}$ such that x is contained in H. Since M is a $\mathbb{P}_k^N$-bundle over C via the first projection, M is irreducible and reduced. By Bertini's theorem, M is generically étale finite over $P = (\mathbb{P}_k^{N+1})^*$ via the second projection. Thus the function field $K(M)$ of M is separable finite over $K(P)$, in particular, $K(M)$ is a simple extension of $K(P)$. So we fix a splitting field Q for this simple extension. Let G_C be the Galois group $\mathrm{Gal}(Q/K(P))$. Then G_C is a subgroup of the full symmetric group S_d and is called the monodromy group of $C \subseteq \mathbb{P}_k^N$, where $d = \deg(C)$. The following is a basic result on the monodromy group of projective curve.

Proposition 1.3. *Let $X \subseteq \mathbb{P}_k^N$ be a generic hyperplane section of nondegenerate projective curve $C \subseteq \mathbb{P}_k^{N+1}$.*
(i) (See [1]). *If $\mathrm{char}(k) = 0$, then $G_C = S_d$.*
(ii) (See [13, (1.8)]). *If either $G_C = S_d$ or $G_C = A_d$, then X is in uniform position.*
(iii) (See [8, (2.5)]). *Assume that X is in uniform position and $\deg(X) \geq N^2 + 2N + 2$. If the equality $\mathrm{reg}(X) = \lceil (\deg(X) - 1)/\mathrm{codim}(X) \rceil + 1$ holds, then X is contained in a rational normal curve in $\mathbb{P}_k^N$.*

We remark here the hypothesis $\deg(X) \geq N^2 + 2N + 2$ is indispensable because of an example of a $(2, 2, 4)$ complete intersection in $\mathbb{P}_k^3$. (See [8, (2.6)].)

In this paper, we focus on the case that X is not in uniform position. So k is assumed to be a field of positive characteristic.

Theorem 1.4. *Let $X \subseteq \mathbb{P}_k^N$ be a generic hyperplane section of nondegenerate projective curve. Assume that X is not in uniform position. If $\deg(X) \geq N^2 + 2N + 2$, then we have $\mathrm{reg}(X) \leq \lceil (\deg(X) - 1)/\mathrm{codim}(X) \rceil$.*

What we have to prove is that $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}_k^N}(t)) \to \mathrm{H}^0(\mathcal{O}_X(t))$ is surjective, that is, $\mathrm{H}^1(\mathcal{I}_X(t)) = 0$, where $t = \lceil (\deg(X) - 1)/N \rceil - 1$. The classical Castelnuovo method plays an important role for the proof of the following lemma, which easily yields the theorem.

Lemma 1.5. *Let $X \subseteq \mathbb{P}_k^N$ be a generic hyperplane section of nondegenerate projective curve. Put $t = \lceil (\deg(X) - 1)/N \rceil - 1$. For any fixed point $P \in X$, there exists a (possibly reducible) hypersurface F of degree t in $\mathbb{P}_k^N$ such that $X \cap F = X \backslash \{P\}$.*

The rest of this paper is devoted to the proof of this lemma. In Section 2, we consider the case of space curves, that is $N = 2$, and in Section 3, the case of curves in $\mathbb{P}^n$ $(n \geq 4)$, that is, $N \geq 3$.

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2. Curve in $\mathbb{P}^3$

In this section, we investigate the extremal examples for the bounds on the Castelnuovo-Mumford regularity described in the introduction for the case $N = 2$, that is, a generic hyperplane section X of space curve C in $\mathbb{P}_k^3$. If $\mathrm{char}(k) = 0$, X is in uniform position, and so we have done. Moreover, in this case, there is an ACM smooth curve $C' \subseteq \mathbb{P}_k^3$ such that $X = C' \cap H$. Thus we describe a free resolution of the defining ideal I_X over $k[x_0, x_1, x_2]$ by the Hilbert-Burch matrix, see [5, 6], and get a detailed information for the regularity of X. This observation comes from the fact that X is "of decreasing type", see [5], in terms of the h-vectors. From now on we assume that a generic hyperplane section X of a nondegenerate space curve C is not in uniform position and k is a field of positive characteristic. The proof of the main result of this section is obtained by the classical Castelnuovo method without using the Hilbert-Burch matrix.

Theorem 2.1. *Let $X \subseteq \mathbb{P}^2_k$ be a generic hyperplane section of nondegenerate projective curve C of degree d in $\mathbb{P}^3_k$. Assume that X is not in general linear position and that $d \geq 10$. For any fixed point $P \in X$, there exists a union F of t lines $L_1, \cdots, L_t$ in $\mathbb{P}^2_k$ such that $X \cap F = X \backslash \{P\}$, where $t = \lceil \frac{d-1}{2} \rceil - 1$. In particular, $\operatorname{reg}(X) \leq \lceil \frac{d-1}{2} \rceil$.*

Proof. Since X is in linear semi-uniform position by [2], the line spanned by any two points of X is contains exactly $v(1)$ points. Since X is not in general linear position by assumption, we have $v = v(1) \geq 3$.

First we consider the case $v \geq 4$. Now let us take any point P of X. We fix a point Q in $X \backslash \{P\}$. Then we take different lines $\ell_1, \cdots, \ell_a$ through the point Q such that the union $\cup_{j=1}^a \ell_j$ covers X. Note that $a \geq 3$. Each line contains the point Q and the other $v-1$ points of X. Thus we see $d = av - a + 1$. We may assume that P is contained in ℓ_a. Then we take $L_1 = \ell_1, \cdots, L_{a-1} = \ell_{a-1}$. Since $(X \cap \ell_a) \backslash \{P, Q\}$ consists of exactly $v-2$ points, we need $v-2$ lines, $L_a, L_{a+1}, \cdots, L_{a+v-3}$, not containing P such that the union $\cup_{j=a}^{a+v-3}$ contains $(X \cap \ell_a) \backslash \{P, Q\}$. Thus the assertion is reduced to showing that $a+v-3 \leq \lceil \frac{av-(a-1)-1}{2} \rceil - 1$ for $d = av-(a-1) \geq 10$. The inequality $a+v-3 \leq \frac{a(v-1)}{2} - 1$ is equivalent to saying that $(a-2)(v-3) \geq 2$, which is easily shown for $v \geq 4$ and $a \geq 3$ with $av - a + 1 \geq 10$ except for $(v, a) = (4, 3)$. For $(v, a) = (4, 3)$, we have $a + v - 3 \leq \lceil \frac{a(v-1)}{2} \rceil - 1$. Hence the assertion is proved.

Next we consider the case $v = 3$. Now let us take any point P of X. Then we take different lines $\ell_1, \cdots, \ell_a$ through the point P such that the union $\cup_{j=1}^a \ell_j$ covers X. Since each line contains 3 points of X, we see $d = 2a + 1$. Now we want to take lines $L_1, \cdots, L_b$ inductively such that $P \notin L_i$ and L_i contains exactly 3 points of $X \backslash (\{P\} \cup (X \cap (\cup_{j=1}^{i-1} L_j)))$ for $i = 1, \cdots, b$. Here we can take $b = \lceil \frac{d-3}{6} \rceil$. In fact, suppose there are lines $L_1, \cdots, L_i$ satisfying the condition. Then $X \cap (\cup_{j=1}^i L_j)$ consists of $3i$ points, and $X \backslash (\{P\} \cup (X \cap (\cup_{j=1}^i L_j)))$ consists of the remaining $d-3i-1$ points. If $3i+1 < d-3i-2$, then there is a line L_{i+1} satisfying the condition, which gives $b = \lceil \frac{d-3}{6} \rceil$. Moreover we want to take lines $L'_1, \cdots, L'_c$ inductively such that $P \notin L'_i$ and L'_i contains at least 2 points of $X \backslash (\{P\} \cup (X \cap ((\cup_{j=1}^a L_j) \cup (\cup_{j=1}^{i-1} L'_j))))$ for $i = 1, \cdots, c$ so that $X \cap ((\cup_{j=1}^b L_j) \cup (\cup_{j=1}^c L'_j)) = X \backslash \{P\}$. On the other hand, the number of the points of $X \backslash (\{P\} \cup (X \cap (\cup_{j=1}^a L_j)))$ is $d - 1 - 3\lceil \frac{d-3}{6} \rceil = 2a - 3\lceil \frac{a-1}{3} \rceil$. So, we can take $c = \lceil \frac{2a - 3\lceil \frac{a-1}{3} \rceil}{2} \rceil$. Thus the assertion is reduced to showing that $\lceil \frac{a-1}{3} \rceil + \lceil \frac{2a-3\lceil \frac{a-1}{3} \rceil}{2} \rceil \leq a - 1$, because $\lceil \frac{d-1}{2} \rceil = a$. For $a = 5, 6, 7, 8$, the inequality holds. Since $2a - 3\lceil \frac{a-1}{3} \rceil \leq a + 1$, the proof of the inequality is reduced to showing that $\lceil \frac{a-1}{3} \rceil + \lceil \frac{a+1}{2} \rceil \leq a - 1$, which is easily shown for $a \geq 9$. Hence the assertion is proved. $\square$

Corollary 2.2. *Let $X \subseteq \mathbb{P}^2_k$ be a generic hyperplane section of nondegenerate projective curve C of degree d in $\mathbb{P}^3_k$. Assume that X is not in uniform position and that $d \geq 10$. For any fixed point $P \in X$, there exists a (possibly reducible) plane curve F of degree t in $\mathbb{P}^2_k$ such that $X \cap F = X \backslash \{P\}$, where $t = \lceil \frac{d-1}{2} \rceil - 1$. In other words,* $\operatorname{reg}(X) \leq \lceil \frac{d-1}{2} \rceil$.

Proof. By (2.1), we have only to consider the case that X is not in uniform position but in general linear position. Let us take any point P of X. Then we take a plane curve F' of degree 3 which contains at least 8 points of $X\backslash\{P\}$ and does not contain P. Since the remaining points of $X \backslash (\{P\} \cup (X \cap F'))$ is in general linear position, we can take lines $L_1, \cdots, L_b$ inductively such that $P \notin L_i$ and L_i contains exactly 2 points of $X \backslash (\{P\} \cup (X \cap (F' \cup (\cup_{j=1}^{i-1} L_j))))$ for $i = 1, \cdots, b$, so that we can take $b = \lceil \frac{d-9}{2} \rceil$. Thus $F = F' \cup (\cup_{j=1}^{b} L_j)$ satisfies $X \cap F = X \backslash \{P\}$ and the degree of F is $\lceil \frac{d-9}{2} \rceil + 3$. Since $\lceil \frac{d-9}{2} \rceil + 3 = t$, the assertion is proved. □

3. Curve in $\mathbb{P}^n$ $(n \geq 4)$

In this section, we consider a generic hyperplane section of nondegenerate projective curve C in $\mathbb{P}^n$ $(n \geq 4)$. Now we begin with describing a useful result of Rathmann [13].

Proposition 3.1. (See [13, (2.5)]). *Let X be a generic hyperplane section in $\mathbb{P}^N$ of a nondegenerate projective curve C of $\mathbb{P}^{N+1}$ for $N \geq 3$. Let G_C be the monodromy group of C. If X is not in uniform position, then either of the following holds:*
(a) $v(1) = 3$, *and G_C is exactly 2-transitive.*
(b) $v(1) = 2$, $v(2) \geq 4$, *and G_C is exactly 3-transitive.*
(c) $\deg(C) = 11, 12, 23$ *or* 24, *and G_C is the Mathieu group M_{11}, M_{12}, M_{23}, M_{24} respectively. Moreover M_{11} and M_{23} are exactly 4-transitive and M_{12} and M_{24} are exactly 5-transitive.*

Now we are in position to state the main theorem of this section.

Theorem 3.2. *Let $X \subseteq \mathbb{P}^N_k$ be a generic hyperplane section of nondegenerate projective curve C of degree d in $\mathbb{P}^{N+1}_k$ for $N \geq 3$. Assume that X is not in uniform position and that $d \geq N^2 + 2N + 2$. For any fixed point $P \in X$, there exists a (possibly reducible) hypersurface F of degree t in $\mathbb{P}^N_k$ such that $X \cap F = X \backslash \{P\}$, where $t = \lceil \frac{d-1}{N} \rceil - 1$. In other words,* $\operatorname{reg}(X) \leq \lceil \frac{d-1}{N} \rceil$.

According to the classification of (3.1) we will prove (3.2). As for the case (c) in (3.1), since $N \geq 4$ for $\deg(C) = 11, 23$ and $N \geq 5$ for $\deg(C) = 12, 24$, we see $N^2 + 2N + 2 \geq 26$. Hence there is no such curves satisfying the degree condition in (3.2). So the proof of the theorem is reduced to the lemmas (3.3), (3.4) and (3.5).

First we consider the case (a) in (3.1).

Lemma 3.3. *Let $X \subseteq \mathbb{P}^N_k$ be a generic hyperplane section of nondegenerate projective curve C of degree d in $\mathbb{P}^{N+1}_k$ for $N \geq 3$. Assume that X is not in uniform position and that $d \geq N^2 + 2N + 2$. Moreover, assume that $v(1) = 3$, that is, G_C is exactly 2-transitive. For any fixed point $P \in X$, there exists (possibly reducible) hypersurface F of degree t in $\mathbb{P}^N_k$ such that $X \cap F = X \backslash \{P\}$, where $t = \lceil \frac{d-1}{N} \rceil - 1$. In other words,* $\operatorname{reg}(X) \leq \lceil \frac{d-1}{N} \rceil$.

Proof. For the case $N \geq 4$, the proof in [4, (2.2)] by the classical Castelnuovo method have given the claim. So, we have only to prove for the case $N = 3$. Since $v(1) = 3$, we have $v(2) \geq 7$ and put $v = v(2)$. Now let us take any point P of X. We fix 2 points Q_1 and Q_2 in $X \backslash \{P\}$. Then we take different 2-planes $F_1, \cdots, F_a$ through the points Q_1 and Q_2 such that the union $\cup_{j=1}^{a} F_j$ covers X. We remark that $a \geq 3$. Since each 2-plane contains exactly v points of X, we see $d = av - 2a + 2$. We may assume that P is contained in F_a. Now we take $L_1 = F_1, \cdots, L_{a-1} = F_{a-1}$. Since $(X \cap F_a) \backslash \{P, Q_1, Q_2\}$ consists of exactly $v - 3$ points, there are 2-planes $F'_1, \cdots, F'_b$ such that $P \notin F'_i$ for $i = 1, \cdots, b$ and the union $\cup_{j=1}^{b} F'_j$ of 2-planes covers $(X \cap F_a) \backslash \{P, Q_1, Q_2\}$, where $b = \lceil \frac{v-3}{2} \rceil$. By taking $L_a = F'_1, L_{a+1} = F'_2, \cdots, L_{a+b-1} = F'_b$, we have $(X \cap (\cup_{j=1}^{a+b-1} L_j)) = X \backslash \{P\}$. Thus we have only to show that $(a-1) + \lceil \frac{v-3}{2} \rceil \leq \lceil \frac{(av-2a+2)-1}{3} \rceil - 1$. The inequality $(a-1) + \frac{v-3}{2} \leq \frac{av-2a+1}{3} - 1$ is equivalent to saying that $(2a-3)(v-5) \geq 4$, which is easily shown for $v \geq 7$ and $a \geq 3$. □

Next we show for the case (b) in (3.1).

Lemma 3.4. *Let $X \subseteq \mathbb{P}_k^3$ be a generic hyperplane section of nondegenerate projective curve C of degree d in $\mathbb{P}_k^4$. Assume that X is not uniform position and that $d \geq 17$. Moreover, assume that $v(1) = 2$, $v(2) \geq 4$, and G_C is exactly 3-transitive. For any fixed point $P \in X$, there exists a union F of t hyperplanes $L_1, \cdots, L_t$ in $\mathbb{P}_k^4$ such that $X \cap F = X \backslash \{P\}$, where $t = \lceil \frac{d-1}{3} \rceil - 1$. In particular,* $\operatorname{reg}(X) \leq \lceil \frac{d-1}{3} \rceil$.

Proof. Let us put $v = v(2) \geq 4$. For the case $v \geq 5$, the proof is proceeded as in (3.3). Here we remark that $d = av - 2a + 2 \geq 17$ and $a \geq 2$. The assertion is reduced to showing that $(a-1) + \lceil \frac{v-3}{2} \rceil \leq \lceil \frac{av-2a+1}{3} \rceil - 1$. The inequality $(a-1) + \frac{v-3}{2} \leq \frac{av-2a+1}{3} - 1$ is equivalent to saying that $(2a-3)(v-5) \geq 4$, which is easily shown for $v \geq 6$ and $a \geq 2$ with $av - 2a + 2 \geq 17$. For the case $v = 5$, we see that $(a-1) + 1 = \lceil \frac{3a+1}{3} \rceil - 1$ gives the inequality.

Next we consider the case $v = 4$. From the table [13, (2.4)] of classification of triple transitive groups, we see that possible degree d (≥ 17) for the projective curve C is either $p^e + 1$ or 2^e, where p is a prime number and e is a positive integer, that is, $d = 17, 18, 20, 22, 24, 26, 28, 30, 32, 33, 38$ or more than 38.

Now let us take any point P of X. Then we want to take 2-planes $L_1, \cdots, L_a$ inductively such that $P \notin L_i$ and L_i contains exactly 4 points of $X \backslash (\{P\} \cup (X \cap (\cup_{j=1}^{i-1} L_j)))$ for $i = 1, \cdots, a$. Here we can take $a = \lceil \frac{d-4}{8} \rceil$. In fact, suppose there are 2-planes $L_1, \cdots, L_i$ satisfying the condition. Then $X \cap (\cup_{j=1}^{i} L_j)$ consists of $4i$ points, and $X \backslash (\{P\} \cup (X \cap (\cup_{j=1}^{i} L_j)))$ consists of the remaining $d - 4i - 1$ points. If $4i + 1 < d - 4i - 3$, then there is a 2-plane L_{i+1} satisfying the condition, which gives $a = \lceil \frac{d-4}{8} \rceil$. Moreover we want to take 2-planes $L'_1, \cdots, L'_b$ for some b inductively such that $P \notin L'_i$ and L'_i contains at least 3 points of $X \backslash (\{P\} \cup (X \cap ((\cup_{j=1}^{a} L_j) \cup (\cup_{j=1}^{i-1} L'_j))))$ for $i = 1, \cdots, b$ so that $X \cap ((\cup_{j=1}^{a} L_j) \cup (\cup_{j=1}^{b} L'_j)) = X \backslash \{P\}$. On the other hand, the number of the points of $X \backslash (\{P\} \cup (X \cap (\cup_{j=1}^{i} L_a)))$ is $d - 1 - 4\lceil \frac{d-4}{8} \rceil$. So, we can take $b = \lceil \frac{d-1-4\lceil \frac{d-4}{8} \rceil}{3} \rceil$. Thus we have only to show that $\lceil \frac{d-4}{8} \rceil + \lceil \frac{d-4\lceil \frac{d-4}{8} \rceil - 1}{3} \rceil \leq \lceil \frac{d-1}{3} \rceil - 1$. Hence we easily obtain this inequality for $d = 26, 28, 32, 33$ or $d \geq 38$.

Finally we check for the case $d = 17, 18, 20, 22, 24, 30$. For the case $d = 18$, let us take any point P of X. First take 2-planes L_1 and L_2 in $\mathbb{P}^4_k$ such that $L_1 \cup L_2$ contains exactly 8 points from $X \backslash \{P\}$. Next take a 3-plane L_3 which contains at least 3 points from $X \backslash (\{P\} \cup (X \cap (L_1 \cup L_2)))$ and do not contain the point P. Since $X \backslash (\{P\} \cup (X \cap (L_1 \cup L_2 \cup L_3)))$ consists of 5 or 6 points, we put $X' = \{Q_1, \cdots, Q_5\}$ or $\{Q_1, \cdots, Q_6\}$. The 2-plane spanned by Q_1, Q_2 and P may contain one point from $\{Q_3, \cdots, Q_6\}$, say $\{Q_4\}$, and the 2-plane spanned by Q_4, Q_5 and P may contain either Q_3 or Q_6, say Q_3. Now let L_4 be the 2-plane spanned by $\{P_1, P_2, P_3\}$, and L_5 be the 2-plane spanned by $\{P_4, P_5, P_6\}$. Then neither L_4 or L_5 contains the point P. Thus the union $\cup_{j=1}^{5} L_j$ of 2-planes covers $X \backslash \{P\}$ and does not contain the point P. The case $d = 17$ is proved as $d = 18$.

For the case $d = 22$, let us take any point P of X. First take 2-planes L_1, L_2, L_3 in $\mathbb{P}^4_k$ such that $L_1 \cup L_2 \cup L_3$ contains exactly 12 points from $X \backslash \{P\}$. Next take a 2-plane L_4 which contains at least 3 points of $X \backslash (\{P\} \cup (X \cap (L_1 \cup L_2 \cup L_3)))$ and does not contain the point P. Since $X' = X \backslash (\{P\} \cup (X \cap (L_1 \cup \cdots \cup L_4)))$ consists of 5 or 6 points, we put $X' = \{Q_1, \cdots, Q_5\}$ or $\{Q_1, \cdots, Q_6\}$. The 2-plane spanned by Q_1, Q_2 and P may contain one point from $\{Q_3, \cdots, Q_6\}$, say $\{Q_4\}$, and the 2-plane spanned by Q_4, Q_5 and P may contain either Q_3 or Q_6, say Q_3. Now let L_5 be the 2-plane spanned by $\{P_1, P_2, P_3\}$, and L_6 be the 2-plane spanned by $\{P_4, P_5, P_6\}$. Then neither L_5 or L_6 contains the point P. Thus the union $\cup_{j=1}^{6} L_j$ of 2-planes covers $X \backslash \{P\}$ and does not contain the point P. The case $d = 20$ is proved as $d = 22$.

Moreover, the case $d = 24, 30$ is much easier to prove, which is left to the readers. □

Lemma 3.5. *Let $X \subseteq \mathbb{P}^N_k$ be a generic hyperplane section of nondegenerate projective curve C of degree d in $\mathbb{P}^{N+1}_k$ for $N \geq 4$. Assume that X is not in uniform position and that $d \geq N^2 + 2N + 2$. Also, assume that $v(1) = 2$, $v(2) \geq 4$, and G_C is exactly 3-transitive. For any fixed point $P \in X$, there exists a (possibly reducible) hypersurface F of degree t in $\mathbb{P}^N_k$ such that $X \cap F = X \backslash \{P\}$, where $t = \lceil \frac{d-1}{N} \rceil - 1$. In other words,* $\operatorname{reg}(X) \leq \lceil \frac{d-1}{N} \rceil$.

Proof. Let us take any point P of X. First we note that $v(i+1) \geq 2v(i) - 1$ for $i \geq 2$. In fact, let us take an i-plane G spanned by linearly independent $i+1$ points of X, and take a point $A_1 \in G$ and a point $A_2 \notin G$. Then we put $X \cap G = \{A_1\} \cup \{B_1, \cdots, B_{v(i)-1}\}$. For any point $B_j \in (X \cap G) \backslash \{A_1\}$, the 2-plane H spanned by A_1, A_2, B_j contains at least one point C_j in $(X \cap H) \backslash \{A_1, A_2, B_j\})$ for all j. Note that $C_j \neq C_{j'}$ for $j \neq j'$. Thus we have $v(i+1) \geq 2v(i) - 1$ for $i \geq 2$. Moreover, since $v(2) \geq 4$, we see that $v(i) \geq 3 \cdot 2^{i-2} + 1$ for $i \geq 2$. Now we put $v = v(N-2)$ and $w = v(N-1)$. Remark that $v \geq 3 \cdot 2^{N-4} + 1$ and $w \geq 2v - 1$. We fix linearly independent $N-1$ points $Q_1, \cdots, Q_{N-1}$ of $X \backslash \{P\}$ such that the $(N-2)$-plane L spanned by $Q_1, \cdots, Q_{N-1}$ does not contain the point P. Then there are different hyperplanes $L_1, \cdots, L_a$ containing L such that the union $\cup_{j=1}^{a} L_j$ covers X. So, we see that $d = a(w - v) + v$. We may assume that P is contained in L_a. Since $X \cap L_a$ is also in linear semi-uniform position in $L_a (\cong \mathbb{P}^{N-1}_k)$, by (1.1) there is a (possibly reducible) hypersurface F' in $\mathbb{P}^N_k$ of degree $\lceil \frac{w-1}{N-1} \rceil$ such that $P \notin F'$ and F' contains $(X \cap L_a) \backslash \{P\}$. Thus the union $F = (\cup_{j=1}^{a-1} L_j) \cup F'$ covers $X \backslash \{P\}$ and $P \notin F$. Hence we have only to show that $a - 1 + \lceil \frac{w-1}{N-1} \rceil \leq \lceil \frac{d-1}{N} \rceil - 1$. In order to show the inequality, it suffices to prove that $a + \frac{w-1}{N-1} \leq \frac{a(w-v)+v}{N}$, which is equivalent to $aN^2 - aN - N \leq (aN-a-N)w - (N-1)(a-1)v$. Since $aN - a - N \geq 0$, we see that $(aN-a-N)w - (N-1)(a-1)v \geq (aN-a-N)(2v-1) - (N-1)(a-1)v = (aN-N-a-1)v - (aN-a-N) \geq (aN-N-a-1)(3\cdot 2^{N-4}+1) - (aN-a-N)$. Thus the assertion is reduced to showing that $3\cdot 2^{N-4} + 1 \geq \frac{aN-N-a-1}{aN^2-2N-a}$, which is easily shown for $N \geq 4$ and $a \geq 2$. □

4. An Application to a Sharp Bound on the Castelnuovo-Mumford Regularity

In this section, we describe an application to a sharp bound on the Castelnuovo-Mumford regularity in order to improve [4, Theorem 3.2].

Let s be a nonnegative integer. Then X is called s-Buchsbaum if the graded S-module $\mathrm{M}^i(X) = \oplus_{\ell\in\mathbb{Z}}\mathrm{H}^i(\mathbb{P}^N_k, \mathcal{I}_X(\ell))$, called the deficiency module of X, is annihilated by $\mathfrak{m}^s$ for $1 \leq i \leq \dim(X)$, see, e.g., [8]. On the other hand, X is called strongly s-Buchsbaum if $X \cap V$ has the s-Buchsbaum property for any complete intersection V of $\mathbb{P}^N_k$ with $\mathrm{codim}(X\cap V) = \mathrm{codim}(X)+\mathrm{codim}(V)$, possibly $V = \mathbb{P}^N_k$. So "strongly s-Buchsbaum" implies "s-Buchsbaum". Further we call the minimal nonnegative integer s, if it exists, such that X is s-Buchsbaum (resp. strongly s-Buchsbaum), as the Ellia-Migliore-Miró Roig number (resp. the strongly Ellia-Migliore-Miró Roig number) of X and denote it by $k(X)$ (resp. $\bar{k}(X)$), see [8]. In case X is not k-Buchsbaum for all $k \geq 0$, then we put $k(X) = \bar{k}(X) = \infty$. Note that $k(X) < \infty$ if and only if $\bar{k}(X) < \infty$, which is equivalent to saying that X is locally Cohen-Macaulay and equi-dimensional.

Upper bounds on the Castelnuovo-Mumford regularity of a projective variety X are given in terms of $\dim(X)$, $\deg(X)$, $\mathrm{codim}(X)$, $k(X)$ and $\bar{k}(X)$.

Proposition 4.1. *Let X be a nondegenerate projective variety in $\mathbb{P}^N_k$. Assume that X is not ACM, that is, $k(X) \geq 1$. Then*

(a) $\mathrm{reg}(X) \leq \lceil(\deg(X)-1)/\mathrm{codim}(X)\rceil + k(X)\dim(X)$.

(b) $\mathrm{reg}(X) \leq \lceil(\deg(X)-1)/\mathrm{codim}(X)\rceil + \bar{k}(X)\dim(X) - \dim(X) + 1$.

Furthermore, assume that $\mathrm{char}(k) = 0$ *and* $\deg(X) \geq \mathrm{codim}(X)^2 + 2\,\mathrm{codim}(X) + 2$. *If the equality, either* $\mathrm{reg}(X) = \lceil(\deg(X)-1)/\mathrm{codim}(X)\rceil + k(X)\dim(X)$ *or* $\mathrm{reg}(X) = \lceil(\deg(X)-1)/\mathrm{codim}(X)\rceil + \bar{k}(X)\dim(X) - \dim(X) + 1$ *holds, then X is a curve on a rational ruled surface.*

Proof. See [4, 8, 7, 11]. □

Now we will study the extremal case for the inequality in (4.1) in positive characteristic. We assume that the variety in question is not ACM, see [10] for the ACM case. The following theorem improves a result of [4].

Theorem 4.2. *Let k be an algebraically closed field of positive characteristic. Let X be a nondegenerate projective variety in $\mathbb{P}^N_k$ with $k(X) \geq 1$. Assume that* $\deg(X) \geq 2\,\mathrm{codim}(X)^2 + \mathrm{codim}(X) + 2$.

(a) *If the equality* $\mathrm{reg}(X) = \lceil(\deg(X)-1)/\mathrm{codim}(X)\rceil + k(X)\dim(X)$ *holds, then X is a curve on a rational ruled surface.*

(b) *If the equality* $\mathrm{reg}(X) = \lceil(\deg(X)-1)/\mathrm{codim}(X)\rceil + \bar{k}(X)\dim(X) - \dim(X) + 1$ *holds, then X is a curve on a rational ruled surface.*

Proof. The proof is proceeded as that of [4, (3.2)] by □

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