TOWARDS A CLASSIFICATION OF BUCHSBAUM VARIETIES WITH A REGULARITY BOUND OF CASTELNUOVO TYPE

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1. Introduction

This paper describes results and problems on a classification of Buchsbaum varieties from the viewpoint of a Castelnuovo-type bound of the Castelnuovo-Mumford regularity $\operatorname{reg} V \leq \lceil(\deg V - 1)/\operatorname{codim} V\rceil + 1$.

The Castelnuovo-Mumford regularity is one of the most important invariants measuring a complexity of the defining equations of a projective variety. There have been various studies on bounding the regularity of a variety.

Let k be an algebraically closed field. Let $S = k[X_0, \cdots, X_N]$ be the polynomial ring over k. A variety $V \subset \mathbb{P}_k^N = \operatorname{Proj} S$ means a nondegenerate irreducible reduced projective scheme over k. For a coherent sheaf $\mathcal{F}$ on $\mathbb{P}_k^N$ and an integer $m \in \mathbb{Z}$, $\mathcal{F}$ is said to be m-regular if $\mathrm{H}^i(\mathbb{P}_k^N, \mathcal{F}(m-i)) = 0$ for all $i \geq 1$. For a projective scheme $X \subseteq \mathbb{P}_k^N$, X is said to be m-regular if the ideal sheaf $\mathcal{I}_X$ is m-regular. The Castelnuovo-Mumford regularity of $X \subseteq \mathbb{P}_k^N$ is the least such integer m and is denoted by $\operatorname{reg} X$. A projective scheme X is m-regular if and only if for every $p \geq 0$ the minimal generators of the pth syzygy module of the defining ideal $I(= \Gamma_* \mathcal{I}_X \subset S)$ of X occur in degree $\leq m + p$, see [1].

For a rational number $m \in \mathbb{Q}$, we write $\lceil m \rceil$ for the minimal integer which is greater than or equal to m and $\lfloor m \rfloor$ for the maximal integer which is less than or equal to m.

The Eisenbud-Goto conjecture $\operatorname{reg} V \leq \deg V - \operatorname{codim} V + 1$ for a nondegenerate projective variety V is one of the most important problems, and it is still open to get the bound for higher dimensional projective varieties. If V is an ACM variety, that is, the coordinate ring of V is Cohen-Macaulay, then a regularity bound $\operatorname{reg} V \leq \lceil(\deg V - 1)/\operatorname{codim} V\rceil + 1$ easily follows from the uniform position principle for a generic hyperplane section of a projective curve.

2000 *Mathematics Subject Classification.* Primary 13H10, 14M05, Secondary 14N25.

Key words and phrases. Castelnuovo-Mumford regularity, Buchsbaum ring, Rational normal scroll, del Pezzo variety.

Partially supported by Grant-in-Aid for Scientific Research (C) (21540044) Japan Society for the Promotion of Science.

In classifying a projective curve in terms of the regularity bound, we make use of an invariant $k(C)$ defined as the minimal nonnegative integer v such that $\mathsf{m}^v \mathrm{M}(C) = 0$, where a graded S-module $\mathrm{M}(C) = \oplus_{\ell \in \mathbb{Z}} \mathrm{H}^1(\mathbb{P}^N_K, \mathcal{I}_C(\ell))$. For a nondegenerate projective curve there is an inequality $\operatorname{reg} C \leq \lceil (\deg C - 1)/\operatorname{codim} C \rceil + \max\{k(C), 1\}$. Furthermore, the following result (1.1) describes the extremal and the next extremal curves with the Castelnuovo-type regularity bound from [4, (1.2)] and [5, (1.2)].

Proposition 1.1. *Let $C \subseteq \mathbb{P}^N_k$ be a nondegenerate projective curve over an algebraically closed field k with $\operatorname{char} k = 0$. Assume that C is not ACM.*

(1) *If $\deg C \geq (\operatorname{codim} C)^2 + 2 \operatorname{codim} C + 2$ and $\operatorname{reg} C = \lceil (\deg C - 1)/\operatorname{codim} C \rceil + k(C)$, then C lies on a rational normal surface scroll.*
(2) *If $\deg C \geq (\operatorname{codim} C)^2 + 4 \operatorname{codim} C + 2$ and $\operatorname{reg} C = \lceil (\deg C - 1)/\operatorname{codim} C \rceil + k(C) - 1$, then C lies either on a rational normal surface scroll or a del Pezzo surface.*

In this paper we consider a Buchsbaum variety. A projective variety $V \subset \mathbb{P}^N_k$ is called a Buchsbaum variety if the coordinate of V is a Buchsbaum ring. A result of Stückrad and Vogel [11] states that $\operatorname{reg} V \leq \lceil (\deg V - 1)/\operatorname{codim} V \rceil + 1$ for a nondegenerate Buchsbaum variety $V \subset \mathbb{P}^N_k$. We describes a classification of the Buchsbaum variety in terms of the regularity bound of Castelnuovo-type.

Theorem 1.2 ([8, 13, 6])**.** *Let $V \subseteq \mathbb{P}^N_k$ be a nondegenerate Buchsbaum variety over an algebraically closed field k with $\operatorname{char} k = 0$.*

(1) *If $\deg V \geq (\operatorname{codim} V)^2 + 2 \operatorname{codim} V + 2$ and $\operatorname{reg} V = \lceil (\deg V - 1)/\operatorname{codim} V \rceil + 1$, then V is a divisor on a variety of minimal degree.*
(2) *If $\deg V \geq (\operatorname{codim} V)^2 + 4 \operatorname{codim} V + 2$ and $\operatorname{reg} V = \lceil (\deg V - 1)/\operatorname{codim} V \rceil$, then V is a divisor on a del Pezzo variety.*

Here we propose the following conjecture. The cases $m = 0, 1$ are stated in (1.2), which is mentioned in detail including a sketch of the proof later in (3.4) and (3.5).

Conjecture 1.3. *Let $V \subset \mathbb{P}^N_k$ be a nondegenerate Buchsbaum variety over an algebraically closed field k. Let $m = \lceil (\deg V - 1)/\operatorname{codim} V \rceil + 1 - \operatorname{reg} V$. Assume $\deg V \gg (\operatorname{codim} V)^2$. Then V is a divisor on a variety Y with $\deg Y \leq \operatorname{codim} Y + 1 + m$.*

I would like to thank the organizers to give me an opportunity to have a talk and submit a paper at the Japan-Vietnam Joint Seminar on Commutaive Algebra, 2010.

2. Buchsbaum variety

Let $V \subset \mathbb{P}^N_k$ be a projective variety. We call V a Buchsbaum variety if the coordinate ring of V has the Buchsbaum property. Before defining a

Buchsbaum variety, we will recall the definition of a Buchsbaum ring with describing basic property according to [10, 12].

Definition 2.1. Let R be a Noetherian local ring with the maximal ideal m. Let M be a finitely generated R-module of $\dim M = d$. The R-module M is a Buchsbaum module if the difference $length_R(M/\mathsf{q}M) - e(\mathsf{q}; M)$ is independent of the choice of a parameter ideal q for M, where $e(\mathsf{q}; M)$ is the multiplicity of q on M. In case $M = R$, R is called a Buchsbaum ring.

Proposition 2.2. ([10, (1.10)], [12, (2.5)]). *Let R be a Noetherian local ring with the maximal ideal m. Let M be a finitely generated R-module of $\dim M = d$. The following conditions are equivalent:*

(a) *The R-module M is Buchsbaum.*
(b) *For any system of parameters $a_1, \cdots, a_d$ for R-module M, the equality $[(a_1, \cdots, a_{i-1})M : a_i] = [(a_1, \cdots, a_{i-1})M : \mathsf{m}]$ holds for $i = 1, \cdots, d$.*
(c) *For any system of parameters $a_1, \cdots, a_d$ for R-module M, $\mathsf{q}\mathrm{H}^i_{\mathsf{m}}(M/\mathsf{q}_jM) = 0$ for all non-negative integers i, j with $i + j < d$, where $\mathsf{q}_j = (a_1, \cdots, a_j)$ and $\mathsf{q} = \mathsf{q}_d$.*

Remark 2.3. Let R be a Noetherian local ring with the maximal ideal m. Let M be a finitely generated R-module of $\dim M = d$. The R-module M is called quasi-Buchsbaum if $\mathsf{m}\mathrm{H}^i_{\mathsf{m}}(M) = 0$ for $i = 0, \cdots, d-1$. From (2.2), the Buchsbaum property implies the quasi-Buchsbaum property.

Definition 2.4. Let $S = k[x_0, \cdots, x_N]$ be a polynomial ring over a field k. Let $\mathsf{m} = S_+$ be the homogeneous maximal ideal of S. Let us consider S as a graded ring with $\deg x_i = 1$ for $i = 0, \cdots, N$. Let I be a homogeneous ideal of S. Then $R = S/I$ is a graded ring. Let us put $\mathsf{n} = \mathsf{m}/I$ and $\dim R = n+1$. The graded ring R is a graded Buchsbaum ring if it satisfies one of the following conditions:

(a) R_{n} is a Buchsbaum ring.
(b) For any homogeneous system of parameters $z_0, \cdots, z_n$ of R, $[(z_0, \cdots, z_{i-1}) : z_i] = [(z_0, \cdots, z_{i-1}) : \mathsf{n}]$ holds for $i = 0, \cdots, n$.
(c) For any linear system of parameters $z_0, \cdots, z_n$ of R, $[(z_0, \cdots, z_{i-1}) : z_i] = [(z_0, \cdots, z_{i-1}) : \mathsf{n}]$ holds for $i = 0, \cdots, n$.

Definition 2.5. Let $S = k[x_0, \cdots, x_N]$ be the polynomial ring over k with the homogeneous maximal ideal m. Let $V (\subset \mathbb{P}^N_k = \operatorname{Proj} S)$ be a projective scheme. Let $I = \Gamma_*\mathcal{I}_{V/\mathbb{P}^N}$ be the defining ideal of V. Let $R = S/I$ be the coordinate ring of V. The deficiency module of V is defined as $\mathrm{M}^i(V) = \mathrm{H}^i_*\mathcal{I}_{V/\mathbb{P}^N} = \oplus_{\ell\in\mathbb{Z}}\mathrm{H}^i(\mathcal{I}_{V/\mathbb{P}^N}(\ell))$ for $i = 1, \cdots, \dim V$, see [3]

The projective scheme V is called a quasi-Buchsbaum scheme if $\mathsf{m}\mathrm{M}^i(V) = 0$ holds for $i = 1, \cdots, \dim V$.

Definition 2.6. Let $V (\subset \mathbb{P}^N_k = \operatorname{Proj} S)$ be a projective scheme with $\dim V = n$. The scheme V is a Buchsbaum scheme if it satisfies one of the following equivalent conditions:

(a) The coordinate ring R of V is a graded Buchsbaum ring.
(b) For any hyperplanes $H_1, \cdots, H_{n-1}$ satisfying that $\dim V_j = \dim V - j$, where $V_{n-j} = V \cap H_1 \cap \cdots \cap H_j$ for $j = 0, \cdots, n-1$, the scheme V_j is quasi-Buchsbaum.

In case V is irreducible and reduced, we call V a Buchsbaum variety.

Remark 2.7. We simply call a Buchsbaum variety in this paper while it is called an arithmetically Buchsbaum variety in [8, 13]. If V is a Buchsbaum variety, then a generic hyperplane section of V is also a Buchsbaum variety.

3. Regularity of Buchsbaum variety

Now let us investigate a Castelnuovo-type bound for the Castelnuovo-Mumford regularity for Buchsbaum varieties. Let us start with stating a result of Stückrad-Vogel [11]. We will explain of the process of the proof in order to find the boundary and the next boundary cases.

Lemma 3.1. *Let V be a Buchsbaum variety of $\mathbb{P}_k^N$ with $n = \dim V \geq 1$ over an algebraically closed field. Let $W = V \cap H$ be a generic hyperplane section. Then we have* $\operatorname{reg} W = \operatorname{reg} V$.

Proof. Let us consider the exact sequence $0 \to \mathcal{I}_{V/\mathbb{P}_k^N}(-1) \xrightarrow{\cdot h} \mathcal{I}_{V/\mathbb{P}_k^N} \to \mathcal{I}_{W/H} \to 0$. Since a graded homomorphism $\mathrm{H}_*^i(\mathcal{I}_{V/\mathbb{P}_k^N})(-1) \xrightarrow{\cdot h} \mathrm{H}_*^i(\mathcal{I}_{V/\mathbb{P}_k^N})$ is zero for $i = 1, \cdots, n$, we have the following exact sequences:

$$0 \to \mathrm{H}_*^i(\mathcal{I}_{V/\mathbb{P}_k^N}) \to \mathrm{H}_*^i(\mathcal{I}_{W/H}) \to \mathrm{H}_*^{i+1}(\mathcal{I}_{V/\mathbb{P}_k^N})(-1) \to 0.$$

$$0 \to \mathrm{H}_*^n(\mathcal{I}_{V/\mathbb{P}_k^N}) \to \mathrm{H}_*^n(\mathcal{I}_{W/H}) \to \mathrm{H}_*^{n+1}(\mathcal{I}_{V/\mathbb{P}_k^N})(-1) \xrightarrow{\cdot h} \mathrm{H}_*^{n+1}(\mathcal{I}_{V/\mathbb{P}_k^N}) \to 0$$

Hence we obtain $\operatorname{reg} V = \operatorname{reg} W$. $\square$

Proposition 3.2 ([11])**.** *Let V be a nondegenerate Buchsbaum variety of $\mathbb{P}_k^N$ over an algebraically closed field. Then* $\operatorname{reg} V \leq \lceil (\deg V - 1)/\operatorname{codim} V \rceil + 1$.

Proof. From (3.1), the inequality follows from the fact that $\operatorname{reg} X \leq \lceil (\deg X - 1)/\operatorname{codim} X \rceil + 1$ for a generic hyperplane section X of a nondegenerate projective curve, which is an easy consequence of the Uniform Position Principle for characteristic zero and also works for positive characteristic case. Let R be the coordinate ring of a zero-dimensional scheme $X \subseteq \mathbb{P}_k^N$. Let $\underline{h} = \underline{h}(X) = (h_0, \cdots, h_s)$ be the h-vector of $X \subseteq \mathbb{P}_k^N$, where $h_i = \dim_k[R]_i - \dim_k[R]_{i-1}$ and s is the largest integer such that $h_s \neq 0$. Note that $s = \operatorname{reg}(X) - 1$. Since X is a generic hyperplane section of a projective curve, we have $h_1 + \cdots + h_i \geq i h_i$ for all $i = 1, \cdots, s-1$, that is, $\mathrm{H}_X(t) \geq \min\{\deg(X), tN+1\}$. Since $\deg(X) = h_0 + \cdots + h_s$ and $\operatorname{codim}(X) = h_1 = N$, we obtain $\lceil (\deg(X) - 1)/\operatorname{codim}(X) \rceil = \lceil (h_1 + \cdots + h_s)/h_1 \rceil \geq s$. $\square$

In general, a nondegenerate projective variety $V \subset \mathbb{P}_k^N$ satisfies $\deg V \geq \operatorname{codim} V + 1$. The projective variety V is called a variety of minimal degree if $\deg V = \operatorname{codim} V + 1$. In this case, the variety V is classified to be a hyperquadric, (a cone over the) Veronese surface, a rational normal scroll, see [2, (3.10)]. A nondegenerate projective variety V is a variety of almost minimal degree if $\deg V = \operatorname{codim} V + 2$, which is classified to be either a normal del Pezzo variety or the image of a variety of minimal degree via a projection.

The regularity of Buchsbaum divisor on a variety of minimal degree can be calculated, and it gives an extremal case as follows.

Lemma 3.3 ([8]). *If a nondegenerate Buchsbaum variety $V \subseteq \mathbb{P}_k^N$ is a divisor on a variety of minimal degree, then* $\operatorname{reg} V = \lceil(\deg V - 1)/\operatorname{codim} V\rceil + 1$.

Now we will describe Buchsbaum varieties with the maximal and the next maximal regularity of Castelnuovo-type.

Theorem 3.4 ([8, 13]). *Let $V \subseteq \mathbb{P}_k^N$ be a nondegenerate Buchsbaum variety over an algebraically closed field k with* $\operatorname{char} k = 0$ *or* $\operatorname{codim} V \geq 5$. *If* $\deg V \geq (\operatorname{codim} V)^2 + 2\operatorname{codim} V + 2$ *and* $\operatorname{reg} V = \lceil(\deg V - 1)/\operatorname{codim} V\rceil + 1$, *then V is a divisor on a variety of minimal degree.*

Theorem 3.5 ([6]). *Let $V \subset \mathbb{P}_k^N$ be a nondegenerate Buchsbaum variety over an algebraically closed field k with* $\operatorname{char} k = 0$. *Assume* $\deg V \geq (\operatorname{codim} V)^2 + 4\operatorname{codim} V + 2$. *If* $\operatorname{reg} V = \lceil(\deg V - 1)/\operatorname{codim} V\rceil$, *then V is a divisor on a del Pezzo variety.*

One of the key points of the proofs of (3.4) and (3.5) is to control the regularity of the zero-dimentsional scheme under the successive generic hyperplane sections.

Lemma 3.6 ([4, 5]). *Let $X \subseteq \mathbb{P}_k^N$ be a generic hyperplane section of a nondegenerate projective curve over an algebraically closed field k with* $\operatorname{char} k = 0$.

(1) *If* $\deg X \geq N^2 + 2N + 2$ *and* $\operatorname{reg} X = \lceil(\deg X - 1)/N\rceil + 1$, *then X lies on a rational normal curve in $\mathbb{P}_k^N$.*
(2) *If* $\deg X \geq N^2 + 4N + 2$ *and* $\operatorname{reg} X = \lceil(\deg X - 1)/N\rceil$, *then X lies on either a rational normal curve or an elliptic normal curve in $\mathbb{P}_k^N$.*

The proof of (3.6) make use of the case $m = 0, 1$ in (3.7). The Eisenbud-Harris conjecture (3.7) are solved for $m = 0$ by Castelnuovo and for $m = 1$ by Eisenbud-Harris, and for $m = 2$ under slightly stronger condition by Petrakiev, see [9].

Conjecture 3.7. *Let X be a set of $d(\geq 2n + 2m - 1)$ points in uniform position in $\mathbb{P}_k^{n-1}$, where $1 \leq m \leq n - 3$. Suppose that $h_X(2) = 2n + m - 2$. Then X lies on a curve C of degree at most $n + m - 2$.*

Sketch of the proof of (3.5). Let $n = \dim V$. Let us take generic hyperplanes $H_1, \cdots, H_n$. Let us define $V_{n-j} = V \cap H_1 \cap \cdots \cap H_j$ and

$L_{n-j} = H_1 \cap \cdots \cap H_j$ for $j = 0, \cdots, n$. Then $\dim V_i = i$ and $L_i \cong \mathbb{P}_k^{N-n+i}$ for $i = 0, \cdots, n$. From (3.1), we have $\operatorname{reg} V_0 = \operatorname{reg} V_1 = \cdots = \operatorname{reg} V_n$. So, $\operatorname{reg} V_0 = \lceil (\deg V_0 - 1)/\operatorname{codim} V_0 \rceil$. By (3.6) and (3.3), V_0 lies on an elliptic normal curve Y_0. The defining equations of an elliptic normal curve consist of quadric equations except for the case Y_0 a plane cubic curve.

Let $c = \operatorname{codim} V = \operatorname{codim} V_0$ and $d = \deg V = \deg V_0$. Then we see $\deg Y_0 = \operatorname{codim} Y_0 + 2 = c + 1$.

Let us consider only the case $c \geq 3$. We want to have a nondegenerate projective surface Y_1 containing V_1 such that $Y_1 \cap H_0 = Y_0$. In order to construct Y_1 we need to lift up to the defining equation of Y_0. Thus we have only to show that $\Gamma(\mathcal{I}_{Y_0}(2)) \cong \Gamma(\mathcal{I}_{V_0}(2))$ and that $\Gamma(\mathcal{I}_{V_1}(2)) \to \Gamma(\mathcal{I}_{V_0}(2))$ is surjective.

Indeed, if there exists a hyperquadric Q such that $V_0 \subseteq Q$ and $Y_0 \nsubseteq Q$, then $V_0 \subseteq Y_0 \cap Q$ and $d \leq 2(c+1)$ by Bezout theorem, which contradicts the assumption $d \geq c^2 + 4c + 2$.

In order to prove that $\Gamma(\mathcal{I}_{V_1}(2)) \to \Gamma(\mathcal{I}_{V_0}(2))$ is surjective we have only to show $\mathrm{H}^1(\mathcal{I}_{V_1}(1)) = 0$.

The exact sequence $\mathrm{H}^1_*(\mathcal{I}_{V_1}(-1)) \xrightarrow{\cdot h} \mathrm{H}^1_*(\mathcal{I}_{V_1}) \to \mathrm{H}^1_*(\mathcal{I}_{V_0})$ leads to an injective map $\mathrm{H}^1_*(\mathcal{I}_{V_1}) \to \operatorname{Soc}(\mathrm{H}^1_*(\mathcal{I}_{V_0}))$ because V_1 is a Buchsbaum variety and $\mathsf{m}\mathrm{H}^1_*(\mathcal{I}_{V_1}) = 0$. So, let us study the structure of $\operatorname{Soc}(\mathrm{H}^1_*(\mathcal{I}_{V_0}))$ in the positive graded part as S-graded module. Since Y_0 is ACM, we have the exact sequence $\mathrm{H}^1_*(\mathcal{I}_{Y_0}) = 0 \to \mathrm{H}^1_*(\mathcal{I}_{V_0}) \to \mathrm{H}^1_*(\mathcal{I}_{V_0/Y_0}) \to \mathrm{H}^2_*(\mathcal{I}_{Y_0})$. Note that $\mathrm{H}^2(\mathcal{I}_{Y_0}(\ell)) \cong \mathrm{H}^1(\mathcal{O}_{Y_0}(\ell)) \cong (\mathrm{H}^0(\mathcal{O}_{Y_0}(-\ell)))' = 0$ for $\ell > 0$. Thus we have $\operatorname{Soc}(\mathrm{H}^1_*(\mathcal{I}_{V_0})) = \operatorname{Soc}(\mathrm{H}^1_*(\mathcal{I}_{V_0/Y_0}))$ in the positive graded part.

Let us only consider the case Y_0 smooth. In this case we see that $\mathcal{I}_{V_0/Y_0} \cong \mathcal{O}_{Y_0}(-V_0)$. By Serre duality the graded S-module $\operatorname{Soc}(\mathrm{H}^1_*(\mathcal{I}_{V_0/Y_0}))$ is isomorphic to the dual of $\Gamma_*(\mathcal{O}_{Y_0}(V_0))/\mathsf{m}\Gamma_*(\mathcal{O}_{Y_0}(V_0))$. Let $\mathcal{F} = \mathcal{O}_{Y_0}(V_0)$. Then we have $\mathrm{H}^1(\mathcal{F} \otimes \mathcal{O}_{Y_0}(m-1)) = 0$ for $d + (c+1)(m-1) \geq 1$. In other words, $\mathcal{F}$ is m-regular for $m \geq (c - d + 2)/(c+1)$. Let us put $m = \lceil (c - d + 2)/(c+1) \rceil$. Then $\Gamma(\mathcal{F} \otimes \mathcal{O}_{Y_0}(\ell)) \otimes \Gamma(\mathcal{O}_{Y_0}(1)) \to \Gamma(\mathcal{F}(\ell+1))$ is surjective for $\ell \geq m$. Hence we obtain $a_-(\operatorname{Soc}(\mathrm{H}^1_* \mathcal{I}_{V_0/Y_0})) \geq -m$. Hence we see that $a_-(\operatorname{Soc}(\mathrm{H}^1_*(\mathcal{I}_{V_0}))) \geq 2$ if $d \geq 3c + 4$. Since $d \geq c^2 + 4c + 2$, we obtain $\mathrm{H}^1(\mathcal{I}_{V_1}(1)) = 0$.

Then for $1 \leq i \leq n-1$. we will proceed to construct inductively a variety Y_{i+1} of almost minimal degree containing V_{i+1} by showing $\mathrm{H}^1(\mathcal{I}_{V_i}(1)) = 0$. Hence the assertion is proved. □

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