Castelnuovo-type bounds on the Castelnuovo-Mumford regularity and Horrocks-type theorems for vector bundles

Chikashi Miyazaki [1]

November 24, 2023

The Castelnuovo-Mumford regularity introduced by Mumford generalizing an idea of Castelnuovo is one of the most important invariants measuring the complexity of the minimal freeresolution of polynomial ideals.

Let $S = k[x_0, \cdots, x_n]$ be a polynomial ring over k with $\mathfrak{m} = (x_0, \cdots, x_n)$. Let M be a finitely generated grade S-module. Then we set $a_i(M) = \sup\{\ell \in \mathbb{Z} | [\mathrm{H}^i_{\mathfrak{m}}(M)]_\ell \neq 0\}$ for i, and especially $a(M) = a_{\dim M}(M)$ is called an a-invariant. The Castelnuovo-Mumford regularity of M is defined as $\operatorname{reg} M = \{a_i(M) + i | i \in \mathbb{Z}\}$. Then we have $\operatorname{reg} M = \max_{i,j}\{\alpha_{i,j}\}$ for the minimal free resolution $0 \to F_s \to \cdots \to F_1 \to F_0 \to M$ of M, where $F_i = \oplus_j S(-\alpha_{i,j})$ is a graded free S-module.

For a projective scheme $V \subseteq \mathbb{P}^n$, the regularity is defined as $\operatorname{reg} V = \operatorname{reg} I$, where I is the defining ideal $I = \Gamma_*(\mathcal{I}_V)(\subseteq S)$ of V. The most famous one is the Eisenbud-Goto conjecture $\operatorname{reg} V \leq \deg V - \operatorname{codim} V + 1$ for a nondegenerate projective variety $V \subseteq \mathbb{P}^n$. The Gruson-Lazarsfeld-Peskine paper (1083) have obtained it for curve cases, describing interesting geometric configuration concerning secant lines. Many important results have followed for higher dimensional cases, and finally the McCullough-Peeva paper (2018) has settled down to give counterexamples of the conjecture, giving great impact.

On the other hand, another bound by Stückrad-Vogel (1987) $\operatorname{reg} V \leq \lceil (\deg V - 1)/\operatorname{codim} V \rceil + 1$ for a nondegenerate Buchsbaum variety had inspired us to obtain $\operatorname{reg} R \leq a(R) + \dim R + 1$ for a Buchsbaum grades ring R, in which Hoa and I investigated a relation between non-vanishing part of intermediate cohomologies and a-invariants of Buchsbaum graded rings in 1995. I call it a Castelnuovo-type bound since Castelnuovo's method for the uniform position principle has an important role for its proof. Further, border examples (and next to border examples) must be divisors on a variety of minimal degree (and a Del Pezzo variety) under certain conditions.

My talk begins with bounds and extremal examples for Castelnuovo-type bounds in order to extend an analogical regularity bound. Beforehand, the k-Buchsbaum property, that is, the intermediate local cohomologies killed by $\mathfrak{m}^k$, has been used to describe the regularity bounds and to give several results, but Hoa's conjecture, $\operatorname{reg} V \leq \lceil (\deg V - 1)/\operatorname{codim} V \rceil + k$ for nondegenerate strongly k-Buchsbaum varities, has not been solved, although sharp examples of the conjecture exist for all $\dim V$ and all k. For that reason, I have been trying to get closer to the conjecture by a commutative algebra method.

Definition 1. Let $f_1, \cdots, f_{d+1}$ be a homogeneous system of parameters of a graded S-module M. We call $f_1, \cdots, f_{d+1}$ as a standard homogeneous system of parameters if $\mathfrak{q}\mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{q}_j M) = 0$ for all nonnegative integers i, j with $i + j \leq d$, where $\mathfrak{q}_j = (f_1, \cdots, f_j)$, $j = 0, \cdots, d+1$ and $\mathfrak{q} = \mathfrak{q}_{d+1}$.

Proposition 2. Let $S = k[x_0, \cdots, x_n]$ be a polynomial ring over a field k. Let M be a finitely generated graded S-module of $\dim M = d+1$ and $\operatorname{depth} M \geq 1$. Let $v \geq 1$ be an integer. If there exists a standard homogeneous system of parameters $f_1, \cdots, f_{d+1}$ of M with $\sum_{i=1}^{d+1} \deg f_i = d+v$, then $a_i(M) + i \leq a(M) + d + 1 + v$.

[1]Partially supported by Grant-in-Aid for Scientific Research (C) (21K03167) Japan Society for the Promotion of Science

Then I will go to another topic, the splittings of vector bundles, very closely related to the Castelnuovo-Mumford regularity theory from the view point of commutative algebra. Horrocks' celebrated theorem says that a vector bundle $\mathcal{E}$ on $\mathbb{P}^n$ having ACM property, that is, $\mathrm{H}^i_*(\mathbb{P}^n, \mathcal{E}) = \oplus_{\ell\in\mathbb{Z}}\mathrm{H}^i(\mathbb{P}^n, \mathcal{E}(\ell)) = 0$ for $1 \leq i \leq n-1$, is isomorphic to a direct sum of line bundles. There are several proofs. Probably best-known proof is due to a dimensional induction based on Grothendieck theorem for $n = 1$. The second proof can also be an immediate consequence of the Auslander-Buchsbaum theorem, that is, "depth + proj.dim = n". The third one comes from the theory of the Castelnuovo-Mumford regularity, which has astonished me to write a paper with Malaspina to extend this result, and I am still working on the track.

Definition 3. A vector bundle $\mathcal{E}$ on $\mathbb{P}^n = \operatorname{Proj} k[x_0, \cdots, x_n]$ is called a Buchsbaum bundle if $(x_0, \ldots, x_n)\mathrm{H}^i_*(\mathbb{P}^n, \mathcal{E}|_L) = 0$, $1 \leq i \leq r-1$ for any r-plane $L(\subseteq \mathbb{P}^n)$, $r = 1, \cdots, n$.

In fact, Chang and Goto have independently obtained that a Buchsbaum bundle $\mathcal{E}$ on $\mathbb{P}^n$ is isomorphic to a direct sum of sheaves of differential forms. Now I will describe a syzygy theoretic method based on the Castelnuovo-Mumford regularity technique. Referring the Horrocks correspondence, and will give a new sight of the structure of vector bundles and 'intermediate cohomologies'. This consideration sheds some light to some vector bundles such as ACM, Buchsbaum bundles and null-correlation bundles, hopefully more. Finally, I am trying to chracterize the null-correlation bundle throough a syzygy theoretic approach.

Remark 4. A Null-corelation bundle $\mathcal{N}$ on $\mathbb{P}^3$ is defined as $0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \Omega_{\mathbb{P}^3}(1) \to \mathcal{N}^\vee \to 0$, which has rank 2, selfdual and simple, having $\mathrm{H}^2_*(\mathcal{N}) \cong k(3)$ and $\mathrm{H}^1_*(\mathcal{N}) \cong k(1)$, quasi-Buchsbaum not Buchsbaum.

References

[1] L. T. Hoa and C. Miyazaki, Bounds on Castelnuovo-Mumford regularity for generalized Cohen-Macaulay graded rings, Math. Ann. 301 (1995), 587 – 598.

[2] C. Miyazaki and W. Vogel, Bounds on cohomology and Castelnuovo-Mumford regularity, J. Algebra 185 (1996), 626 – 642.

[3] C. Miyazaki, Sharp bounds on Castelnuovo-Mumford regularity, Trans. Amer. Math. Soc. 352 (2000), 1675 – 1686.

[4] C. Miyazaki, Buchsbaum varieties with next to sharp bounds on Castelnuovo-Mumford regularity, Proc. Amer. Math. Soc. 139 (2011), 1909 – 1914.

[5] F. Malaspina and C. Miyazaki, Cohomological property of vector bundles on biprojective spaces, Ric. Mat. 67(2018), 963–968.

[6] C. Miyazaki, Bounds on Castelnuovo-Mumford regularity for graded modules and projective varieties, Beitr. Algebra Geom. 60 (2019), 57 – 65.

[7] C. Miyazaki, Syzygy theoretic approach to Horrocks-type criteria for vector bundles, in preparation.

Kumamoto University
Faculty of Advanced Science and Technology
e-mail: cmiyazak@educ.kumamoto-u.ac.jp