Syzygy Theoretic Approach to Horrocks-type Criteria for Vector Bundles

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Abstract

This paper studies a variant of Horrocks criteria for vector bundles mainly through a syzygy theoretic approach. In this spirit we begin with describing various proofs of the splitting criteria for ACM and Buchsbaum bundles, giving new sights of the structure theorem. Our main result gives a structure theorem of quasi-Buchsbaum bundles on $\mathbb{P}^n$, which characterizes the null-correlation bundle. Also, the quasi-Buchsbaum bundles on $\mathbb{P}^3$ with simple cohomologies are classified in terms of standard system of parameters.

1 Introduction

The purpose of this paper is to study Horrocks-type criteria for vector bundles on the projective space. Horrocks' celebrated theorem [10] says that a vector bundle on the projective space without intermediate cohomologies is isomorphic to a direct sum of line bundles. There are several proofs, say, Okonek-Schneider-Spindler[21] or Matsumura [15], due to inductive arguement. The original proof by Horrocks is modernly described as a categorical equivalence, e.g., Walter[25] and Malaspina-Rao[14]. In this paper, we are pursuing research into this topic through the Castelnuovo-Mumford regularity. This is our starting point, highlighted in Section 2 and 3. In these sections we will give not only somewhat extended introductions but also prepare methods in order to apply to the main results of Section 4 and 5.

Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^n = \operatorname{Proj} S$, where $S = k[x_0, \cdots, x_n]$ and $\mathfrak{m} = (x_0, \cdots, x_n)$. Let us wirte $M = \Gamma_*(\mathcal{E}) = \oplus_{\ell \in \mathbb{Z}} \Gamma(\mathcal{E}(\ell))$ as a graded S-module. Note that $\dim M = n+1$, $\mathrm{H}^0_{\mathfrak{m}}(M) = \mathrm{H}^1_{\mathfrak{m}}(M) = 0$, $\mathcal{E} = \widetilde{M}$ and $\mathrm{H}^i_*(\mathcal{E}) = \mathrm{H}^{i+1}_{\mathfrak{m}}(M)$ for $i \geq 1$. A vector bundle $\mathcal{E}$ has an ACM property if M is a Cohen-Macaulay graded S-module. In commutative algebra, the Horrocks theorem asserts that a Cohen-Macaulay graded S-modlule is graded free. In this direction we will study the Buchsbaum property and the quasi-Buchsbaum property for graded S-modules. In particular, we give a structure theorem of a graded S-module M with $\mathrm{H}^2_{\mathfrak{m}}(M) \cong \mathrm{H}^n_{\mathfrak{m}}(M) \cong k$ and $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for

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$i \neq 2, n, n+1$. In Section 3, we explain the structure theorem of Buchsbaum vector bundles on the projective space, which was given by Goto and Chang independently. A Buchsbaum vector bundle on the projective space is obtained to be isomorphic to a direct sum of sheaves of differential p-form with balanced twist. The methods play an important role in investigating quasi-Buchsbaum bundles.. After a survey of three standard proofs, we will give a new syzygy theoretic proof of the Chang-Goto theorem.

In Section 4, we study a quasi-Buchsbaum vector bundle on projective space. A vector bundle $\mathcal{E}$ on $\mathbb{P}^n = \operatorname{Proj} S$ is quasi-Buchsbaum if $\mathfrak{m}\mathrm{H}^i_*(\mathcal{E}) = 0$ for $1 \le i \le n-1$ as graded S-modules. There are some structure theorems for quasi-Buchsbaum bundles of rank 2, say Ellia-Sarti [8], Chang [5] and Kumar-Rao [12]. Our main result, Theorem 4.9, describes intensively a vector bundle on $\mathbb{P}^n$ with $\mathrm{H}^1_*(\mathcal{E}) \cong \mathrm{H}^{n-1}_*(\mathcal{E}) \cong k$ and $\mathrm{H}^i_*(\mathcal{E}) = 0$ for $2 \le i \le n-2$, which characterizes the null-correlation bundle and provides interesting classification.

In Section 5, we will go on a syzygy theoretic approach for quasi-Buchsbaum bundles on $\mathbb{P}^3$, which I believe gives a striking method to study a free resolution of null-correlation bundles, providing a different flavor from Section 4. Focusing on $\mathbb{P}^3$, that is, $S = k[x_0, x_1, x_2, x_3]$, we will name 'pseudo-Buchsbaum' and 'nonstandard-Buchsbaum' for quasi-Buchsbaum bundles in terms of standard system of parameters, mostly studied in the theory of generalized Cohen-Macaulay modules. We will show that a nonstandard-Buchsbaum bundle $\mathcal{E}$ on $\mathbb{P}^3$ with $\dim_k \mathrm{H}^1_*(\mathcal{E}) = \dim_k \mathrm{H}^2_*(\mathcal{E}) = 1$ is isomorphic to a null-correlation bundle.

In Section 6, we remark an application of a syzygy theoretic method to vector bundles on multiprojective space.

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2 Horrocks criterion for ACM bundles on projective space

This section describes a survey of four proofs. Among them, the method of the third proof using the Castelnuovo-Mumford regularity penetrates the philosophy of the whole paper.

Definition 2.1. A vector bundle $\mathcal{E}$ on $\mathbb{P}^n$ is called an ACM bundle if $\mathrm{H}^i_*(\mathcal{E}) = 0$ for $1 \le i \le n-1$,

Theorem 2.2 (Horrocks [10])**.** *An ACM bundle $\mathcal{E}$ of rank r on $\mathbb{P}^n$ is isomorphic to a direct sum of line bundles, that is, $\mathcal{E} \cong \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(\ell_i)$ for some $\ell_1, \cdots, \ell_r \in \mathbb{Z}$.*

The first proof, probably best-known, is due to an induction on n, see, e.g., [21, (I, 2.3.1)]. For $n = 1$ it is a consequence of Grothendieck theorem. In case $n \ge 2$, since $\mathcal{E}|_H$ is ACM on $H \cong \mathbb{P}^{n-1}$ for a hyperplane H of $\mathbb{P}^n$, we have $\mathcal{E}|_H \cong \oplus \mathcal{O}_H(\ell_i)$ by the hypothesis of induction. Let us put $\mathcal{F} = \oplus \mathcal{O}_{\mathbb{P}^n}(\ell_i)$. Then we extend $\psi : \mathcal{F}|_H \cong \mathcal{E}|_H$ to a morphism $\varphi : \mathcal{F} \to \mathcal{E}$ by the exact sequence $\mathrm{Hom}_{\mathbb{P}^n}(\mathcal{F}, \mathcal{E}) \to \mathrm{Hom}_H(\mathcal{F}|_H, \mathcal{E}|_H) \to \mathrm{Ext}^1_{\mathbb{P}^n}(\mathcal{F}, \mathcal{E}(-1)) = 0$ from the ACM assumption. Since $\det \varphi \in \Gamma(\mathbb{P}^n, (\wedge^r \mathcal{F})^\vee \otimes (\wedge^r \mathcal{E})) \cong k$ does not vanish on H, we see φ is an isomorphism. □

The second proof is a consequence of the Auslander-Buchsbaum theorem [1], which says that a finitely generated module M over a Noetherian local ring R with $\operatorname{proj\,dim} M < \infty$ satisfies $\operatorname{depth} M + \operatorname{proj\,dim} M = \operatorname{depth} R$, see, e.g., [15, (19.1)]. We will sketch a graded analog. Let us put $S = k[x_0, \cdots, x_n]$ and $M = \Gamma_*(\mathcal{E})$. For a graded S-module M, we will show $\operatorname{depth}_S M + \operatorname{proj\,dim}_S M = n+1$. Let us take a minimal free resolution of a graded S-module M: $0 \to F_r \xrightarrow{\varphi_{r-1}} F_{r-1} \to \cdots \to F_1 \xrightarrow{\varphi_0} F_0 \to M \to 0$, where $r = \operatorname{proj\,dim}_S M$ and $F_t = \oplus_j S(-\ell_{tj})$ are graded free S-modules. In case $r = 0$, that is, M is graded free, it is clear. Note that $\operatorname{depth}_S M = \inf\{i | \operatorname{Ext}^i_S(k, M) \neq 0\}$. For $r = 1$, φ'_0 is zero in an exact sequence: $0 \to \operatorname{Ext}^n_S(k, M) \to \operatorname{Ext}^{n+1}_S(k, F_1) \xrightarrow{\varphi'_0} \operatorname{Ext}^{n+1}_S(k, F_0)$ from the minimality of the free resolution, and we see $\operatorname{depth}_S M = n$. For $r \geq 2$, we easily have the assertion from an exact sequence $0 \to \operatorname{Ker} \varphi_0 \to F_0 \to M \to 0$ by inductive arguement. In particular, if $\mathcal{E}$ is ACM, that is, $\operatorname{depth}_S M = n+1$, then M is a graded free S-module. □

The third proof illustrates an interesting application of basic properties of the Castelnuovo-Mumford regularity, see, e.g., [2, 13]. In fact, the Horrocks theorem immediately follows from Lemma 2.4.

Definition and Proposition 2.3. A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is called m-regular if $\mathrm{H}^i(\mathbb{P}^n, \mathcal{F}(m-i) = 0$ for $i \geq 1$. If $\mathcal{F}$ is m-regular, then $\mathcal{F}$ is $(m+1)$-regular and globally generated, see [20, 7]. The Castelnuovo-Mumford regularity $\operatorname{reg} \mathcal{F}$ is the minimal integer m such that $\mathcal{F}$ is m-regular.

Lemma 2.4. *Let $\mathcal{E}$ a vector bundle on $\mathbb{P}^n$. Assume that $\operatorname{reg} \mathcal{E} = a_n(\mathcal{E}) + n$, in other words, $\mathrm{H}^n(\mathcal{E}(m-1-n)) \neq 0$ for $m = \operatorname{reg} \mathcal{E}$. Then $\mathcal{O}_{\mathbb{P}^n}(-m)$ is a direct summand of $\mathcal{E}$.*

Proof. Let us take $m = \operatorname{reg} \mathcal{E}$ for a vector bundle $\mathcal{E}$, which gives a surjective map $\psi : \mathcal{O}^{\oplus}_{\mathbb{P}^n}(-m) \to \mathcal{E}$ for a globally generated vector bundle $\mathcal{E}(m)$. From the ACM property, we see that $\mathrm{H}^n(\mathbb{P}^n, \mathcal{E}(m-n-1)) \neq 0$. By Serre duality we have $\mathrm{H}^0(\mathcal{E}^\vee(-m)) \neq 0$, which gives a nonzero map $\varphi : \mathcal{E} \to \mathcal{O}_{\mathbb{P}^n}(-m)$. Since $\varphi \circ \psi$ is a nonzero map, $\mathcal{O}_{\mathbb{P}^n}(-m)$ is a direct summand of $\mathcal{E}$. □

The fourth proof is based on an idea of Horrocks' origical proof [10]. For reader's convenience we briefly explain the Horrocks correspondence following Walter[25] and Malaspina-Rao[14].

Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^n = \operatorname{Proj} S$, $S = k[x_0, \cdots, x_n]$. Then $M = \Gamma_* \mathcal{E}$ is a graded S-module. The graded $S^\vee$-module $M^\vee$, negatively graded, is finitely generated and finite projective dimension. Since $\operatorname{depth} M^\vee \geq 2$, we have an exact sequence $0 \to P^{n-1\vee} \to \cdots \to P^{0\vee} \to M^\vee \to 0$, where $P^{i\vee}$ is a dual of a graded S-free module for $i = 1, \cdots, n-1$. Then we have a complex of graded S-modules $0 \to M \to P^0 \to \cdots \to P^{n-1} \to 0$, and an exact sequence $0 \to \mathcal{E} \to \mathcal{P}^0 \to \cdots \to \mathcal{P}^{n-1} \to 0$ on $\mathbb{P}^n$. Then we see that $\mathrm{H}^i_*(\mathcal{E}) \cong \mathrm{H}^i(Q^\bullet)$, $1 \leq i \leq n-1$, precisely $\tau_{<n} \mathbb{R}\Gamma_* \mathcal{E} \cong Q^\bullet$, where $Q^\bullet$ is a complex $0 \to P^0 \to \cdots \to P^{n-1} \to 0$. By connecting a complex $0 \to M \to P^0 \to \cdots \to P^{n-1} \to 0$ and the minimal free resolution $0 \to P^{-n} \to \cdots \to P^{-1} \to M \to 0$ of a graded S-module M, we have an enlarged complex $P^\bullet : 0 \to P^{-n} \to \cdots \to P^0 \to \cdots \to P^{n-1} \to 0$, where $\mathrm{H}^i(P^\bullet)$ an S-module of finite length, and especially $\mathrm{H}^i(P^\bullet) = 0$, $i \notin \{1, \cdots, n-1\}$. In other words, any vector bundle $\mathcal{E}$

on $\mathbb{P}^n$ yields an object $\tau_{>0}\tau_{<n}\mathbb{R}\Gamma_*(\mathcal{E}) \cong P^\bullet$ of the derived category of bounded complexes of graded free S-modules.

Let $\underline{\mathbb{VB}}$ be the category of vector bundles on $\mathbb{P}^n$ modulo stable equivalence. Here vector bundles $\mathcal{E}$ and $\mathcal{F}$ on $\mathbb{P}^n$ are stable equivalent if there are direct sums of line bundles $\mathcal{L}$ and $\mathcal{M}$ such that $\mathcal{E} \oplus \mathcal{L} \cong \mathcal{F} \oplus \mathcal{M}$. Let us write **FinL** for the full subcategory of $C^\bullet \in Ob(D^\flat(\mathrm{S}-\mathbf{Mod}))$ such that $\mathrm{H}^i(C^\bullet)$ is a finite over S and $\mathrm{H}^i(C^\bullet) = 0$, $0 < i < n$. Thus we obtain Theorem 2.5, see ([25, (0.4)], [10]).

Theorem 2.5 (Horrocks, Walter, Malaspina-Rao)**.** *A functor* $\tau_{>0}\tau_{<n}\mathbb{R}\Gamma_* : \underline{\mathbb{VB}} \to \mathbf{FinL}$ *gives an equivalence of the categories. Inverse functor is* $\mathrm{Syz} : \mathbf{FinL} \to \underline{\mathbb{VB}}$.

From the Horrocks correspondence (2.5), the vanishing of the intermediate cohomologies of a vector bundle $\mathcal{E}$ on $\mathbb{P}^n$, that is, $\tau_{>0}\tau_{<n}\mathbb{R}\Gamma_*(\mathcal{E}) = 0$ implies that $\mathcal{E}$ is isomorphic to a direct sum of line bundles. □

3 Buchsbaum Bundles on Projective Space

This section investigates a survey on the Chang-Goto structure theorem (3.4) of Buchsbaum bundles on projective space, preparing basic facts and technique for our main results in the following sections. We will do the groundwork to extend their results towards the structure theorem of quasi-Buchsbaum bundles. The first proof [9] somehow easy-to-follow is based on 'surjectivity criterion' for Buchsbaum modules and technical lemma (3.8). The second proof [4] studies a map from $\Omega^p_{\mathbb{P}^n}(p)$ to $\mathcal{O}_{\mathbb{P}^n}$ in Case B, which has driven us to consider our research on the null-correlation bundle. The method will be applied in Section 4. After briefing Yoshino's proof [26] by the Horrocks correspondence, we explain a syzygy theoretic method as the fourth proof of (3.4), which will be made progress in the theory developed in Section 5.

Definition and Proposition 3.1 ([22, 23])**.** A graded S-module M with $\dim M = d$ is called as a Buchsbaum module it the following equivalent conditions are satisfied.

(i) $\ell(M/\mathfrak{q}M) - e(\mathfrak{q}; M)$ does not depend on the choice of any homogeneous parameter ideal $\mathfrak{q} = (y_1, \cdots, y_d)$.

(ii) For any homogeneous system $y_1, \cdots, y_d$ of parameters
$\mathfrak{m}\mathrm{H}^j_\mathfrak{m}(M/(y_1, \cdots, y_i)M) = 0$ for $0 \le i \le d-1$, $0 \le j \le d-i-1$.

(iii) $\tau_{<d}\mathbb{R}\Gamma_\mathfrak{m}(M)$ is isomorphic to a complex of k-vector spaces in $D^\flat(\mathrm{S}-\mathbf{Mod})$.

Definition 3.2. Let $S = k[x_0, \cdots, x_n]$ be the polynomial ring over a field k with $\mathfrak{m} = (x_0, \cdots, x_n)$. A vector bundle $\mathcal{E}$ on $\mathbb{P}^n = \mathrm{Proj}\, S$ is called a Buchsbaum bundle if $\mathfrak{m}\mathrm{H}^i_*(\mathcal{E}|_L) = 0$, $1 \le i \le r-1$ for any r-plane $L (\subseteq \mathbb{P}^n)$, $r = 1, \cdots, n$.

Remark 3.3. A vector bundle $\mathcal{E}$ on $\mathbb{P}^n = \mathrm{Proj}\, S$ is Buchsbaum if and only of a graded S-module $M = \Gamma_*(\mathcal{E})$ is Buchsbaum.

The Koszul complex $K_\bullet = K_\bullet((x_0, \cdots, x_n); S)$ gives the minimal free resolution of a graded S-module $k = S/\mathfrak{m}$. Then $\Omega^{p-1}_{\mathbb{P}^n} = \widetilde{E_p}$, where E_p is the p-th syzygy of a graded S-module k. Note that $E_0 \cong k$, $E_1 \cong \mathfrak{m}$ and $E_{n+1} \cong S(-n-1)$.

Theorem 3.4 (Chang [4], Goto [9]). *A Buchsbaum bundle $\mathcal{E}$ on $\mathbb{P}^n$ is isomorphic to a direct sum of sheaves of differential form, that is, $\mathcal{E} \cong \oplus_{i=1}^r \Omega_{\mathbb{P}^n}^{k_i}(\ell_i)$.*

In other words, a graded S-module $M = \Gamma_*(\mathcal{E})$ is isomorphic to $\oplus E_{p_i}(\ell_i)$.

First proof of (3.4)

We use descending induction on $t = \operatorname{depth}_S M \geq 2$. It is clear for the Cohen-Macaulay case, that is, $t = n + 1$. Let us assume that $t \leq n$. From a presentation $0 \to N \xrightarrow{f} F \xrightarrow{g} M \to 0$, where F is graded free, we see N is Buchsbaum and $\operatorname{depth}_S N = t + 1$. Then we have $N \cong \oplus E_{p_i}(\ell_i)$. By the dual sequence $0 \to M^\vee \to F^\vee \to N^\vee \xrightarrow{\partial} \operatorname{Ext}_S^1(M, S) \to 0$, we have short exact sequences

$$0 \to M^\vee \to F^\vee \to W \to 0 \quad \text{and} \quad 0 \to W \to N^\vee \xrightarrow{\partial} \operatorname{Ext}_S^1(M, S) \to 0.$$

Now we will prove that W is isomorphic to a direct sum of some copies of $E_p(\ell)$'s. Then $\widetilde{M^\vee}$ is isomorphic to a direct sum of sheaves of differential p-form with some twist by Lemma 3.8, and so is $\mathcal{E}$ as desired.

In order to describe the structure of W, let us put $N = N' \oplus N''$, where $N' \cong \oplus_{t+1 \leq k_i \leq n} E_{p_i}(\ell_i)$ and $N'' \cong \oplus E_{n+1}(\ell'_j)$. Note that $\operatorname{Ext}_S^1(M, S)$ is a k-vector space, that is, $\mathfrak{m}\operatorname{Ext}_S^1(M, S) = 0$ from the Buchsbaum property. So, we have only to show $\partial(N'^\vee) = 0$, that is, suffice to prove $\partial(E_j) = 0$ for $j = 1, \cdots, n$, and by local duality, equivalently, $\mathrm{H}_\mathfrak{m}^n(M) \to \mathrm{H}_\mathfrak{m}^{n+1}(N) \to \mathrm{H}_\mathfrak{m}^{n+1}(N')$ is zero. From the commutative diagram with exact rows

$$\begin{array}{ccccc} \mathrm{H}^n(x_0, \cdots, x_n; M) & \to & \mathrm{H}^{n+1}(x_0, \cdots, x_n; N) & \to & \mathrm{H}^{n+1}(x_0, \cdots, x_n; N') \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}_\mathfrak{m}^n(M) & \to & \mathrm{H}_\mathfrak{m}^{n+1}(N) & \to & \mathrm{H}_\mathfrak{m}^{n+1}(N') \end{array}$$

and the surjectivity of the left downarrow from the Buchsbaum property, what we need to show is $\mathrm{H}^{n+1}(x_0, \cdots, x_n; N') \to \mathrm{H}_\mathfrak{m}^{n+1}(N')$ is zero, which follows from Remark 3.5. □

Remark 3.5. Let $S = k[x_0, \cdots, x_n]$ be the polynomial ring. Let E_j be the j-th syzygy module. Then the natural map $\mathrm{H}^{n+1}(x_0, \cdots, x_n; E_j) \to \mathrm{H}_\mathfrak{m}^{n+1}(E_j)$ is zero for $1 \leq j \leq n$. Indeed, we give another proof of [9, (2.11)]. A graded S-module $\mathrm{H}^{n+1}(x_0, \cdots, x_n; E_j)(\cong \operatorname{Ext}_S^{n+1}(k, E_p)) \cong (E_p/\mathfrak{m}E_p)(n + 1)$ has nonzero elements only in degree $-n + p$. On the other hand, $\mathrm{H}_\mathfrak{m}^{n+1}(E_p) \cong \operatorname{Hom}_k(E_{n-p}, k)$ has nonzero elements only in degree $\leq -n+p-1$. Thus the assertion is proved. □

Remark 3.6. The structure theorem also works for $t = \operatorname{depth}_S M \leq 1$, see [9]. Indeed, in case $t = 0$, it can be reduced to the case $t \geq 1$ because $\mathrm{H}_\mathfrak{m}^0(M) \cap \mathfrak{m}M = (0)$ implies that $\mathrm{H}_\mathfrak{m}^0(M)$ is a direct summand of M. In case $t = 1$, an exact sequence $0 \to M \to M^{\vee\vee} \to \operatorname{Ext}_S^1(W, S) \to 0$ gives the assertion as in the first proof of (3.4).

Second proof of (3.4)

Let us denote $i(\mathcal{E})(= n+1-\operatorname{depth} M^\vee)$ as the maximal integer i such that $\mathrm{H}_*^p(\mathcal{E}) = 0$, $i+1 \leq p \leq n-1$. We use induction on $i = i(\mathcal{E})$. It is clear for $i = 0$. Let us assume $i \geq 1$. If $\mathrm{H}_*^1(\mathcal{E}) \neq 0$, we have a short exact sequence of vector bundles $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{M} \to 0$, where $\mathcal{F}$ is a vector bundle with $\mathrm{H}_*^1(\mathcal{F}) = 0$ and $\mathcal{M}$ is a direct sum of line bundles on $\mathbb{P}^n$ by Lemma 3.7.

The minimal generator of $\Gamma_*(\mathcal{F}^\vee)$ give a short exact sequence $0 \to \mathcal{F} \to \mathcal{L} \to \mathcal{K} \to 0$, where $\mathcal{L}$ is a sum of line bundles. Then $\mathcal{K}$ is Buchsbaum with $i(\mathcal{K}) = i(\mathcal{E}) - 1$. Thus we have $\mathcal{K}$ is isomorphic to a direct sum of $\Omega^{p_j}_{\mathbb{P}^n}(k_j)$'s, and so is $\mathcal{F}$ by Lemma 3.8.

Note that $\mathrm{Hom}(\Omega^p_{\mathbb{P}^n}(\ell), \mathcal{O}_{\mathbb{P}^n}) \neq 0$ if and only if $\ell \leq p$. We may assume $k_j \leq p_j$ if $p_j \geq 2$. and $k_j < 0$ if $p_j = 0$, where $\mathcal{F} = \oplus \Omega^{p_j}_{\mathbb{P}^n}(k_j)$ and $\mathcal{L} = \oplus \mathcal{O}_{\mathbb{P}^n}(-c_i)^{\oplus \gamma_i}$, $0 = c_1 < \cdots < c_s$.

Case A. Let us consider the case $k_j < p_j$ for all $p_j \geq 0$. Then we will show that the exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{L} \to 0$ has a sequence $0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}^{n+1}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n} \to 0$ as a direct summand. As in [4, page 330 Case 1], we use only the quasi-Buchsbaum property, $\mathfrak{m}\mathrm{H}^i_*(\mathcal{E}) = 0$ for $1 \leq i \leq n-1$, to prove the assertion.

Case B. Let us consider the case $k_j = p_j$ for some j, that is, $\mathcal{F}$ has a direct summand of the form $\Omega^q_{\mathbb{P}^n}(q)$ for some $q > 1$. We want to show that $\Omega^q_{\mathbb{P}^n}(q) \to \mathcal{O}_{\mathbb{P}^n}$ is zero in the map $\mathcal{F} \to \mathcal{L}$, which gives the assertion. As in [4, page 331 Case 2], we use the Buchsbaum property of $\mathcal{E}$. □

Lemma 3.7. *Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^n$ with $\mathrm{H}^1_*(\mathcal{E}) \neq 0$. Then there exists a short exact sequence of vector bundles $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{L} \to 0$, where $\mathcal{F}$ is a vector bundle with $\mathrm{H}^1_*(\mathcal{F}) = 0$ and $\mathcal{L}$ is a direct sum of line bundles on $\mathbb{P}^n$.*

Proof. Let $s = \dim \mathrm{H}^1_*(\mathcal{E}) > 0$. A nonzero element of $\mathrm{H}^1(\mathcal{E}(-\ell_1))$ gives a short exact sequences $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^n}(\ell_1) \to 0$. Then $\dim \mathrm{H}^1_*(\mathcal{E}_1) = \dim \mathrm{H}^1_*(\mathcal{E}) - 1$. By repeating this process, we have vector bundles $\mathcal{E}_i$ satisfying short exact sequences $0 \to \mathcal{E}_i \to \mathcal{E}_{i+1} \to \mathcal{O}_{\mathbb{P}^n}(\ell_{i+1}) \to 0$, where $\mathcal{E}_0 = \mathcal{E}$ and $\dim \mathrm{H}^1_*(\mathcal{E}_{i+1}) = \dim \mathrm{H}^1_*(\mathcal{E}_i) - 1$ for $i = 0, \cdots, s-1$. Since the exact sequence $0 \to \mathcal{E}_i/\mathcal{E} \to \mathcal{E}_{i+1}/\mathcal{E} \to \mathcal{O}_{\mathbb{P}^n}(\ell_{i+1}) \to 0$ is inductively shown to split, we see $\mathcal{F}/\mathcal{E} \cong \oplus_{i=1}^s \mathcal{O}_{\mathbb{P}^n}(\ell_i)$ by taking $\mathcal{F} = \mathcal{E}_i$, as desired. □

Lemma 3.8. (cf. [4, (1.3)], [9, (3.5.2)]) *Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^n$ with $\mathrm{H}^1_*(\mathcal{E}) = 0$. Assume that there is an exact sequence $0 \to \mathcal{E} \to \mathcal{L} \to \mathcal{F} \to 0$, where $\mathcal{L}$ is a direct sum of line bundles not being any summand of $\mathcal{E}$, and $\mathcal{F} = \oplus_{p_j \geq 1} \Omega^{p_j}_{\mathbb{P}^n}(k_j)$. Then we have $\mathcal{E} \cong \oplus_{p_j \geq 1} \Omega^{p_j+1}_{\mathbb{P}^n}(k_j)$.*

Proof. We may assume $\mathcal{F} = \mathcal{F}' \oplus (\oplus_{q \geq 1} \Omega^q_{\mathbb{P}^n}(q+1)^{\oplus})$, where $\mathcal{F}' = \oplus_{p_j \leq k_j} \Omega^{p_j}(k_j)$, and $\mathcal{L}$ has no direct summand of positive degree. Since $\mathrm{H}^1_*(\mathcal{E}) = 0$ and a global section $\Omega^q_{\mathbb{P}^n}(q+1)$ is lifted up to a section of $\mathcal{L}$, there is a direct summand $\mathcal{O}^N_{\mathbb{P}^n}$ of $\mathcal{L}$. Then an exact sequence $0 \to \oplus_{q \geq 1} \Omega^{q+1}_{\mathbb{P}^n}(q+1)^{\oplus} \to \mathcal{O}^N_{\mathbb{P}^n} \to \oplus_{q \geq 1} \Omega^q_{\mathbb{P}^n}(q+1)^{\oplus} \to 0$ gives a direct summand of the sequence $0 \to \mathcal{E} \to \mathcal{L} \to \mathcal{F} \to 0$, which gives the assertion by repeating this process. □

Third proof of (3.4)

By (3.1) (iii), $\tau_{>0}\tau_{<n}\mathbb{R}\Gamma_*(\mathcal{E})(\cong \tau_{<n+1}\mathbb{R}\Gamma_{\mathfrak{m}}(M))$ is isomorphic to a compex of k-vector spaces, where $M = \Gamma_*(\mathcal{E})$ is a graded S-module. Since $\tau_{>0}\tau_{<n}\mathbb{R}\Gamma_*(\Omega^p_{\mathbb{P}^n})$ is isomorphic to a complex of one k-vector space, we see that a Buchsbaum bundle $\mathcal{E}$ is isomorphic to a direct sum of the sheaves of differential form $\Omega^p_{\mathbb{P}^n}(\ell)$ under stable equivalence from the categorical equivalence (2.5). □

Fourth proof of (3.4)

First we will give a summary of the Buchsbaum criterion in terms of spectral sequence [16, 17, 19] in order to apply to the syzgy theoretic proof of the structure theorem. For a graded S-module $M = \Gamma_*(\mathcal{E})$, we consider a Koszul complex $K_\bullet = K_\bullet((x_0, \cdots, x_n); S)$ and a Čech complex $L^\bullet = (0 \to M \to C^\bullet(\mathcal{U}; \mathcal{E})[-1])$, where $\mathcal{U}$ is an affine covering of $\mathbb{P}^n$. Then we take a double complex $\mathrm{Hom}_S(K_\bullet, L^\bullet)$, which yields a spectral sequence $\{\mathrm{E}_r^{i,j}\}$ such that

$$\mathrm{E}_1^{i,j} = \mathrm{H}_i((x_0, \cdots, x_n); \mathrm{H}^j_{\mathfrak{m}}(M)) \Rightarrow \mathrm{H}^{i+j} = \mathrm{H}^{i+j}((x_0, \cdots, x_n); M).$$

The natural map $\mathrm{H}^j = \mathrm{H}^j((x_0, \cdots, x_n); M) \to \mathrm{E}_1^{0,j} = \mathrm{H}^j_{\mathfrak{m}}(M)$ is surjective for $0 \le j \le n$ from the theory of Buchsbaum ring (cf.[17, 23]), and $d_r^{i,j} : \mathrm{E}_r^{i,j} \to \mathrm{E}_r^{i+r,j-r+1}$ is zero for $j \le n$, $r \ge 1$ ([16, (1.11)]).

Keeping the construction above in mind, we will give an analog of the spectral sequence. From the Koszul complex $K_\bullet = K_\bullet((x_0, \cdots, x_n); S)$, we have exact sequences:

(i) $\bar{N}^\bullet \,:\, 0 \to \Omega^p_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}^{\oplus a_p}(-p) \to \cdots \to \mathcal{O}_{\mathbb{P}^n}^{\oplus a_1}(-1) \to \mathcal{O}_{\mathbb{P}^n} \to 0$

(ii) $\bar{\bar{N}}^\bullet \,:\, 0 \to \mathcal{O}_{\mathbb{P}^n}(-n-1) \to \mathcal{O}_{\mathbb{P}^n}^{\oplus a_n}(-n) \to \cdots \to \mathcal{O}_{\mathbb{P}^n}^{\oplus a_{p+1}}(-p-1) \to \Omega^p_{\mathbb{P}^n} \to 0,$

where $\bar{N}^{-i} = \mathcal{O}_{\mathbb{P}^n}^{\oplus a_i}(-i)$ for $0 \le i \le p$, $\bar{N}^{-p-1} = \Omega^p_{\mathbb{P}^n}$ and $\bar{\bar{N}}^{-i} = \mathcal{O}_{\mathbb{P}^n}^{\oplus a_i}(-i)$ for $p+1 \le i \le n+1$, $\bar{\bar{N}}^{-p} = \Omega^p_{\mathbb{P}^n}$, $a_r = \begin{pmatrix} n+1 \\ r \end{pmatrix}$. Then the exact sequences

(iii) $0 \to \mathcal{E} \to \mathcal{E}^{\oplus a_1}(1) \to \cdots \to \mathcal{E}^{\oplus a_p}(p) \to \mathcal{E} \otimes \Omega^{p\vee}_{\mathbb{P}^n} \to 0$

(iv) $0 \to \mathcal{E}^\vee(-n-1) \to \mathcal{E}^{\vee \oplus a_n}(-n) \to \cdots \to \mathcal{E}^{\vee \oplus a_{p+1}}(-p-1) \to \mathcal{E}^\vee \otimes \Omega^p_{\mathbb{P}^n} \to 0$

give maps $\varphi : \mathrm{H}^0(\mathcal{E} \otimes \Omega^{p\vee}_{\mathbb{P}^n}) \to \mathrm{H}^p(\mathcal{E})$ and $\psi : \mathrm{H}^0(\mathcal{E}^\vee \otimes \Omega^p_{\mathbb{P}^n}) \to \mathrm{H}^{n-p}(\mathcal{E}^\vee(-n-1))$.

Lemma 3.9 (cf. [13])**.** *Under the condition above, assume that there is a nonzero element $s \in \mathrm{H}^p(\mathcal{E})$ and a corresponding element $t \in \mathrm{H}^{n-p}(\mathcal{E}^\vee(-n-1))$ by Serre duality satisfying that s and t can be lifted up to $\mathrm{H}^0(\mathcal{E} \otimes \Omega^{p\vee}_{\mathbb{P}^n})$ and $\mathrm{H}^0(\mathcal{E}^\vee \otimes \Omega^p_{\mathbb{P}^n})$ by φ ans ψ respectively. Then $\Omega^p_{\mathbb{P}^n}$ is a direct summand of $\mathcal{E}$.*

Proof. For $s(\neq 0) \in \mathrm{H}^p(\mathcal{E})$ there exists $f \in \mathrm{H}^0(\mathcal{E} \otimes \Omega^{p\vee}_{\mathbb{P}^n})$ such that $\varphi(f) = s(\neq 0) \in \mathrm{H}^p(\mathcal{E})$ Let us take $s \in \mathrm{H}^m(\mathcal{E})$ and $g \in \mathrm{H}^0(\mathcal{E}^\vee \otimes \Omega^p_{\mathbb{P}^n})$ corresponding to $t \in \mathrm{H}^{n-p}(\mathcal{E}^\vee(-n-1))$ and $\psi(g) = t(\neq 0) \in \mathrm{H}^{n-p}(\mathcal{E}^\vee(-n-1))$ respectively. Then we regard f and g as elements of $\mathrm{Hom}(\Omega^p_{\mathbb{P}^n}, \mathcal{E})$ and $\mathrm{Hom}(\mathcal{E}, \Omega^p_{\mathbb{P}^n})$ respectively. From a commutative diagram

$$\begin{array}{ccccc} \mathrm{H}^0(\mathcal{E} \otimes \Omega^{p\vee}_{\mathbb{P}^n}) \otimes \mathrm{H}^0(\mathcal{E}^\vee \otimes \Omega^p_{\mathbb{P}^n}) & \to & \mathrm{H}^0(\Omega^{p\vee}_{\mathbb{P}^n} \otimes \Omega^p_{\mathbb{P}^n}) & \cong & \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}) \\ \downarrow & & & & \downarrow \\ \mathrm{H}^p(\mathcal{E}) \otimes \mathrm{H}^{n-p}(\mathcal{E}^\vee(-n-1)) & \to & & & \mathrm{H}^n(\mathcal{O}_{\mathbb{P}^n}(-n-1)), \end{array}$$

the natural map $\mathrm{H}^0(\mathcal{E} \otimes \Omega^{p\vee}_{\mathbb{P}^n}) \otimes \mathrm{H}^0(\mathcal{E}^\vee \otimes \Omega^p_{\mathbb{P}^n}) \to \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n})$ gives an isomorphism $g \circ f$. Thus we obtain that $\Omega^p_{\mathbb{P}^n}$ is a direct summand of $\mathcal{E}$. □

Let us come back to the proof of (3.4). Let us consider a double complex $\mathrm{Hom}_S(\bar{N}^\bullet, L^\bullet)$. From the spectral sequence theory of Buchsbaum modules, we have a spectral sequence $\{\mathrm{F}_r^{i,j}\}$ which satisfies that $\mathrm{F}_1^{i,j} = \mathrm{H}_*^{j-1}(\mathcal{E}(i))^{\oplus a_i}$ for $0 \le i \le p$ and $j \ge 2$ and $\mathrm{F}_1^{p+1,j} = \mathrm{H}_*^{j-1}(\mathcal{E} \otimes \Omega^{p\vee}_{\mathbb{P}^n})$. Note that the maps $d_r^{i,j} : \mathrm{F}_r^{i,j} \to \mathrm{F}_r^{i+r.j-r+1}$ is zero for i, j, r with $1 \le r \le j \le n-1$ and $0 \le i \le p-r$ from the comparison between $\{\mathrm{E}_r^{i,j}\}$ and $\{\mathrm{F}_r^{i,j}\}$. Thus we have a surjective map $\varphi : \mathrm{H}^0(\mathcal{E} \otimes \Omega^{p\vee}_{\mathbb{P}^n}) \to \mathrm{H}^p(\mathcal{E})$. Similarly, a double complex constructed from the Čech resolution of $\mathcal{E}^\vee \otimes \bar{\bar{N}}^\bullet$ also gives a surjective map $\psi : \mathrm{H}^0(\mathcal{E}^\vee \otimes \Omega^p_{\mathbb{P}^n}) \to \mathrm{H}^{n-p}(\mathcal{E}^\vee(-n-1))$ from the spectral sequence criterion of Buchsbaum modules. Hence we have the assertion by (3.9). □

4 Towards Structure Theorem of quasi-Buchsbaum bundles on projective space — Null-Correlation Bundles

In the previous sections we have described cohomological and ring-theoretic criteria for vector bundles through the Cohen-Macaulay and Buchsbaum property, which characterizes sheaves of differential p-forms under stable equivalence. In this section we study quasi-Buchsbaum vector bundles and gives a characterization of null-correlation bundles on $\mathbb{P}^n$. Our main result (4.9) gives a classification of indecomposable vector bundles $\mathcal{E}$ on $\mathbb{P}^n$ with $\dim_k \mathrm{H}^1_*(\mathcal{E}) = \dim_k \mathrm{H}^{n-1}_*(\mathcal{E}) = 1$. In particular, if $\operatorname{rank} \mathcal{E} \leq n-1$, then $\mathcal{E}$ is isomorphic to a null-correlation bundle on an odd-dimensional projective space. The proof is based on (3.7) in the second proof and the regularity technique (4.7).

On the other hand, later in Section 5, we will take a syzygy theoretic way to this topic. We wish to extend this criterion to characterize some interesting examples of algebraic vector bundles such as the Horrocks-Mumford bundle.

Throughout this section we assume $\operatorname{char} k \neq 2$ to simplify a standardization of skew-symmetric matirices.

Let us define null-correlation bundles, see, e.g., [21], and also see [6] for generalized null-correlation bundles. Let $\mathbb{P}^n = \operatorname{Proj} S$, $S = k[x_0, \cdots, x_n]$, $\mathfrak{m} = (x_0, \cdots, x_n)$ with n odd. Let us write an element of $\Gamma(\Omega_{\mathbb{P}^n}(2))$ explicitly. Since $\Gamma(\Omega_{\mathbb{P}^n}(2))$ is the kernel of $\Gamma(\mathcal{O}_{\mathbb{P}^n}(1))^{\oplus n+1} \to \Gamma(\mathcal{O}_{\mathbb{P}^n}(2))$ in the Euler sequence, we have an element $(a_{00}x_0 + \cdots + a_{0n}x_n, \cdots, a_{n0}x_0 + \cdots + a_{nn}x_n)$ of $\Gamma(\mathcal{O}_{\mathbb{P}^n}(1))^{\oplus n+1}$ satisfying that $\sum_{ij} a_{ij}x_ix_j = 0$, where $a_{ij} \in k$, $i, j = 0, \cdots, n$. Then we have a skew-symmetric $(n+1) \times (n+1)$-matrix $A = (a_{ij})$, which gives a map $\mathcal{O}_{\mathbb{P}^n} \to \Omega_{\mathbb{P}^n}(2)$. Now assume that $\operatorname{rank} A = n+1$. Then the cokernel of this map defines a vector bundle of rank $n-1$. By standardizing the skew-symmetric matrix, we may take $A = \begin{pmatrix} O & B \\ -B & O \end{pmatrix}$, where B is a diagonal matrix $\operatorname{diag}(\lambda_1, \cdots, \lambda_{(n+1)/2})$ with $\lambda_i \neq 0$. By taking the dual and twisting of $\mathcal{O}_{\mathbb{P}^n} \to \Omega_{\mathbb{P}^n}(2)$, we have a surjective morphism $\varphi : \mathcal{T}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n}(1)$. Then a null-correlation bundle $\mathcal{N}$ is defined as $\operatorname{Ker} \varphi$, which gives a short exact sequence

$$0 \to \mathcal{N} \to \mathcal{T}_{\mathbb{P}^n}(-1)(\cong \Omega^{n-1}_{\mathbb{P}^n}(n)) \to \mathcal{O}_{\mathbb{P}^n}(1) \to 0.$$

Thus we see that a skew-symmetric $(n+1) \times (n+1)$-matrix A of rank $n+1$ defines a null-correlation bundle $\mathcal{N}$, which does not depend on a choice of A via a coordinate change. For example, $(x_1, -x_0, x_3, -x_2, \cdots, x_n, -x_{n-1}) \in \Gamma(\Omega_{\mathbb{P}^n}(2))$ gives an injective morphism $\mathcal{O}_{\mathbb{P}^n} \to \Omega_{\mathbb{P}^n}(2)$, which defines a null-correlation bundle, see, e.g., [21].

The intermediate cohomologies appear only in $\mathrm{H}^1(\mathcal{N}(-1))(\cong k)$ and $\mathrm{H}^{n-1}(\mathcal{N}(-n))(\cong k)$. Also, note that $\mathrm{H}^0(\mathcal{N}(\ell)) \neq 0$ only in case $\ell > 0$ and $\mathrm{H}^n(\mathcal{N}(\ell)) \neq 0$ only in case $\ell < -n-1$. The dual $\mathcal{N}^\vee$ has the same cohomology table as $\mathcal{N}$, and $\mathcal{N}^\vee \cong \wedge^{n-2}\mathcal{N}$, especially, $\mathcal{N}$ is self-dual for $n = 3$. Thus we see that $\operatorname{reg} \mathcal{N} = 1$ and $\operatorname{reg} \mathcal{N}^\vee = 1$.

Definition 4.1. A vector bundle $\mathcal{E}$ on $\mathbb{P}^n$ is called a quasi-Buchsbaum bundle if $\mathfrak{m}\mathrm{H}^i_*(\mathcal{E}) = 0$, $1 \leq i \leq n-1$.

Remark 4.2. A Buchsbaum vector bundle is a quasi-Buchsbaum vector bundle. A null-correlation bundle is quasi-Buchsbaum but not Buchsbaum.

Question 4.3. Is there a structure theorem of quasi-Buchsbaum bundle on the projective space? Can we classify vector bundles $\mathcal{E}$ on $\mathbb{P}^n$ satisfying that $\mathrm{H}^1_*(\mathcal{E}) \cong \mathrm{H}^{n-1}_*(\mathcal{E}) \cong k$ and $\mathrm{H}^i_*(\mathcal{E}) = 0$, $2 \le i \le n-2$?

There is an answer to this question for stable vector bundles of rank 2 on $\mathbb{P}^3_{\mathbb{C}}$. Barth's Restriction Theorem [3] plays an important role for their proof.

Proposition 4.4 (Ellia-Sarti(1999))**.** *Let $\mathcal{E}$ be a stable vector bundle of rank 2 on $\mathbb{P}^3_{\mathbb{C}}$. Then $\mathcal{E}$ is quasi-Buchsbaum if and only if $\mathcal{E}$ is a null-correlation bundle.*

Remark 4.5. Let us take a map $\varphi : \mathcal{O}_{\mathbb{P}^3} \to \Omega_{\mathbb{P}^3}(2)$ corresponding to a skew-symmetric 4×4-matrix $A = (a_{ij})$. Then we have

$$P^{-1}AP = \begin{pmatrix} 0 & \lambda & 0 & 0 \\ -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu \\ 0 & 0 & -\mu & 0 \end{pmatrix} \quad \text{for an invertible matrix } P.$$

So, there are 3 types as the cokernel of φ

$$\text{(i) } A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{(ii) } A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{(iii) } A = 0$$

A null-correlation bundle $\mathcal{N}$ is defined as $\mathcal{N}^\vee(1) \cong \operatorname{Coker} \varphi$ in case (i).

In general, let us consider a map $\varphi : \mathcal{O}_{\mathbb{P}^n} \to \Omega_{\mathbb{P}^n}(2)$. Here n is not necessarily an odd number. The map corresponds to a skew-symmetric matrix $(n+1) \times (n+1)$- matrix $A = (a_{ij})$, and the rank of A can be only an even number.. In case $\operatorname{rank} A = 2m$, there is an invertible matrix P such that $P^{-1}AP = \begin{pmatrix} 0 & B & 0 \\ -B & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ with a $m \times m$-diagonal matrix $B = \operatorname{diag}(\lambda_1, \cdots, \lambda_m)$, $\lambda_i \neq 0$. □

Definition 4.6. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. Then we define

$$a_i(\mathcal{F}) = \min\{\ell \in \mathbb{Z} | \mathrm{H}^i(\mathcal{E}(\ell - i) = 0\}$$

for $i \ge 1$. Note that $\operatorname{reg} \mathcal{F} = \max\{a_i(\mathcal{F}) + i | i \ge 1\}$.

Lemma 4.7. *Let $\mathcal{E}$ be a quasi-Buchsbaum but not Buchsbaum bundle on $\mathbb{P}^n$ such that $\mathrm{H}^i_*(\mathcal{E}) = 0$ for $2 \le i \le n-2$. Then there is a quasi-Buchsbaum bundle $\mathcal{F}$ with $a_1(\mathcal{F}) = -1$, $a_{n-1}(\mathcal{F}) = -n+1$ and $a_n(\mathcal{F}) \le -n$ such that $\mathcal{E}(\ell) \cong \mathcal{F} \oplus (\oplus_i \mathcal{O}_{\mathbb{P}^n}(\ell'_i)) \oplus \oplus_j(\Omega^1_{\mathbb{P}^n}(\ell''_j)) \oplus \oplus_k(\Omega^{n-1}_{\mathbb{P}^n}(\ell'''_k))$ for some ℓ, ℓ'_i, ℓ''_j and ℓ'''_k.*

Proof. By twisting if necessary, we may assume that $\operatorname{reg} \mathcal{E} = 1$. In other words, $a_i(\mathcal{E}) \le -i$ and $a_i(\mathcal{E}) = -i$ for some $i = 1, n-1, n$.

Case I. If $a_n(\mathcal{E}) = -n$, then $\mathcal{O}_{\mathbb{P}^3}(-1)$ is a direct summand of $\mathcal{E}$ by Lemma 2.4, that is, $\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}_{\mathbb{P}^3}(-1)$ for some vector bundle $\mathcal{E}'$ having the quasi-Buchsbaum property. Then we may reduce to a vector bundle $\mathcal{E}'$ of lower rank.

Case II. If $a_{n-1}(\mathcal{E}) = -n+1$, then a nonzero element $s \in \mathrm{H}^{n-1}(\mathcal{E}(-n+1))$ can be lifted up to $\mathrm{H}^0(\mathcal{E} \otimes \Omega_{\mathbb{P}^n}^{n-1\vee}(-n+1))$, and the corresponding element $t \in \mathrm{H}^1(\mathcal{E}^\vee(-2))$ by Serre duality can be also lifted up to $\mathrm{H}^0(\mathcal{E}^\vee \otimes \Omega_{\mathbb{P}^n}^{n-1}(n-1))$. Indeed, as in the fourth proof of (3.4), exact sequences

$$0 \to \mathcal{E}(-n+1) \to \mathcal{E}^{\oplus}(-n+2) \to \cdots \to \mathcal{E}^{\oplus} \to \mathcal{E} \otimes \Omega_{\mathbb{P}^n}^{n-1\vee}(-n+1) \to 0,$$
$$0 \to \mathcal{E}^\vee \to \mathcal{E}^{\vee\oplus}(-1) \to \mathcal{E}^\vee \otimes \Omega_{\mathbb{P}^n}^{n-1}(n-1) \to 0$$

give surjective maps

$$\mathrm{Hom}(\Omega_{\mathbb{P}^n}^{n-1}(n-1), \mathcal{E}) = \mathrm{H}^0(\mathcal{E} \otimes \Omega_{\mathbb{P}^n}^{n-1\vee}(-n+1)) \to \mathrm{H}^{n-1}(\mathcal{E}(-n+1)),$$
$$\mathrm{Hom}(\mathcal{E}, \Omega_{\mathbb{P}^n}^{n-1}(n-1)) = \mathrm{H}^0((\mathcal{E})^\vee \otimes \Omega_{\mathbb{P}^n}^{n-1}(n-1)) \to \mathrm{H}^1(\mathcal{E}^\vee(-2)),$$

which gives $f : \Omega_{\mathbb{P}^n}^{n-1}(n-1) \to \mathcal{E}$ and $g : \mathcal{E} \to \Omega_{\mathbb{P}^n}^{n-1}(n-1)$ corresponding to s and t respectively. Hence we see $\Omega_{\mathbb{P}^n}^{n-1}(n-1)$ is a direct summand of $\mathcal{E}$ because $g \circ f$ is an isomorphism. By Lemma 3.9, $\mathcal{E} \cong \mathcal{E}' \oplus \Omega_{\mathbb{P}^n}^{n-1}(n-1)$ for some vector bundle $\mathcal{E}'$ having the quasi-Buchsbaum property. Then we may reduce to a vector bundle $\mathcal{E}'$ of lower rank again.

Case III. If $a_1(\mathcal{E}) = -1$ and $a_{n-1}(\mathcal{E}) < -n+1$, a nonzero element $s \in \mathrm{H}^1(\mathcal{E}(-1))$ can be lifted up to $\mathrm{H}^0(\mathcal{E} \otimes \Omega_{\mathbb{P}^n}^{1\vee}(-1))$, and the corresponding element $t \in \mathrm{H}^{n-1}(\mathcal{E}^\vee - n))$ by Serre duality can be also lifted up to $\mathrm{H}^0(\mathcal{E}^\vee \otimes \Omega_{\mathbb{P}^n}^1(1))$ by the same way as in Case II. Thus we have a direct summand $\Omega_{\mathbb{P}^n}^1(1)$ of $\mathcal{E}$ as desired. □

Thus we have only to consider an indecomposable quasi-Buchsbaum (not Buchsbaum) bundle with $a_1(\mathcal{E}) = -1$, $a_{n-1}(\mathcal{E}) = -n+1$ and $a_3(\mathcal{E}) \leq -n$ in order to investigate quasi-Buchsbaum bundles.

By virtue of the observation (4.7), the following remark is a detailed study of [5] in order to have an application to (4.9).

Remark 4.8. Let $\mathcal{E}$ be an indecomposable quasi-Buchsbaum bundle on $\mathbb{P}^n$ satisfying that $\mathrm{H}^i_*(\mathcal{E}) = 0$, $2 \leq i \leq n-2$. From the observation (4.7), there are increasing numbers $e_1 < \cdots < e_r$ such that $\mathrm{H}^1(\mathcal{E}(\ell)) = \mathrm{H}^{n-1}(\mathcal{E}(\ell - n + 1)) = 0$ for any $\ell \neq e_1, \cdots, e_r$. Let us put $s_i = \dim \mathrm{H}^1(\mathcal{E}(e_i)) > 0$ and $t_i = \dim \mathrm{H}^{n-1}(\mathcal{E}(e_i - n + 1)) > 0$. By the proof of (3.7), we have an exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{L} \to 0$ with $\mathrm{H}^1_*(\mathcal{F}) = 0$, where $\mathcal{L} = \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^3}(-e_i)^{\oplus s_i}$. Since $\mathrm{H}^i_*(\mathcal{F}) = 0$, $1 \leq i \leq n-2$ and $\mathrm{H}^{n-1}_*(\mathcal{F}) \cong \mathrm{H}^{n-1}_*(\mathcal{E})$, we see that $\mathcal{F}$ is Buchsbaum. From the structure theorem (3.4) we have an isomorphism $\mathcal{F} \cong \oplus_{i=1}^r \Omega_{\mathbb{P}^n}^{n-1}(-e_i + n - 1)^{\oplus t_i} \oplus \mathcal{M}$, where $\mathcal{M}$ is a direct summand of line bundles. As in the case of the second proof of (3.4), the exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{L} \to 0$ have 2 types, Case A and B. In Case A, the quasi-Buchsbaum property of $\mathcal{E}$ implies that $\mathcal{E}$ has a direct summand $\Omega_{\mathbb{P}^n}^1(\ell)$ as a direct summand, which gives $\mathcal{E} \cong \Omega_{\mathbb{P}^n}^1(\ell)$. So, we have only to focus on Case B. Then we have an exact sequence

$$0 \to \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(e_i)^{\oplus s_i} \to \oplus_{i=1}^r \Omega_{\mathbb{P}^n}(e_i + n - 1)^{\oplus t_i} \oplus \mathcal{M}^\vee \to \mathcal{E}^\vee \to 0.$$

In particular, $\mathcal{E}$ has a subbundle $\mathcal{G}$ as a direct summand such that $0 \to \mathcal{O}_{\mathbb{P}^3}(e_r)^{\oplus s_r} \to \Omega_{\mathbb{P}^3}(e_r + 2)^{\oplus t_r} \oplus \mathcal{L}' \to \mathcal{F}^\vee \to 0$, where $\mathcal{L}'$ is a direct sum of line bundles.

Now we present a classification theorem for quasi-Buchsbaum bundles by restricting the assumption.

Theorem 4.9. *Let $\mathcal{E}$ be an indecomposable quasi-Buchsbaum bundle on $\mathbb{P}^n$ with n odd. Assume that $\mathrm{H}^i_*(\mathcal{E}) = 0$, $2 \le i \le n-2$ and $\dim_k \mathrm{H}^1_*(\mathcal{E}) = \dim_k \mathrm{H}^{n-1}_*(\mathcal{E}) = 1$. Then $\mathcal{E}$ is isomorphic to one of the follows with some twist:*

(i) *Null-correlation bundle with n odd*

(ii) $0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\psi} \Omega_{\mathbb{P}^n}(1) \oplus \mathcal{O}_{\mathbb{P}^n}^{\oplus n-2m} \to \mathcal{E}^\vee \to 0,$

where ψ is given by $\varphi : \mathcal{O}_{\mathbb{P}^n} \to \Omega_{\mathbb{P}^n}(2)$ of rank $2m$ in (4.5).

Proof. We may assume that $\mathcal{E}$ is not Buchsbaum and $\mathrm{H}^1(\mathcal{E}(-1)) \cong \mathrm{H}^{n-1}(\mathcal{E}(-n)) \cong k$. From (4.8), we have an exact sequnce $0 \to \mathcal{E} \to \Omega^{n-1}_{\mathbb{P}^n}(n) \oplus \mathcal{L}' \to \mathcal{O}_{\mathbb{P}^n}(1) \to 0$, where $\mathcal{L}'$ is a direct sum of line bundles, because $\mathrm{H}^1_*(\mathcal{E}) \cong k(1)$ and $\mathrm{H}^{n-1}_*(\mathcal{E}) \cong k(n)$. Then we have a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \Omega^1_{\mathbb{P}^n}(2) \oplus \mathcal{L} \to \mathcal{E}^\vee(1) \to 0,$$

where $\mathcal{L} = \oplus_{i=1,\cdots,r}\mathcal{O}_{\mathbb{P}^n}(\ell_i)$. Clearly $\ell_i \ge 1$ for any i. A map $\varphi : \mathcal{O}_{\mathbb{P}^n} \to \Omega^1_{\mathbb{P}^n}(2)$ in the exact sequence is classified as in (4.5) according to the $\operatorname{rank}\varphi = 2m$, explicitly φ is written as $(x_1, -x_0, x_3, -x_2, \cdots, x_{2m-1}, -x_{2m-2}, 0, \cdots, 0)$ by coordinate change. A map $\mathcal{O}_{\mathbb{P}^n} \to \mathcal{L} = \oplus\mathcal{O}_{\mathbb{P}^n}(\ell_i)$ is defined by homogeneous polynomials f_i of degree ℓ_i, and $I = (x_0, \cdots, x_{2m-1}, f_1, \cdots, f_r)$ must be minimally generated and have no zero-points in $\mathbb{P}^n$. By taking the dual and a cohomology sequence, we have an exact sequence

$$\mathrm{H}^0_*((\Omega^1_{\mathbb{P}^n})^\vee(-2) \oplus (\oplus\mathcal{O}_{\mathbb{P}^n}(-\ell_i))) \xrightarrow{\alpha} \mathrm{H}^0_*(\mathcal{O}_{\mathbb{P}^n}) = S \to \mathrm{H}^1_*(\mathcal{E}(-1)) \to 0.$$

Since $\mathfrak{m}\mathrm{H}^1_*(\mathcal{E}(-1)) = 0$, we have $\mathfrak{m} \subseteq \operatorname{Im}\alpha$. Note that $\operatorname{Im}\alpha$ is generated by $x_0, \cdots, x_{2m-1}$ and $f_1, \cdots, f_r$. Hence we obtain $r = n-2m$ and $\ell_1 = 1$ as desired. □

Corollary 4.10. *Let $\mathcal{E}$ be an indecomposable vector bundle on $\mathbb{P}^n$ with $\dim_k \mathrm{H}^1_*(\mathcal{E}) = \dim_k \mathrm{H}^{n-1}_*(\mathcal{E}) = 1$ and $\mathrm{H}^i_*(\mathcal{E}) = 0$ for $2 \le i \le n-2$. If $\operatorname{rank}\mathcal{E} \le n-1$, then n is odd and $\mathcal{E}$ is isomorphic to a null-correlation bundle on $\mathbb{P}^n$.*

Remark 4.11. Let $\mathcal{N}$ be a null-correlation bundle on $\mathbb{P}^n$, with n odd. Then it is clear $\mathcal{N}^\vee \cong \mathcal{N}$ for $n = 3$ because $\operatorname{rank}\mathcal{N} = 2$. However, we also see that $\mathcal{N}^\vee \cong \wedge^{n-2}\mathcal{N} \cong \mathcal{N}$ from the structure theorem (4.10).

Example 4.12. Let $\varphi_1 : \mathcal{O}_{\mathbb{P}^3} \to \Omega^1_{\mathbb{P}^3}(2)(\subset \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 4})$ by $\varphi_1(1) = (x_1, -x_0, 0, 0)$ and $\varphi_2 : \mathcal{O}_{\mathbb{P}^3} \to \Omega^1_{\mathbb{P}^3}(2)(\subset \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 4})$ by $\varphi_2(1) = (0, 0, , x_3, -x_2)$. Then $\varphi = \varphi_1 + \varphi_2 : \mathcal{O}_{\mathbb{P}^3} \to \Omega^1_{\mathbb{P}^3}(2) \oplus \Omega^1_{\mathbb{P}^3}(2)$ gives $\mathcal{F} = \operatorname{Coker}\varphi$ as a vector bundle of rank 5. We easily have that $\mathcal{F}$ is an indecomposable quasi-Buchsbaum but not Buchsbaum bundle with $\mathrm{H}^1_*(\mathcal{F}) \cong k(2) \oplus k(2)$ and $\mathrm{H}^2_*(\mathcal{F}) \cong k$.

Question 4.13. Classify vector bundles $\mathcal{E}$ on $\mathbb{P}^n$ satisfying that there is an integer p with $2 \le p \le n-2$ such that $\mathrm{H}^1_*(\mathcal{E}) \cong k(1)$, $\mathrm{H}^p_*(\mathcal{E}) \cong k(p+1)$ and $\mathrm{H}^i_*(\mathcal{E}) = 0$ for $2 \le i \le p-1$ and $p+1 \le i \le n-1$.

5 Syzygy theoretic approach to quasi-Buchsbaum bundles on $\mathbb{P}^3$

Let $S = k[x_0, \cdots, x_n]$ be a polynomial ring over a field k with $\deg x_i = 1$, $i = 0, \cdots, n$. Let M be a finitely generated graded S-module of $\dim M = d+1$. Assume that M has a finite local cohomology, that is, $\ell(\mathrm{H}^i_{\mathfrak{m}}(M)) < \infty$, equivalently $\widetilde{M}$ is locally free on $\mathrm{Proj}\, S$.

Definition 5.1 ([24]). Let $f_1, \cdots, f_e$ be a part of homogeneous system of parameters for an S-module M. We call $f_1, \cdots, f_e$ as a standard system if $\mathfrak{q}\mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{q}_j M) = 0$ for all nonnegative integers i, j with $i + j \leq d$, where $\mathfrak{q}_j = (f_1, \cdots, f_j)$, $j = 0, \cdots, e$ and $\mathfrak{q} = \mathfrak{q}_e$.

Proposition 5.2 ([24]). *Let $y_0, \cdots, y_d$ be a homogeneous system of parameters for an S-module M. Then $y_0, \cdots, y_d$ is standard if and only if the natural maps from the Koszul cohomologies $\mathrm{H}^i((y_0, \cdots, y_d); M) \to \mathrm{H}^i_{\mathfrak{m}}(M)$ for surjective for $0 \leq i \leq d$*

Remark 5.3 ([23, 24]). In general, a graded S-module M is Buchsbaum if and only if any homogeneous system of parameters for the S-module M is standard.

Now let us consider a quasi-Buchsbaum bundle $\mathcal{E}$ on $\mathbb{P}^3$. Let $S = k[x_0, x_1, x_2, x_3]$ with $\mathfrak{m} = (x_0, x_1, x_2, x_3)$ and $M = \Gamma_*(\mathcal{E})$. Note that M has the property $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for $i = 0, 1$ $\mathfrak{m}\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for $i = 2, 3$. Then x_0, x_1, x_2, x_3 is a homogeneous system of parameters for a graded S-module M. Let us define a standard ideal, and we need not consider $\mathrm{H}^0_{\mathfrak{m}}$ and $\mathrm{H}^1_{\mathfrak{m}}$ local cohomologies through saturation and sheafification of a graded S-module. So, we give slightly different definition of standard ideals from the usual in this paper.

Definition 5.4. A homogeneous ideal $I \subset S$ is called standard for M if any part of homogeneous system of parameters $y_1, y_2 (\in I)$ is standard, that is, $y_1\mathrm{H}^2_{\mathfrak{m}}(M/y_2M) = y_2\mathrm{H}^2_{\mathfrak{m}}(M/y_1M) = 0$.

Remark 5.5. A graded S-module M is Buchsbaum if and only if $\mathfrak{m}$ is standard by (5.3).

Definition 5.6. Let M be a graded S-module with $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for $i = 0, 1$ as above. Then we call M as follows:

(i) M is pseudo-Buchsbaum if there is a part of linear standard system of parameters y_1, y_2, that is, $y_1\mathrm{H}^2_{\mathfrak{m}}(M/y_2M) = y_2\mathrm{H}^2_{\mathfrak{m}}(M/y_1M) = 0$, but $\mathfrak{m}$ is not standard.

(ii) M is nonstandard-Buchsbaum if M is neither Buchsbaum nor pseudo-Buchsbaum.

Let $L^\bullet$ be an exact sequence $0 \to \mathcal{O}_{\mathbb{P}^n}(-3) \to \mathcal{O}_{\mathbb{P}^n}(-2)^4 \to \mathcal{O}_{\mathbb{P}^n}(-1)^6 \to \Omega^1_{\mathbb{P}^3}(1) \to 0$ arising from the Koszul complex $K_\bullet(x_0, \cdots, x_3; S)$, where $L^{-4} = \mathcal{O}_{\mathbb{P}^n}(-3)$, $L^{-3} = \mathcal{O}_{\mathbb{P}^n}(-2)^4$, $L^{-2} = \mathcal{O}_{\mathbb{P}^n}(-1)^6$ and $L^{-1} = \Omega^1_{\mathbb{P}^3}(1)$. Let $C^\bullet$ be the Čech resolution $C^i = C^i(\mathfrak{U}; \mathcal{E})$, where $\{\mathfrak{U}\} = \{\mathrm{D}_+(x_i) | 0 \leq i \leq 3\}$ is an affine open covering of $\mathbb{P}^3$. Then we have a double complex $C^{\bullet\bullet} = L^\bullet \otimes C^\bullet$, see the diagram below, giving a spectral sequence $\{E_r^{p,q}\}$ with $E_1^{-4,q} = \mathrm{H}^q_*(\mathcal{E}(-3))$ and so forth. Since $\mathcal{E}$ is quasi-Buchsbaum, we have a map

$$d_2^{-4,2} : E_2^{-4,2} = \mathrm{H}^2_*(\mathcal{E}(-3)) \to E_2^{-2,1} = \mathrm{H}^1_*(\mathcal{E}(-1))^6.$$

Let us explain the above map explicitly. In fact, $d_2^{-4,2} : \mathrm{H}^2_*(\mathcal{E}(-3)) \to \mathrm{H}^1_*(\mathcal{E}(-1))^6$ is written in the following diagram:

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{H}^1_*(\mathcal{E}(-2)) & \to & \mathrm{H}^1_*(\mathcal{E}(-2)|_H) & \to & \mathrm{H}^2_*(\mathcal{E}(-3)) & \to & 0 \\ & & \downarrow x_j & & \downarrow x_j & & \downarrow x_j & & \\ 0 & \to & \mathrm{H}^1_*(\mathcal{E}(-1)) & \to & \mathrm{H}^1_*(\mathcal{E}(-1)|_H) & \to & \mathrm{H}^2_*(\mathcal{E}(-2)) & \to & 0 \end{array}$$

from a short exact sequence $0 \to \mathcal{E}(-1) \overset{x_i}{\to} \mathcal{E} \to \mathcal{E}|_H \to 0$, where $H = \{x_i = 0\}$. Since mutliplication maps $\mathrm{H}^1(\mathcal{E}(-2)) \overset{x_j}{\to} \mathrm{H}^1(\mathcal{E}(-1))$ and $\mathrm{H}^2(\mathcal{E}(-3)) \overset{x_j}{\to} \mathrm{H}^2(\mathcal{E}(-2))$ are zero, by snake lemma we obtain a map $\varphi_{x_i \wedge x_j} : \mathrm{H}^2(\mathcal{E}(-3)) \to \mathrm{H}^1(\mathcal{E}(-1))$, $i \neq j$, see [17, 19]. In this way we see a map $d_2^{-4,2}$ is written as

$$\bigoplus_{0 \le i < j \le 3} \varphi_{x_i \wedge x_j} : \mathrm{H}^2(\mathcal{E}(-3)) \to \bigoplus_{0 \le i < j \le 3} \mathrm{H}^1(\mathcal{E}(-1)).$$

Thus we have a map from $u \in C^2(\mathcal{E}(-3))$ with $\alpha(u) = d^{-3,1}(v)$ to $\beta(v)$ in the Čech diagram below.

$$\begin{array}{ccccccccccc} 0 & \to & C^2(\mathcal{E}(-3)) & \overset{\alpha}{\to} & C^2(\mathcal{E}(-2))^4 & \to & C^2(\mathcal{E}(-1))^6 & \to & C^2(\mathcal{E} \otimes \Omega^1_{\mathbb{P}^3}(1)) & \to & 0 \\ & & \uparrow & & \uparrow d^{-3,1} & & \uparrow & & \uparrow & & \\ 0 & \to & C^1(\mathcal{E}(-3)) & \to & C^1(\mathcal{E}(-2))^4 & \overset{\beta}{\to} & C^1(\mathcal{E}(-1))^6 & \to & C^1(\mathcal{E} \otimes \Omega^1_{\mathbb{P}^3}(1)) & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow d^{-2,0} & & \uparrow & & \\ 0 & \to & C^0(\mathcal{E}(-3)) & \to & C^0(\mathcal{E}(-2))^4 & \to & C^0(\mathcal{E}(-1))^6 & \overset{\gamma}{\to} & C^0(\mathcal{E} \otimes \Omega^1_{\mathbb{P}^3}(1)) & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & 0 & & 0 & & \end{array}$$

In case $\beta(v)$ were an element of $d^{-2,0}(C^0(\mathcal{E}(-1))^6)$, $\bar{u} \in \mathrm{H}^2(\mathcal{E}(-3))$ could be lifted to $\mathrm{H}^0(\mathcal{E} \otimes \Omega^1_{\mathbb{P}^3}(1))$. Then $\Omega^1_{\mathbb{P}^3}(1)$ is isomorphic to a direct summand of $\mathcal{E}$.

From this viewpoint, we consider the case $\beta(v) \notin d^{-2,0}(C^0(\mathcal{E}(-1))^6)$.

Now we state our main result of this section.

Theorem 5.7. *Let $\mathcal{E}$ be an indecomposable quasi-Buchsbaum bundle on $\mathbb{P}^3$ with $\dim_k \mathrm{H}^i_*(\mathcal{E}) = 1$ for $i = 1, 2$. If $\mathcal{E}$ is nonstandard Buchsbaum, then $\mathcal{E}$ is isomorphic to a null-correlation bundle with some twist.*

Remark 5.8. By (4.9) and (5.7), the indecomposable quasi-Buchsbaum bundles on $\mathbb{P}^3$ $\dim_k \mathrm{H}^1_*(\mathcal{E}) = \dim_k \mathrm{H}^2_*(\mathcal{E}) = 1$ are classified with some twist as

(a) Null-correlation bundle as Nonstandard-Buchsbaum.

(b) Bundle of (ii) in (4.9) as Psuedo-Buchsbaum.

Proof of (5.7). From the observation (4.7), we may assume that $\mathrm{H}^2(\mathcal{E}(-3)) \cong k$ and $\mathrm{H}^1(\mathcal{E}(-1)) \cong k$. Then there exists a nonzero element $s \in \mathrm{H}^2(\mathcal{E}(-3))$.

Before going back to the track of the proof, we explain what prevents $\mathcal{E}$ from being isomorphic to the sheaf of differential forms. As in the fourth proof of (3.4), from the exact sequence:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{E}(-3) & \to & \mathcal{E}(-2)^4 & \to & \mathcal{E}(-1)^6 & \to & \mathcal{E}(-3)\otimes\Omega^{2\vee}_{\mathbb{P}^3}\ (\cong \mathcal{E}\otimes\Omega^1_{\mathbb{P}^3}(1)) \to 0 \\ & & & & \searrow & & \nearrow & & \\ & & & & & \mathcal{E}(-3)\otimes\Omega^{1\vee}_{\mathbb{P}^3} & & & \\ & & & & \nearrow & & \searrow & & \\ & & & & 0 & & 0 & & \end{array}$$

we have a composition of maps $\mathrm{H}^0(\mathcal{E}(-3)\otimes\Omega^{2\vee}_{\mathbb{P}^3}) \to \mathrm{H}^1(\mathcal{E}(-3)\otimes\Omega^{1\vee}_{\mathbb{P}^3}) \to \mathrm{H}^2(\mathcal{E}(-3))$. Since $s(\neq 0)\in \mathrm{H}^2(\mathcal{E}(-3))$ cannot be lifted to an element of $\mathrm{H}^0(\mathcal{E}(-3))$, we cannot have a map $\Omega^2_{\mathbb{P}^3}\to\mathcal{E}(-3)$ in this way.

Now let us construct a null-correlation bundle $\mathcal{N}$ with a skew-symmetric matrix $A=(a_{ij})$, $0\le i,j\le 3$ in order to give non-zero maps $f:\mathcal{N}\to\mathcal{E}$ and $g:\mathcal{E}\to\mathcal{N}$ regarded as elements of $\mathrm{H}^0(\mathcal{E}\otimes\mathcal{N}^\vee)$ and $\mathrm{H}^0(\mathcal{E}^\vee\otimes\mathcal{N})$ such that $g\circ f$ is isomorphic, which implies that $\mathcal{N}$ is a direct summand of $\mathcal{E}$.

First let us show there is a nonzero element of $\Gamma(\mathcal{E}\otimes\mathcal{N}^\vee)$ by lifting an element of $\mathrm{H}^2(\mathcal{E}(-3))$. In the following short exact sequence corresponding to a null-correlation bundle with a skew-symmetric matrix A, we take the minimal free resolution of $\Omega^1_{\mathbb{P}^3}(1)$ such as

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \\ 0 & \to & \mathcal{O}_{\mathbb{P}^3}(-1) & \to & \Omega^1_{\mathbb{P}^3}(1) & \to & \mathcal{N}^\vee(\cong\mathcal{N}) \to 0. \\ & & \uparrow & & \uparrow & & \\ & & \mathcal{O}_{\mathbb{P}^3}(-1) & & \mathcal{O}_{\mathbb{P}^3}(-1)^6 & & \\ & & \uparrow & & \uparrow & & \\ & & 0 & & \mathcal{O}_{\mathbb{P}^3}(-2)^4 & & \\ & & & & \uparrow & & \\ & & & & \mathcal{O}_{\mathbb{P}^3}(-3) & & \\ & & & & \uparrow & & \\ & & & & 0 & & \end{array}$$

Here we use the word 'free resolution' if the sequence maintains the exactness after taking $\Gamma_*(\,\cdot\,)$. Thus we have a free resolution of $\mathcal{N}$ by taking a mapping cone

$$0\to\mathcal{O}_{\mathbb{P}^3}(-3)\to\mathcal{O}_{\mathbb{P}^3}(-2)^4\oplus\mathcal{O}_{\mathbb{P}^3}(-1)\to\mathcal{O}_{\mathbb{P}^3}(-1)^6\to\mathcal{N}^\vee\to 0,$$

which yields the minimal free resolution

$$0\to\mathcal{O}_{\mathbb{P}^3}(-3)\to\mathcal{O}_{\mathbb{P}^3}(-2)^4\to\mathcal{O}_{\mathbb{P}^3}(-1)^5\to\mathcal{N}^\vee\to 0,$$

This resolution connects short exact sequences $0\to\mathcal{O}_{\mathbb{P}^3}(-3)\to\mathcal{O}_{\mathbb{P}^3}(-2)^4\to\Omega^2_{\mathbb{P}^3}(1)\to 0$ and $0\to\Omega^2_{\mathbb{P}^3}(1)\to\mathcal{O}_{\mathbb{P}^3}(-1)^5\to\mathcal{N}^\vee\to 0$.

What we have to do is to construct a null-correlation bundle $\mathcal{N}$ having a nonzero element $g\in\mathrm{H}^0(\mathcal{E}\otimes\mathcal{N}^\vee)$ such that $\varphi(g)=s(\neq 0)$, where

$$\varphi:\mathrm{H}^0(\mathcal{E}\otimes\mathcal{N}^\vee)\to\mathrm{H}^1(\mathcal{E}\otimes\Omega^2_{\mathbb{P}^3}(1))\to\mathrm{H}^2(\mathcal{E}(-3))$$

arising from an exact sequence

$$0\to\mathcal{E}(-3)\to\mathcal{E}(-2)^4\oplus\mathcal{E}(-1)\to\mathcal{E}(-1)^6\to\mathcal{E}\otimes\mathcal{N}^\vee\to 0.$$

Let us take $u \in C^2(\mathcal{E}(-3))$ as an element $s(\neq 0) \in \mathrm{H}^2(\mathcal{E}(-3))$ which gives $v \in C^1(\mathcal{E}(-2))^4$ with $d^{-3,1}(v) = \alpha(u)$ and $\beta(v) \notin d^{-2,0}(C^0(\mathcal{E}(-1))^6)$ in chasing an element in the Čech diagram $C^{\bullet\bullet}$ before. As explained before, $\bigoplus_{0\leq i<j\leq 3} \varphi_{x_i\wedge x_j} : \mathrm{H}^2(\mathcal{E}(-3)) \to \bigoplus_{0\leq i<j\leq 3} \mathrm{H}^1(\mathcal{E}(-1))$ gives a map from $u \in C^2(\mathcal{E}(-3))$ to $\beta(v) \notin d^{-2,0}(C^0(\mathcal{E}(-1))^6)$ in the Čech diagram. Then we give a map $\gamma : \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}(-1)^6$ by choosing an element of k^6 such that the image contains $\beta(v)$. Thus we have constructed a null-correlation bundle $\mathcal{N}$ obtained from $0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3}(-2)^4 \to \mathcal{O}_{\mathbb{P}^3}(-1)^6 \to \Omega^1_{\mathbb{P}^3}(1) \to 0$ and $\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\gamma} \mathcal{O}_{\mathbb{P}^3}(-1)^6$.

By adding a superfluous component, we have $(\beta - \gamma)(v) = d^{-2,0}(w)$ for some $w \in C^0(\mathcal{E}(-1))^6$, Thus we have a cycle $\delta(w)$ of $C^0(\mathcal{E} \otimes \mathcal{N}^\vee)$. More explicitly, $\varphi_{x_i\wedge x_j}$ gives a skew-symmetric matrix $A = (a_{ij})$, where $a_{ij} = \varphi_{x_i\wedge x_j}(1) \in k$. Since $\mathcal{E}$ is nonstandard-Buchsbaum, $\operatorname{rank} A = 4$. In this way, the following double complex $D^{\bullet\bullet}$ constructed by adding an extra component enables a nonzero element of $\mathrm{H}^2(\mathcal{E}(-3))$ to be lifted to $\mathrm{H}^0(\mathcal{E} \otimes \mathcal{N}^\vee)$ as desired.

$$
\begin{array}{ccccccccccc}
0 & \to & C^2(\mathcal{E}(-3)) & \to & C^2(\mathcal{E}(-2))^4 \oplus C^2(\mathcal{E}(-1)) & \to & C^2(\mathcal{E}(-1))^6 & \to & C^2(\mathcal{E}\otimes\mathcal{N}^\vee) & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & C^1(\mathcal{E}(-3)) & \to & C^1(\mathcal{E}(-2))^4 \oplus C^1(\mathcal{E}(-1)) & \overset{\beta-\gamma}{\to} & C^1(\mathcal{E}(-1))^6 & \to & C^1(\mathcal{E}\otimes\mathcal{N}^\vee) & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow d^{-2,0} & & \uparrow & & \\
0 & \to & C^0(\mathcal{E}(-3)) & \to & C^0(\mathcal{E}(-2))^4 \oplus C^0(\mathcal{E}(-1)) & \to & C^0(\mathcal{E}(-1))^6 & \overset{\delta}{\to} & C^0(\mathcal{E}\otimes\mathcal{N}^\vee) & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

In other words, there is a nonzero element $g \in \mathrm{H}^0(\mathcal{E} \otimes \mathcal{N}^\vee)$ such that $\varphi(g) = s$, where $\varphi : \mathrm{H}^0(\mathcal{E}\otimes\mathcal{N}^\vee) \to \mathrm{H}^1(\mathcal{E}\otimes\Omega^2(1)) \to \mathrm{H}^2(\mathcal{E}(-3))$. Comparing the double complex $C^{\bullet\bullet}$ and $D^{\bullet\bullet}$ with the corresponding spectral sequences $\{E_r^{p,q}\}$ and $\{F_r^{p,q}\}$ respectively, we have shown that $d_2^{-4,2} : E_2^{-4,2} \to E_2^{-2,1}$ is not zero but $d_2^{-4,2} : F_2^{-4,2} \to F_2^{-2,1}$ is zero.

From an exact sequence $0 \to \mathcal{N} \to \Omega^2(1) \to \mathcal{O}_{\mathbb{P}^3}(1) \to 0$, we take an element $t \in \mathrm{H}^3(\mathcal{E}\otimes\mathcal{N}(-4))$ from $s \in \mathrm{H}^2(\mathcal{E}(-3))$. For a dual element $s' \in \mathrm{H}^1(\mathcal{E}^\vee(-1))$ corresponding to $s \in \mathrm{H}^2(\mathcal{E}(-3))$ by Serre duality $\varphi(g) = s$, there exists a nonzero element $f \in \mathrm{H}^0(\mathcal{E}^\vee\otimes\mathcal{N})$ such that $\psi(f) = s'$, where $\psi : \mathrm{H}^0(\mathcal{E}^\vee \otimes \mathcal{N}) \to \mathrm{H}^1(\mathcal{E}^\vee(-1))$ arising from $0 \to \mathcal{E}^\vee(-1) \to \mathcal{E}^\vee \otimes \Omega^2(1) \to \mathcal{E}^\vee \otimes \mathcal{N} \to 0$. From a commutative diagram

$$
\begin{array}{ccccc}
\mathrm{H}^0(\mathcal{E}\otimes\mathcal{N}^\vee)\otimes\mathrm{H}^0(\mathcal{E}^\vee\otimes\mathcal{N}) & \to & \mathrm{H}^0(\mathcal{N}^\vee\otimes\mathcal{N}) & \cong & \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}) \\
\downarrow & & & & \downarrow \\
\mathrm{H}^2(\mathcal{E}(-3))\otimes\mathrm{H}^1(\mathcal{E}^\vee(-1)) & \to & & & \mathrm{H}^3(\mathcal{O}_{\mathbb{P}^3}(-4)),
\end{array}
$$

the natural map $\mathrm{H}^0(\mathcal{E}\otimes\mathcal{N}^\vee)\otimes\mathrm{H}^0(\mathcal{E}^\vee\otimes\mathcal{N}) \to \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3})$ gives an isomorphism $g\circ f$. Hence we obtain that $\mathcal{N}$ is a direct summand of $\mathcal{E}$. □

Remark 5.9. In the proof of (5.7), we have taken a skew-symmetric matrix A of rank 4 because $\mathcal{E}$ is nonstandard-Buchsbaum. In general, $\operatorname{rank} A = 0, 2, 4$ corresponds with Buchsbaum, pseudo-Buchsbaum and nonstandard-Buchsbaum, respectively. Also, there exists no $\operatorname{rank} A = 3$ cases, because the rank of skew-symmetric matrix is even.

Example 5.10. If $\mathcal{E}$ were pseudo-Buchsbaum in the proof of (5.7), the skew-symmetric matrix A would has rank 2, see (5.9). In this case, we can similarly take a map

$$\gamma' : \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}(-1)^6 \oplus \mathcal{O}_{\mathbb{P}^3}^2$$

by $(\beta - \gamma')(v)$ being liftable such that the corresponding injective map

$$\varphi : \mathcal{O}_{\mathbb{P}^3}(-1) \to \Omega^1_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}^2$$

gives a vector bundle $\mathcal{F} = \operatorname{Coker}\varphi$. In fact, $\mathcal{F}$ has a free resolution (exact even after taking $\Gamma_*(\cdot)$)

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3}(-2)^4 \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}(-1)^6 \oplus \mathcal{O}_{\mathbb{P}^3}^2 \to \mathcal{F} \to 0.$$

Then a syzygy theoretic method shows that an indecomposable pseudo-Buchsbaum vector bundle on $\mathbb{P}^3$ with $\dim_k \mathrm{H}^1_*(\mathcal{E}) = \dim_k \mathrm{H}^2_*(\mathcal{E}) = 1$ is isomorphic to a vector bundle $\mathcal{F}(\ell)$ for some $\ell \in \mathbb{Z}$.

Remark 5.11. A vector bundle $\mathcal{E}$ on $\mathbb{P}^3$ with $\mathrm{H}^1_*(\mathcal{E}) \cong k(1)$ and $\mathrm{H}^2_*(\mathcal{E}) \cong k(3)$ is isomorphic to either a null-correlation bundle, a bundle $\mathcal{F}$ in (5.10), or $\Omega^1_{\mathbb{P}^3}(1) \oplus \Omega^2_{\mathbb{P}^3}(3)$ 'under stable equivalence', that is, without a direct sum of line bundles. These bundles are self-dual, and correspond with 'nonstandard-Buchsbaum', 'pseudo-Buchsbaum' and 'Buchsbaum'.

6 Vector bundles on multiprojective spaces

Is there a vector bundle $\mathcal{E}$ on $X = \mathbb{P}^m \times \mathbb{P}^n$ satisfying that $\mathrm{H}^i(X, \mathcal{E} \otimes \mathcal{O}_X(\ell_1, \ell_2)) = 0$, $1 \le i \le m+n-1$ for any $(\ell_1, \ell_2) \in \mathbb{Z}\times\mathbb{Z}$? In fact there are no such vector bundles obtained from the basic property of the Castelnuovo-Mumford regularity on multiprojective space.

Definition and Proposition 6.1 ([2])**.** A coherent sheaf $\mathcal{F}$ on $X = \mathbb{P}^m \times \mathbb{P}^n$ is 0-regular if $\mathrm{H}^i(X, \mathcal{F}(j_1, j_2)) = 0$ for $i \ge 1$, $j_1 + j_2 = -i$, $-m \le j_1 \le 0$, $-n \le j_2 \le 0$.

Then $\mathcal{F}|_{H\times\mathbb{P}^n}$ is 0-regular on $H \times \mathbb{P}^n (\cong \mathbb{P}^{m-1} \times \mathbb{P}^n)$ for a generic hyperplane H of $\mathbb{P}^m$, and $\mathcal{F}(m_1, m_2)$ is 0-regular for any $m_1 \ge 0$, $m_2 \ge 0$, and $\mathcal{F}$ is globally generated.

Let us consider a vector bundle $\mathcal{E}$ on $X = \mathbb{P}^m \times \mathbb{P}^n$ without intermediate cohomology. Let t be the minimal integer such that $\mathcal{F} = \mathcal{E}(t, t)$ is 0-regular. Since the intermediate cohomologies vanishes, we see that $\mathrm{H}^{m+n}(X, \mathcal{F}(-m-1, -n-1)) \neq 0$. By Serre duality we have $\mathrm{H}^0(\mathcal{F}^\vee) \neq 0$, which gives a nonzero map $\varphi : \mathcal{F} \to \mathcal{O}_X$. On the other hand, a globally generated bundle $\mathcal{F}$ gives a surjective map $\psi : \mathcal{O}_X^{\oplus} \to \mathcal{F}$. Since $\varphi \circ \psi$ is a nonzero map, $\mathcal{O}_X$ is a direct summand of $\mathcal{F}$. However, we see $\mathrm{H}^m(X, \mathcal{O}_X(-m-1, 0)) \neq 0$, which contradicts with the assumption of $\mathcal{E}$.

The following result is an immediate consequence of [18, (2.5)], given by a syzygy theoretic approach implies that an ACM bundle on a smooth quadric surface $Q = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ is isomorphic to a direct summand of $\mathcal{O}_X$, $\mathcal{O}_X(-1, 0)$, $\mathcal{O}_X(0, -1)$ with some twist, a special case of [11].

Proposition 6.2. *Let $\mathcal{E}$ be a vector bundle on $X = \mathbb{P}^n \times \mathbb{P}^n$. Then the following conditions are equivalent:*

(a) (i) $\mathrm{H}^i(X, \mathcal{E}(\ell_1, \ell_2)) = 0$ *for any* $\ell_1, \ell_2 \in \mathbb{Z}$ *with* $|\ell_1 - \ell_2| \leq n$, *and* $i = 1, \cdots, n-1, n+1, \cdots, 2n-1$.

(ii) $\mathrm{H}^n(X, \mathcal{E}(\ell, \ell)) = 0$ *for any* $\ell \in \mathbb{Z}$.

(b) *A vector bundle $\mathcal{E}$ is isomorphic to a direct sum of line bundles of the form $\mathcal{O}_X(u, v)$ for some u, v with $|u - v| \leq n$.*

Question 6.3. Find a generalization of cohomological criteria of (6.2) for vector bundles on $\mathbb{P}^n \times \cdots \times \mathbb{P}^n$.

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