Castelnuovo-Mumford 正則量とシジジーに関連する話題について (サーベイ講演)

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Outline

Castelnuovo-Mumford Regularity Basics

Notation

k : an algebraically closed field
$S = k[x_0, \cdots, x_n]$: the polynomial ring over k
$\mathfrak{m} = S_+ = (x_0, \cdots, x_n)$
$\mathbb{P}^n = \mathrm{Proj}\, S$

Definition and Proposition (Mumford)

$\mathcal{F}$: a coherent sheaf on $\mathbb{P}^n$, $m \in \mathbb{Z}$

$\mathcal{F}$ is m-regular $\iff \mathrm{H}^i(\mathbb{P}^n, \mathcal{F}(m-i)) = 0,\ i \geq 1$

$\iff \mathrm{H}^i(\mathbb{P}^n, \mathcal{F}(j)) = 0,\ i \geq 1,\ i+j \geq m \ \Rightarrow \mathcal{F}(m)$ is generated by global sections

- $\mathrm{reg}\, \mathcal{F} := \min\{m \in \mathbb{Z} \mid \mathcal{F}$ is m-regular $\}$
- $X \subseteq \mathbb{P}^n$: a projective scheme
  $\mathrm{reg}\, X := \mathrm{reg}\, \mathcal{I}_X$: Castelnuovo-Mumford regularity

Castelnuovo-Mumford Regularity Basics

Definition and Proposition (Continued)

If $\mathcal{F}$ is m-regular on $\mathbb{P}^n$, then we have

(1) $\mathcal{F}$ is $(m+1)$-regular, and

(2) $\Gamma(\mathcal{F}(m)) \otimes \Gamma(\mathcal{O}_{\mathbb{P}^n}(1)) \to \Gamma(\mathcal{F}(m+1))$ is surjective.

Since $\Gamma(\mathcal{F}(\ell)) \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{F}(\ell)$ is surjective for $\ell \gg 0$, we have $\Gamma(\mathcal{F}(m)) \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{F}(m)$ is surjective.

Remark

In order to extend the definitions of Castelnuovo-Mumford regularity, say multi-graded, weighted, Grassmaniann, or globally generated ample line bundle, we shoud keep in mind whether the properties above work or not.

- D. Maclagan and G. Smith, Multigraded Castelnuovo-Mumford regularity, J. Reine. Angew. Math. 571 (2004).

Castelnuovo-Mumford Regularity Basics

Definition

M: a finitely generated graded S-module
$a_i(M) = \max\{\ell \in \mathbb{Z} | [\mathrm{H}^i_{\mathfrak{m}}(M)]_\ell \neq 0\}$, $i = 0, \cdots, n+1$
$\mathrm{reg}\, M = \max\{a_i + i | i = 0, \cdots, n+1\}$: Castelnuovo-Mumford regularity
$d(M)$: the maximal degree of the minimal generators of M

Notation

$X \subseteq \mathbb{P}^n$: a projective scheme
$I := \Gamma_* \mathcal{I}_X = \oplus_{\ell \in \mathbb{Z}} \Gamma(\mathbb{P}^N, \mathcal{I}_X(\ell))$: the defining ideal of X
$R := S/I$: the coordinate ring of X

Remark

$d(I) \leq \mathrm{reg}\, I$
$\mathrm{reg}\, X = \mathrm{reg}\, \mathcal{I}_X = \mathrm{reg}\, R + 1 = \mathrm{reg}\, I$

Castelnuovo-Mumford Regularity Basics

Let us take a minimal free resolution of I as graded S-module.
The Syzygy Theorem gives the finiteness.

$$0 \to F_s \to \cdots \to F_1 \to F_0 \to I \to 0$$

where, $F_i = \oplus_j S(-\alpha_{i,j})$ is a graded free S-module.
Each map $F_{i+1} \to F_i$ is written as a matrix which components are homogeneous polynomials.

Theorem (cf. Eisenbud-Goto, Bayer-Mumford)

$\operatorname{reg} X = \max_{i,j}\{\alpha_{i,j} - i\}$

The Castelnuovo-Mumford regularity measures the complexity of the defining ideals of the projective varieties.

Castelnuovo-Mumford Regularity Basics

Proposition (Eisenbud-Goto)

A graded S-module is m-regular if and only if $M_{\geq m} := \oplus_{\ell \geq m} M_\ell$ has an m-linear resolution, that is, the minimal free resolution of $M_{\geq m}$:

$$0 \to F_s \to \cdots \to F_1 \to F_0 \to M_{\geq m} \to 0,$$

where $F_i = \oplus S(-m-i)$ is a graded free S-module for $i = 0, \cdots, s$.

Proof

From an exact sequence $0 \to M_{\geq m} \to M \to M/M_{\geq m} \to 0$, we have an exact sequence

$$0 \to \mathrm{H}^0_{\mathfrak{m}}(M_{\geq m}) \to \mathrm{H}^0_{\mathfrak{m}}(M) \to M/M_{\geq m} \to \mathrm{H}^1_{\mathfrak{m}}(M_{\geq m}) \to \mathrm{H}^1_{\mathfrak{m}}(M) \to 0$$

and an isomorphisim $\mathrm{H}^i_{\mathfrak{m}}(M_{\geq m}) \cong \mathrm{H}^i_{\mathfrak{m}}(M)$, $i \geq 2$.
Thus we have $M_{\geq m}$ is m-regular, which gives the minimal free resolution as desired.

Castelnuovo-Mumford Regularity Basics

Remark

From Lazarsfeld's book, they have an exact sequence

$$0 \to \mathcal{F}_1 \to \Gamma(\mathcal{F}(m)) \otimes \mathcal{O}_{\mathbb{P}^n}(-m) \to \mathcal{F}(m) \to 0,$$

where $\mathcal{F}_1$ is $(m+1)$-regular. By repeating this process we have a linear resolution:

$$\cdots \to \cdots \to \oplus\mathcal{O}_{\mathbb{P}^n}(-m-2) \to \oplus\mathcal{O}_{\mathbb{P}^n}(-m-1) \to \oplus\mathcal{O}_{\mathbb{P}^n}(-m) \to \mathcal{F} \to 0.$$

Castelnuovo-Mumford Regularity Basics

Definition

Let M be a finitely generated graded S-module.
Let $\mathbf{F}_\bullet$ be the minimal free resolution, where $F_i = \oplus_j S(-j)^{\beta_{ij}}$ and

$$\beta_{ij} = \dim_k[\mathrm{Tor}_i^S(M,k)]_j.$$

$\beta_{i,j}$ is called as Betti number, and Betti table is described $\beta_{i,i+j}$ in position (i,j).

Remark

- $\mathrm{proj.dim}_S M = \max\{i|\beta_{ij} \neq 0\}$
- $\mathrm{reg}\, M = \max\{j|\beta_{i,i+j} \neq 0\}$
- Poincaré series of M is $\mathrm{P}(M,t) = \sum_i h_M(i)t^i = \frac{\sum_i (-1)^i \beta_{ij} t^j}{(1-t)^{n+1}}$

Castelnuovo-Mumford Regularity Basics

Example

$X \subseteq \mathbb{P}^N$: a complete intersection of type $(d_1, \cdots, d_r)$

$$0 \to S(-d_1 - \cdots - d_r) \to \cdots \to \oplus_{j=1,\cdots,r} S(-d_j) \to S \to S/I \to 0$$

The Koszul complex arising from the defining equations gives the minimal free resolution of the defining ideal I.

$$\operatorname{reg} X = d_1 + \cdots + d_r - r + 1$$

Castelnuovo-Mumford Regularity Basics

Example

C : a rational normal curve

$$\mathbb{P}^1 \ni (s : t) \longrightarrow (s^3 : s^2t : st^2 : t^3) \in \mathbb{P}^3$$

The defining ideal I of C is an ideal generated by 2×2-minors of the matrix $A = \begin{bmatrix} x & y & z \\ y & z & w \end{bmatrix}$ in $S = k[x, y, z, w]$.

Let $f = yw - z^2$, $g = yz - xw$, $h = xz - y^2$.

Then the minimal free resolution of $I = (f, g, h)$ is

$$0 \to S(-3) \oplus S(-3) \xrightarrow{{}^tA} S(-2) \oplus S(-2) \oplus S(-2) \xrightarrow{[f\ g\ h]} S \to S/I \to 0$$

In this case we have $\mathrm{reg}\, C = 2$

Castelnuovo-Mumford Regularity Basics

Example

(1) In case a $(2,3)$-complete intersection X in $\mathbb{P}^4$, $\operatorname{reg} C = 4$ from the Betti table.

	0	1	2
0	1	-	-
1	-	-	-
2	-	1	-
3	-	1	-
4	-	-	1

(2) In case a twisted cubic curve C in $\mathbb{P}^3$, $\operatorname{reg} C = 2$ from Betti table.

	0	1	2
0	1	-	-
1	-	-	-
2	-	3	2

General References

Reference

- D. Bayer and D. Mumford, What can be computed in Algebraic Geometry? Computational algebraic geometry and commutative algebra, CUP 1993.

  https://arxiv.org/abs/alg-geom/9304003
- D. Eisenbud, The geometry of Syzygies, Springer GTM 229, 2005
- R. Lazarsfeld, Positivity I, Chaper 1, Section 8, Springer, 2004.
- E. Miller and D. Perkinson, Eight Lectures on Monomial Ideals by B. Sturmfels, CoCoA Summer School 1999.

  https://services.math.duke.edu/~ezra/Queens/cocoa.pdf
- D. Eisenbud and S. Goto, Linear free resolutions and minimal multiplicity. J. Algebra 88 (1984).

Regulariy Conjecture

Remark

(1) $\operatorname{reg} X \geq 1$

(2) If $X (\subseteq \mathbb{P}^n)$ is nondegenerate, that is, X is not contained in any hyperplane of $\mathbb{P}^n$, then $\operatorname{reg} X \geq 2$.

Conjecture (Regularity Conjecture by Eisenbud-Goto)

X : a nondegenerate projective variety $\Rightarrow \operatorname{reg} X \leq \deg X - \operatorname{codim} X + 1$?

Remark

'Irreducible' and 'Reduced' are necessary.

(1) Skew lines in $\mathbb{P}^3$, $I = (x, y) \cap (z, w) \subset k[x, y, z, w]$

(2) A double line in $\mathbb{P}^3$, $I = (xw - yz, x^2, xy, y^2) \subset k[x, y, z, w]$

$\operatorname{reg} I = \deg S/I = \operatorname{ht} I = 2$.

Regularity Conjecture

Fact

(1) $\dim X = 1$: Gruson-Lazarsfeld-Peskine, 1983

(2) $\dim X = 2$, smooth, $\operatorname{char} \mathrm{k} = 0$: Lazarsfeld, 1987

(3) $\dim X = 3$, smooth, $\operatorname{char} \mathrm{k} = 0$,
$$\operatorname{reg} X \leq \deg X - \operatorname{codim} X + 2$$: Kwak, 1998

(4) $\dim X \leq 14$, smooth, $\operatorname{char} \mathrm{k} = 0$, $n = \dim X$,
$$\operatorname{reg} X \leq \deg X - \operatorname{codim} X + 1 + (n-1)(n-2)/2$$
: Chiantini-Chiarli-Greco, 2000

(5) $\dim X \geq 3$, singular, Conterexamples : McCullough-Peeva, 2018

(6) Toric variety with $\operatorname{codim} X = 2$: Peeva-Sturmfels, 1998

Gruson-Lazarsfeld-Peskine

Theorem

Let $C \subseteq \mathbb{P}^n$ be a nondegenerate projective curve of degree d. Then $\operatorname{reg} C \leq d + 2 - n$. The equality holds if and only if

(1) $d = n$, that is, a rational normal curve

(2) $d = n + 1$

(3) $d > n + 1$, and C has a $(d + 2 - n)$-secant line.

Theorem

Let $C \subseteq \mathbb{P}^n$ be a nondegenerate projective curve of degree d. If $g = p_g(C) \geq 1$, then $\operatorname{reg} C \leq d + 1 - n$ unless C is an elliptic normal curve.

Gruson-Lazarsfeld-Peskine

Reference

L. Gruson, C. Peskine and R. Lazarsfeld, On a theorem of Castelnuovo, and the equations defining space curves, Invent. Math. 72(1983)

Lemma

Let $p : \widetilde{C} \to C \subseteq \mathbb{P}^n$ be the normalization of C. Let $\mathcal{M} = p^*\Omega_{\mathbb{P}^n}(1)$. Assume $H^1(\widetilde{C}, \wedge^2\mathcal{M} \otimes \mathcal{A}) = 0$ for some $\mathcal{A} \in \mathrm{Pic}\,\widetilde{C}$. Then $\mathrm{reg}\, C \leq \mathrm{h}^0(\mathcal{A})$.

Lemma

Let $p : \widetilde{C} \to C \subseteq \mathbb{P}^n$. Let $d = \deg p^*\mathcal{O}_{\mathbb{P}^n}(1)$. Then there exists an ample line bundle $\mathcal{A}$ such that $\mathrm{h}^0(\mathcal{A}) = d + 2 - n$ and $\mathrm{h}^1(\wedge^2\mathcal{M} \otimes \mathcal{A}) = 0$.

Gruson-Lazarsfeld-Peskine

Sketch of Proof

Let $\mathcal{O}_{\widetilde{C}}(1) = p^*\mathcal{O}_{\mathbb{P}^n}(1)$ and $V = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}(1)) \subseteq \mathrm{H}^0(\mathcal{O}_{\widetilde{C}}(1))$.
Let $\pi : \widetilde{C} \times \mathbb{P}^n \to \widetilde{C}$ and $f : \widetilde{C} \times \mathbb{P}^n \to \mathbb{P}^n$ be the projections.
Let Γ be the graph of $p : \widetilde{C} \to \mathbb{P}^n$.

$$\begin{array}{ccccccccc} 0 & \to & \pi^*\mathcal{M} & \to & V \otimes \mathcal{O}_{\widetilde{C}\times\mathbb{P}^n} & \to & \pi^*\mathcal{O}_{\widetilde{C}}(1) & \to & 0 \\ & & & & \| & & & & \\ 0 & \to & f^*\Omega_{\mathbb{P}^n}(1) & \to & V \otimes \mathcal{O}_{\widetilde{C}\times\mathbb{P}^n} & \to & f^*\mathcal{O}_{\widetilde{C}}(1) & \to & 0, \end{array}$$

The graph $\Gamma(\subseteq \widetilde{C} \times \mathbb{P}^n)$ is defined by $\pi^*\mathcal{M} \to f^*\mathcal{O}_C(1)$.
Then we have the exact sequence

$$\pi^*\mathcal{M} \otimes f^*\mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\widetilde{C}\times\mathbb{P}^n} \to \mathcal{O}_\Gamma \to 0.$$

After tensoring with $\pi^*\mathcal{A}$, we take the Koszul resolution

$$\pi^*(\wedge^2\mathcal{M}\otimes\mathcal{A})\otimes f^*\mathcal{O}_{\mathbb{P}^n}(-2) \to \pi^*(\mathcal{M}\otimes\mathcal{A})\otimes f^*\mathcal{O}_{\mathbb{P}^n}(-1) \to \pi^*\mathcal{A} \to \mathcal{O}_\Gamma\otimes\pi^*\mathcal{A} \to 0.$$

Gruson-Lazarsfeld-Peskine

Sketch of Proof

Then we have an exact sequence

$$\mathrm{H}^0(\mathcal{M} \otimes \mathcal{A}) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{u} \mathrm{H}^0(\mathcal{A}) \otimes \mathcal{O}_{\mathbb{P}^n} \to p_*\mathcal{A} \to 0.$$

Let $\mathcal{J}(\subseteq \mathcal{O}_{\mathbb{P}^n})$ be the Fitting ideal of $p_*\mathcal{A}$, that is, $\mathcal{J} = \mathrm{Im} \wedge^{n_0} u$, $n_0 = \mathrm{h}^0(\mathcal{A})$.

Note that $\mathrm{Supp}\, p_*\mathcal{A} = C$.

Then we have the Eagon-Northcott complex of u

$$\cdots \to \mathcal{O}_{\mathbb{P}^n}(-n_0 - 2)^{\oplus} \to \mathcal{O}_{\mathbb{P}^n}(-n_0 - 1)^{\oplus} \to \mathcal{O}_{\mathbb{P}^n}(-n_0)^{\oplus} \xrightarrow{\varepsilon} \mathcal{J} \to 0$$

such that ε is surjective and the complex is exact away from C, which gives $\mathcal{J}$ is n_0-regular, that is, $\mathcal{I}_X$ is n_0-regular,

Gruson-Lazarsfeld-Peskine

Proposition

Let $\mathcal{E}$ and $\mathcal{F}$ be locally free sheaves of $\operatorname{rank}\mathcal{E} = e$ and $\operatorname{rank}\mathcal{F} = f$ on a scheme X. Let $u : \mathcal{E} \to \mathcal{F}$. Then there is a complex

$$0 \to \wedge^e \mathcal{E} \otimes S^{e-f}(\mathcal{F}^*) \to \cdots \to \wedge^{f+1}\mathcal{E} \otimes S^1(\mathcal{F}^*) \to \wedge^f \mathcal{E} \to \wedge^f \mathcal{F} \to 0,$$

which is called as the Eagon-Northcott complex. If $u : \mathcal{E} \to \mathcal{F}$ is surjective, then the complex is exact.

Reference

H. Clemens, J. Kollár and S. Mori, Higher Dimensional Complex Geometry, Asterisque 166, SMF, 1088,
L.ecture 24: A Theorem of Gruson-Lazarsfeld-Peskine and a Lemma of Lazarsfeld by L. Ein.

Generic Projection Method

Theorem

Let X be a nondegenerate smooth projective variety of $\mathbb{P}^N_{\mathbb{C}}$. If $n = \dim X \leq 14$, then $\mathrm{reg}\, X \leq \deg X - \mathrm{codim}\, X + 1 + (n-2)(n-1)/2$.

Setup

Let $p : X(\subseteq \mathbb{P}^N_{\mathbb{C}}) \to \mathbb{P}^{n+1}_{\mathbb{C}}$ be a generic projection.

Take a cordinate $(x_0 : \cdots : x_{n+1} : x_{n+2} : \cdots , x_N) \to (x_0 : \cdots : x_{n+1})$, we have the canonical maps:

- $\psi_0 : \mathcal{O}_{\mathbb{P}^n} \to p_* \mathcal{O}_X$: a canonical map
- $\psi_1 = \displaystyle\sum_{n+2 \leq j \leq N} \phi_{x_j} : \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus} \to p_* \mathcal{O}_X$, where $\phi_{x_j} : \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{x_j} p_* \mathcal{O}_X$
- $\psi_2 = \displaystyle\sum_{0 \leq i \leq j \leq N} \phi_{x_i x_j} : \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus} \to p_* \mathcal{O}_X$, where $\phi_{x_i x_j} : \mathcal{O}_{\mathbb{P}^n}(-2) \xrightarrow{x_i x_j} p_* \mathcal{O}_X$

Generic Projection Method

Setup

Take $w = \psi_0 + \psi_1 + \psi_2 : \mathcal{G} = \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus} \oplus \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus} \to p_* \mathcal{O}_X$.

Lemma

Let $\mathcal{F} = \mathcal{G} \oplus \mathcal{O}_{\mathbb{P}^{n+1}_{\mathbb{C}}}(-3) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{n+1}_{\mathbb{C}}}(-n)$.
If there is a surjective morphism $v : \mathcal{F} \to p_* \mathcal{O}_X$ such that $v|_{\mathcal{G}} = w$,
then $\operatorname{reg} X \leq d - N + n + 1 + (n-1)(n-2)/2$.

Lemma

If $p : X(\subseteq \mathbb{P}^N_{\mathbb{C}}) \to \mathbb{P}^{n+1}_{\mathbb{C}}$ is 'good', there exists a surjective morphism $\mathcal{F} \to p_* \mathcal{O}_X$.

Generic Projection Method

Definition and Theorem

Let $p : X(\subseteq \mathbb{P}^N_{\mathbb{C}}) \to \mathbb{P}^{n+1}_{\mathbb{C}}$ be a projection. Let $S_j = \{z \in \mathbb{P}^{n+1}_{\mathbb{C}} | \deg p^{-1}(z) = j\}$. The projection p is said to be good if $\dim S_j \leq \max\{-1, n-j+1\}$ for all j.

(Mather's Theory) If $n = \dim X \leq 14$, then p is good.

Reference

- R. Lazarsfeld, A sharp Castelnuovo bound for smooth surfaces, Duke Math. J. 55(1987)
- S. Kwak, Castelnuovo regularity for smooth subvarieties of dimensions 3 and 4. J. Algebraic Geom. 7 (1998)
- L.. Chiantini, N. Chiarli and S. Greco, Bounding Castelnuovo-Mumford regularity for varieties with good general projections, J. Pure Appl. Algebra 152(2000)

Generic Projection Method

Example (Behesti-Eisenbud 2010)

Lazarsfeld has shown that the fibers $p^{-1}(z)$ of a generic projection $p : X(\subseteq \mathbb{P}^N_{\mathbb{C}}) \to \mathbb{P}^{n+1}_{\mathbb{C}}$, $n = \dim X$, can have exponentially greater degree.

Generic Projection Method seems not to work for higher dimensional cases of Eisenbud-Goto conjecture.

Reference

- R. Behesti and D. Eisenbud, Fibers of generic projections, Compositio Math., 146(2010).
- J. Mather, Generic projections. Ann. of Math. 98(1973)

$\mathcal{O}_X$-regularity

Remark

Let $X \subset \mathbb{P}^N$ be a nondegenerate projective variety. Let $m(\geq 2)$ be an integer. Then $\operatorname{reg} X \leq m$ if and only if the following conditions are satisfied:

(1) $X \subset \mathbb{P}^N$ is $(m-1)$-normal, that is, $\Gamma(\mathcal{O}_{\mathbb{P}^N}(m-1)) \to \Gamma(\mathcal{O}_X(m-1))$ is surjective.

(2) $\operatorname{reg} \mathcal{O}_X \leq m-1$.

Theorem (Noma, Kwak-Park)

Let X be a nondegenerate smooth projective variety in $\mathbb{P}^N$ over an algebraically closed field k of $\operatorname{char} k = 0$. Then $\mathcal{O}_X$ is $(\deg X - \operatorname{codim} X)$-regular.

In other words, $\mathrm{H}^i(\mathcal{I}_X(m-i)) = 0$ for $i \geq 2$, where $m = \deg X - \operatorname{codim} X + 1$.

$\mathcal{O}_X$-regularity

Sketch of Proof

Let us put $n = \dim X$, $d = \deg X$, and $c = \mathrm{codim}\, X = N - n$.
Let us consider a generic inner projection $p : X(\subset \mathbb{P}^N) \cdots \longrightarrow \bar{X}(\subset \mathbb{P}^{n+1})$.
Note that $\deg \bar{X} = d - c + 1$. Let us define the double point divisor from the inner projection (cf. Bayer-Mumford Technical Appendix Section 3, 4):

$$D_{\mathrm{inn}} = -K_X + (d - n - c - 1)H.$$

Then D_{inn} is semiample and by Kodaira Vanishing, we have $\mathrm{reg}\, \mathcal{O}_X \leq d - c$.

Reference

- A. Noma, Generic inner projections of projective varieties and an application to the positivity of double point divisors., Trans. AMS, 366 (2014)
- Sijong Kwak and Jinhyung Park, A bound for Castelnuovo-Mumford regularity by double point divisors, Adv. Math. 364 (2020)

Asymptotic property of Castelnuovo-Mumford regularity

Theorem (Bertram-Ein-Lazarsfeld 1991)

Let V be a smooth projective variety of $\mathbb{P}^n_{\mathbb{C}}$ scheme-theoretically defined by hypersurfaces of degrees $d_1 \geq \cdots \geq d_r$. Then $\mathrm{H}^i(\mathbb{P}^n, \mathcal{I}_V^q(\ell)) = 0$ for $\ell \geq d_1 q + d_2 + \cdots + d_r - n$.

Remark

The proof is difficult and obtained from the Kawamata-Viehweg vanishing theorem. The result means $\operatorname{reg} I^m \leq d_1 m + b$ for some b if a polynomilal ideal I defines a smooth projective variety over $\mathbb{C}$.

Reference

- A. Bertram, L. Ein and R. Lazarsfeld, Vanishing theorems, a theorem of Severi, and the equations defining projective varieties, J. Amer. Math. Soc. 4 (1991), 587 – 602.

Asymptotical Linearity of Regularity

The asymptotical linearity of the regularity had been believed to be true.

- K. Chandler, Regularity of the powers of an ideal, Comm. Algebra, 25 (1997), 3773 – 3776.
- I. Swanson, Powers of ideals: Primary decompositions, Artin-Rees lemma and regularity, Math. Ann. 307 (1997), 299 – 313.

Theorem (Cutkosky-Herzog-Trung 1999, Kodiyalam 2000)

Let I be a homogeneous ideal of the polynomial ring $S = k[x_1, \cdots, x_n]$. Then the regularity of I^m is asymptotically linear function, that is, there are integers d, b, s such that $\operatorname{reg} I^m = dm + b$ for any $m \geq s$.

Remark

The striking theorem are proved independently by Cutkosky-Herzog-Trung and Kodiyalam. Later, there are several attempts to obtain d and b, and s.

Asymptotical Linearity of Regularity

Reference

- S. D. Cutkosky, J. Herzog and N. V. Trung, Asymptotic behaviour of the Castelnuovo-Mumford regularity, Compositio Math. 118 (1999), 243 – 261.
- V. Kodiyalam, Asymptotic behavior of Castelnuovo-Mumford regularity, Proc. Amer. Math. Soc. 128 (2000), 407 –411.

Remark

We describe a proof of the Cutkosky-Herzog-Trung, Kodiyalam theorem.

The proof has 3 steps,

Step 1 surprisingly inceldes the Bertram-Ein-Lazarsfeld theorem.

Step 3 is the most complicated, depending on the method of Kodiyalam.

Asymptotical Linearity of Regularity

Sketch of Proof

STEP I. To prove $\operatorname{reg} I^m \leq Am + B, m \gg 0$ for some constant A, B.

Suppose I is minimally generated by $f_1, \cdots, f_r$ with $\deg f_i = d_i$. Let $R = k[X_1, \cdots, X_n, T_1, \cdots, T_r]$ with bigrading $\deg X_i = (1, 0)$, $\deg T_j = (d_j, 1)$.

For a bigraded R-module $M = \oplus M_{(d,\ell)}$, $M^{(m)}$ is defined as $\oplus_d M_{(d,m)}$.

A bigraded R-algebra $R(I) = S[It]$ by $X_i \to x_i$, $T_j \to f_j t_j$ has $S[It]^{(m)} \cong I^m$.

$R(-a, -b)^{(m)} \cong R^{(m-b)}(-a) \cong \oplus_{\ell_1+\cdots+\ell_r=m-b} S(-\ell_1 d_1 - \cdots - \ell_r d_r - a)$.

By Hilbert syzygy theorem, a graded R-modules $S[It]$ has a grade free resolution $0 \to F_u \to \cdots \to F_0 \to S[It] \to 0$, where $F_i = \oplus_{j=1}^{t_i} R(-a_{ij}, -b_{ij})$.

By taking $(-)^{(m)}$, we have the free resolution $0 \to F_u^{(m)} \to \cdots F_0^{(m)} \to I^m \to 0$.
Here $F_i^{(m)} \cong \oplus_{j=1}^{t_i} \oplus_{\ell_1+\cdots\ell_s=m-b_{ij}} S(-\ell_1 d_1 - \cdots - \ell_s d_s - a_{ij})$.
Thus we have $\operatorname{reg} I^m \leq Am + B$, where $A = \max d_i$ and $B = \max\{a_{ij} - Ab_{ij} - i\}$.

Asymptotic Linearity of Regularity

Sketch of Proof

STEP II. Let J be a reduction of I, that is, $I^q = JI^{q-1}$ for some q. As in **STEP I**, through the surjective map $R \to S[Jt]$, the Rees algebra $S[It]$ is a finitely generated R-module. Then $\operatorname{reg} I^m \leq d(J)m + b$ for $m \gg 0$ from **STEP I**

Let d be the minimum of $d(J)$ such that J is a reduction of I. We want to show

$$dm \leq \operatorname{reg} I^m \leq dm + b, m \gg 0.$$

There exists $f \in I$ of (the largest) degree p such that $f^m \notin \mathfrak{m}I^m$ for all $m \geq 1$, which implies $d(I^m) \geq pm$.

So, we have only to show $p \geq d$, that is, there exists a reduction J of I with $d(J) \leq p$. In fact, let us take a minimal generator $f_1, \cdots, f_r, \cdots, f_t$ of I with $\deg f_1 \leq \cdots \leq \deg f_r = p$ and otherwise $\deg f_i > p$.
Since $I^n = JI^{n-1} + (f_{r+1} + \cdots + f_t)^n \subset JI + \mathfrak{m}I^n$ for $n \gg 0$, J is a reduction of I by Nakayama's lemma, and $\operatorname{reg} I^m \geq d(I^m) \geq dm$.

Asymptotic Linearity of Regularity

Sketch of Proof

STEP III. From **STEP I** and **STEP II**, we have $\operatorname{reg} I^m = dm + b_n$ for $m \gg 0$. We will show that b_m is constant for $m \geq 0$.

Take a reduction $J = (f_1, \cdots, f_r)$ as in **STEP II**.

Let $R = k[X_1, \cdots, X_n, T_1, \cdots, T_r]$ and consider $S[Jt] \subset S[It]$. Take the Koszul complex of the bigraded R-module $S[It]$ with respect to $T_1, \cdots, T_r$.

Since the homology modules are annihilated by a power of $(T_1, \cdots, T_r)$, by taking $(-)^{(m)}$ for $m \gg 0$, we have

$$0 \to I^{m-r}(-d_1 - \cdots - d_r) \to \cdots \to I^{m-1}(-d_1) \oplus \cdots \oplus I^{m-1}(-d_r) \to I^m \to 0,$$

which implies $\operatorname{reg} I^m \leq \max\{\operatorname{reg} I^{m-1} + \max\{d_i\}, \operatorname{reg} I^{m-2} + \max\{d_i + d_j\} + \cdots + \operatorname{reg} I^{m-r} + (d_1 + \cdots + d_r)\}$.

Asymptotic Linearity of Regularity

Sketch of Proof

Since $\operatorname{reg} I^k \leq dk + b_k$ and $d_{i_1} + \cdots + d_{i_s} \leq sd$, we have

$$\begin{aligned} \operatorname{reg} I^m &= dm + b_m \\ &\leq \max\{d(m-1) + b_{m-1} + d, d(m-2) + b_{m-2} + 2d - 1, \\ &\quad \cdots, d(m-r) + b_{m-r} + (m-r)d - (m-r-1)\} \end{aligned}$$

Thus we have $b_m \leq \max\{b_{m-1}, b_{m-2} - 1, \cdots, b_{m-r} - (r-1)\}$.
Hence b_m is nonincreasing for $m \gg 0$, and b_m is constant for $m \gg 0$.

Asymptotic property and Geometry

Remark

Contrary to the ideal case, $\lim_{m\to\infty} \mathrm{reg}\, \frac{{\mathcal{I}_X}^m}{m}$ can be taken an irrational number.

- S. D. Cutkosky, L. Ein and R. Lazarsfeld. Positivity and complexity of ideal sheves, Math. Ann. 321 (2001), 213 –234.

Theorem (Eisenbud-Harris 2010)

Let $\varphi : X \to \mathbb{P}^n$ be a linear projection whose center does not meet X, defined by a linear subspace V. Let $I \subset S$ be the ideal generated by V.

$$\max\{\mathrm{reg}\, \varphi^{-1}(x) | x \in \mathbb{P}^n\} = b + 1,$$

where b is the least integer $\mathfrak{m}^{t+b} \subseteq I^t$ for $t \gg 0$.

- D. Eisenbud and J. Harris, Power of ideals and fibers of morphisms, Math. Res. Lett. 17 (2010), 269 – 275.

Asymptotic property and Geometry

Lemma

Let $\varphi : X \to \mathbb{P}^n$ be a finite morphism. Set $\mathcal{L} = \varphi^*\mathcal{O}_{\mathbb{P}^n}(1)$ amd $V = \varphi^*(\Gamma(\mathcal{O}_{\mathbb{P}^n}(1)) \subset \Gamma(\mathcal{L})$. Let $\mathcal{M}$ be a coherent sheaf on X and $W \subset \Gamma(\mathcal{M})$.
The following are equivalent:

(1) For $t \gg 0$, the map $\mathrm{Sym}_t(V) \otimes W \to \Gamma(\mathcal{L}^t \otimes \mathcal{M})$ is surjective.

(2) For every closed point $x \in \mathbb{P}^n$, the restriction map $W \to \Gamma(\mathcal{M}|_{\varphi^{-1}(x)})$ is surjective.

Sketch of Proof

(1) and (2) are equivalent to (3) $\mu : W \otimes \mathcal{O}_{\mathbb{P}^n} \to \varphi_*\mathcal{M}$ is surjective.

Indeed, (3) means that $W \otimes \mathrm{Sym}_t(V) \to \Gamma(\varphi_*\mathcal{M}(t))$ is surjective for $t \gg 0$. Also, $\Gamma(\varphi_*\mathcal{M}(t)) = \Gamma(\mathcal{M} \otimes \mathcal{L}^t)$, so (3) is equivalent to (1).

On the other hand, (2) $\Leftrightarrow$ (3) follows from the restriction of μ at $x \in \mathbb{P}^n$ and the finiteness of φ.

Asymptotic property and Geometry

Sketch of Proof (Eisenbud-Harris Theorem)

Let us take $\mathcal{M} = \mathcal{O}_{\mathbb{P}^n}(e)$ and $W = \Gamma(\mathcal{O}_{\mathbb{P}^n}(e))$ in Lemma.

(1) means there is an integer q such that $\mathrm{Sym}_t(V) \otimes \Gamma(\mathcal{O}_{\mathbb{P}^n}(e)) \to \Gamma(\mathcal{O}_{\mathbb{P}^n}(t+e))$ is surjective for $t \geq q$, in other words, $\mathfrak{m}^{t+e} \subset I^t$.

(2) means $\Gamma(\mathcal{O}_{\mathbb{P}^n}(e)) \to \Gamma(\mathcal{O}_{\mathbb{P}^n}(e)|_{\varphi^{-1}(x)})$ is surjective, equivalently $\mathrm{H}^1(\mathcal{I}_{\varphi^{-1}(x)}(e)) = 0$, that is, $\mathrm{reg}\, \varphi^{-1}(x) \leq e+1$.

Example

Let $\varphi : \mathbb{P}^1 = \mathrm{Proj}\, k[x,y] \to \mathbb{P}^n$ be a finite morphism by a linear system $V \subset \Gamma(\mathcal{O}_{\mathbb{P}^1}(d))$.
Let I be an ideal of $k[x,y]$ generated by V.
Then $\mathrm{reg}\, I^m = dm + r - 1$ for $q \gg 0$, where r is the number of the fibers.

Asymptotic property and Geometry

Mather's generic projection theorem means the following in commutative algebra.

Proposition

Let R be a standard graded algebra with $\dim R = n + 1$ over $\mathbb{C}$, and $\mathrm{Proj}\, R$ is smooth. If $I = (f_1, \cdots, f_{n+2})$ is an ideal generated $n + 2$ generic forms of degree d, and $n \leq 14$, then $\mathfrak{m}^{t+n} \subset I^t$ for all $t \gg 0$.

Conjecture (Beheshti-Eisenbud 2008)

The regularity of a every fiber of a generic projection of a smooth projective variety X to $\mathbb{P}^{n+c}$, $c \geq 1$ is bounded by $1 + n/c$, where $\dim X = n$.
Let R be the coordinate ring of X. This conjecture is equivalent to $\mathfrak{m}^{t+\lceil n/c \rceil} \subset I^t$ for $t \gg 0$ for an ideal I generated by $n + 1 + c$ general linear forms.

Reference

- Z. Ran, Unobstructedness of filling secants and the Gruson-Peskine general projection theorem. Duke Math. J. 164 (2015), 697722.

References

Reference

- H. T. Hà, Asymptotic linearity of regularity and a^*-invariant of powers of ideals. Math. Res. Lett. 18 (2011), 1 – 9.
- D. Eisenbud and B. Ulrich, Notes on regularity stabilization. Proc. Amer. Math. Soc. 140 (2012), no. 4, 1221 – 1232.
- M. Chardin, Regularity stabilization for the powers of graded $\mathfrak{m}$-primary ideals. Proc. Amer. Math. Soc. 143 (2015), 3343 – 3349.
- M. Chardin, Power of ideals and the cohomology of stalks and fibers of morphisms, Algebra Number Theory, 7 (2013), 1 – 18.
- S. Bisui, H. T. Hà and A. C. Thomas, Fiber invariants of projective morphisms and regularity of powers of ideals. Acta Math. Vietnam. 45 (2020), 183 – 198.

Regularity Bounds for Buchsbaum Variety

Definition

Let $X \subset \mathbb{P}^n$ be a projective scheme, where $\mathbb{P}^n = \mathrm{Proj}\, S$ and S is a polynomial ring with maximal ideal $\mathfrak{m}$.

(1) X is ACM if $\mathrm{H}^i(\mathcal{I}_X(\ell)) = 0$ for $1 \le i \le \dim X$ and ℓ.

(2) X is Buchsbaum if for all r-planes L (successive hyperplane sections) with $\dim X \cap L = \dim X - \mathrm{codim}\, L$,
$\mathfrak{m}\mathrm{H}^i_*(\mathcal{I}_{X\cap L}) = 0$ for $1 \le i \le \dim X \cap L$.

Theorem (Eisenbud-Goto 1984; Stückrad-Vogel 1988)

(1) Assume X is an ACM variety, i.e., R is Cohen-Macaulay, then
$\mathrm{reg}\, X \le \deg X - \mathrm{codim}\, X + 1$.

(2) Assume X is a Buchsbaum variety, i.e., R is Buchsbaum, then
$\mathrm{reg}\, X \le \lceil (\deg X - 1)/\mathrm{codim}\, X \rceil + 1$

Regularity Bounds for Buchsbaum Variety

Theorem (Trung-Valla 1988, Nagel 1995; Yanagawa 1997, Nagel 1999; Miyazaki 2011)

(1) An ACM variety X with $\deg X \gg 0$ and $\operatorname{reg} X = \lceil (\deg X - 1)/\operatorname{codim} X \rceil + 1$ is a divisor on a variety of minimal degree.

(2) A Buchsbaum variety X with $\deg X \gg 0$ and $\operatorname{reg} X = \lceil (\deg X - 1)/\operatorname{codim} X \rceil + 1$ is a divisor on a variety of minimal degree.

(3) A Buchsbaum variety X with $\deg X \gg 0$ and $\operatorname{reg} X = \lceil (\deg X - 1)/\operatorname{codim} X \rceil$ is a divisor either on a variety of minimal degree or on a Del Pezzo variety.

Regularity Bounds for Buchsbaum Variety

Remark

If $X \subseteq \mathbb{P}^n$ is a variety of minimal degree, that is, $\deg X = \operatorname{codim} X + 1$, then X is either (a) a quadric hypersurface, (b) the Veronese surface in $\mathbb{P}^5$, (c) a rational normal scroll or (d) their cone.

Definition (from Fujita's Book)

$X \subseteq \mathbb{P}^n$ is called a Del Pezzo variety if

- $\deg X = \operatorname{codim} X + 2$
- $X \cap L$ is an elliptic normal curve for a generic $(\operatorname{codim} X + 1)$-plane L
- only Gorenstein singularities, $\omega_X \cong \mathcal{O}_X(1-n)$
- $\mathrm{H}^q(X, \mathcal{O}_X(\ell)) = 0$ for all ℓ and $1 \leq q \leq \dim X - 1$

Classification in terms of Regularity Bound

Reference

- Stückrad and W. Vogel, Castelnuovo bounds for locally Cohen-Macaulay schemes. Math. Nachr. 136 (1988)
- L. T. Hoa and C. Miyazaki, Bounds on Castelnuovo-Mumford regularity for generalized Cohen-Macaulay graded rings. Math. Ann. 301 (1995).
- U. Nagel and P. Schenzel, Degree bounds for generators of cohomology modules and Castelnuovo-Mumford regularity. Nagoya Math. J. 152 (1998)
- C. Miyazaki, Buchsbaum varieties with next to sharp bounds on Castelnuovo-Mumford regularity. Proc. AMS 139 (2011).

Regularity Bound

Sketch of Proof

$V \subseteq \mathbb{P}^{n+\dim V}$: a Buchsbaum variety
$C = V \cap H_1 \cap \cdots \cap H_{\dim V - 1}$: a successive generic hyperplane section
$X = C \cap H \subseteq H (\cong \mathbb{P}^n)$: a generic hyperplane section.

$\operatorname{reg} V = \operatorname{reg} C = \operatorname{reg} X$

$$\begin{aligned} \operatorname{reg} X &= \min\{m \mid \mathrm{H}^1(\mathcal{I}_X(m-1)) = 0\} \\ &= \min\{t \mid \Gamma(\mathcal{O}_{\mathbb{P}^N}(t)) \twoheadrightarrow \Gamma(\mathcal{O}_X(t))\} + 1 \end{aligned}$$

X is in uniform position in $\operatorname{char} k = 0$ (linear semi-uniform position in $\operatorname{char} k > 0$). Take a union of hyperplanes F such that $F \cap X = X \setminus \{P\}$ for any $P \in X$ in $\operatorname{char} k = 0$

Uniform Position Principle

Proposition

$\operatorname{reg} X \leq \lceil (d-1)/n \rceil + 1$

Sketch of Proof

In case $\operatorname{char} k = 0$, X is in uniform position. Castelnuovo's method

$P \in X$, $\ell = \lceil (d-1)/n \rceil$
Devide the points $X \backslash \{P\}$ into ℓ groups.

$X \backslash \{P\} = \{P_1, \cdots, P_n | P_{n+1}, \cdots, P_{2n} | \cdots | P_{(\ell-1)n+1}, \cdots, P_{d-1}\}$
Take ℓ hyperplanes: $H_i = \langle P_{n(i-1)+1}, \cdots, P_{ni} \rangle \not\ni P$, $1 \leq i \leq \ell$.

Let us take a union of hyperplanes $F = H_1 \cup \cdots \cup H_\ell$.
Then we have $F \cap X = X \backslash \{P\}$ and $\Gamma(\mathcal{O}_H(\ell)) \to \Gamma(\mathcal{O}_X(\ell))$ is surjective.

Generic Hyperplane Section of Projective Curve

Sketch of Proof

In case $\operatorname{char} k > 0$, X is not necessarily in uniform position.

R: the the coordinate ring of X
$\underline{h} = (h_0, \cdots, h_s)$ be the h-vector of R
$h_i = \dim_k [R]_i - \dim_k [R]_{i-1}$, where s is the largest integer such that $h_s \neq 0$.
$h_0 = 1$, $h_1 = (n+1) - 1 = n$, $\deg X = h_0 + \cdots + h_s = d$ and $s = \operatorname{reg} X - 1$.

Lemma (Uniform Position Lemma(Griffiths-Harris, Ballico))

- $\operatorname{char} k = 0$, $h_i \geq h_1$, $i = 1, \cdots, s-1$
- $\operatorname{char} k > 0$, $h_1 + \cdots + h_i \geq ih_1$, $i = 1, \cdots, s-1$

Reference

E. Ballico, On singular curves in the case of positive characteristic. Math. Nachr. 141 (1989)

Generic Hyperplane Section of Projective Curve

Proposition

(1) $\operatorname{reg} X \leq d - n + 1 (= \deg X - \operatorname{codim} X + 1)$

(2) $\operatorname{reg} X \leq \lceil (d-1)/n \rceil + 1$

Sketch of Proof

(1) Since $h_i \geq 1$ for $0 \leq i \leq s$ and $h_1 = n$, we have

$$\operatorname{reg} X = s + 1 \leq h_0 + h_1 + \cdots + h_s - n + 1 = d - n + 1.$$

(2) Since $h_0 + \cdots + h_s = d$ and $h_1 + \cdots + h_{s-1} \geq (s-1)h_1$,

$$\operatorname{reg} X - 2 + h_s/h_1 = (s-1) + h_s/h_1 \leq (h_1 + \cdots + h_{s-1})/h_1 + h_s/h_1 = (d-1)/n.$$

Thus we have $\operatorname{reg} X - 1 \leq \lceil (d-1)/n \rceil$ as desired.

Castelnuovo, Eisenbud-Harris

Lemma (Castelnuovo, Eisenbud-Harris)

Let $X \subset \mathbb{P}^n$ be a generic hyperplane section of a curve.

(1) If $\deg X \geq 2n+1$ and $h_2 = h_1$, then X lies on a rational normal curve.

(2) If $\deg X \geq 2n+3$ and $h_2 = h_1 + 1$, then X lies on an elliptic normal curve.

Lemma (char k =0 for simplicity)

(1) $\deg X \geq n^2 + 2n + 2$ and $\operatorname{reg} X = \lceil (\deg X - 1)/n \rceil + 1$
$\Rightarrow X$ lies on a rational normal curve.

(2) $\deg X \geq n^2 + 4n + 2$ and $\operatorname{reg} X = \lceil (\deg X - 1)/n \rceil$
$\Rightarrow X$ lies on an elliptic normal curve.

Castelnuovo, Eisenbud-Harris

Conjecture (Harris)

For $1 \leq m \leq n-1$, if $\deg X \geq 2n+2m-1$ and $h_2 = h_1 + m - 1$, X lies on a curve of degree at most $n+m-1$.

Remark

What should we do in positive characteristic case?
$C \subseteq \mathbb{P}^{n+1}$
$X = C \cap H \subseteq H(\cong \mathbb{P}^n)$: a generic hyperplane section

If X is not in uniform position (it may happen only if $\mathrm{char}\, k = p > 0$) and $\deg X \gg 0$, then $\mathrm{reg}\, X \ll \lceil (d-1)/N \rceil + 1$?

Sketch of the Proofs

Sketch of Proof

In case Z is generated by quadratic equations.

$$\begin{array}{ccccccc} X = C \cap H & \subset & Z & \subset & H(\cong \mathbb{P}^n) \\ C & \subset & & \subset & \mathbb{P}^{n+1} \end{array}$$

We have to show

(1) $\Gamma(\mathcal{I}_{Z/H}(2)) \cong \Gamma(\mathcal{I}_{X/H}(2))$.

(2) $\Gamma(\mathcal{I}_{C/\mathbb{P}^{n+1}}(2)) \to \Gamma(\mathcal{I}_{X/H}(2))$ is surjective.

Keypoints

- Uniform Position Lemma, Castelnuovo's Lemma, Eienbud-Harris' Lemma
- Socle Lemma

Socle Lemma

Theorem (Socle Lemma(Huneke-Ulrich J. Alg. Geom. 1993))

Let $S = k[x_0, \cdots, x_n]$ be the polynomial ring over a field k, $\mathrm{char}\, k = 0$.
Let M be a finitely generated graded S-module.
For a generic element $h \in [S]_1$,

$$0 \to \mathrm{Ker}\,\varphi \to M(-1) \xrightarrow{\varphi} M \to \mathrm{Coker}\,\varphi \to 0$$

where $\varphi = \cdot h$.

If $\mathrm{Ker}\,\varphi \neq 0$, then $a_-(\mathrm{Ker}\,\varphi) > a_-(\mathrm{Soc}(\mathrm{Coker}\,\varphi))$,
where $\mathrm{Soc}(N) = [0 : \mathfrak{m}]_N$ and $a_-(N) = \min\{i | [N]_i \neq 0\}$ for a finitely generated graded S-module N.

A Small Theorem

Definition

C : a projective curve in $\mathbb{P}^n$
$M(C) = \mathrm{H}^1{}_*\mathcal{I}_C = \oplus_{\ell\in\mathbb{Z}}\mathrm{H}^1(\mathcal{I}_C(\ell))$: Hartshorne-Rao module
$k(C) = \min\{v \geq 0 \mid \mathfrak{m}^v M(C) = 0\}$

Proposition

Let C be a nondegenerate non-ACM space curve in $\mathbb{P}^3$ over an algebraically closed field of characterstic 0. Then $\mathrm{reg}\, C \leq \lceil(\deg C - 1)/\mathrm{codim}\, C\rceil + k(C)$.

If $\deg C \geq 10$ and $\mathrm{reg}\, C = \lceil(\deg C - 1)/\mathrm{codim}\, C\rceil + k(C)$, then C is a divisor of either $(a, a+2)$ or $(a, a+3)$ on a smooth quaric surface $\mathbb{P}^1 \times \mathbb{P}^1$.

Counterexamples

Theorem (McCullough-Peeva)

Over any field k the Castelnuovo-Mumford regularity of nondegenerate homogeneous prime ideals is not bounded by any polynomial function of the mulplicity.

Corollary

There is a nondegenerate projective variety X in $\mathbb{P}^n$ such that $\mathrm{reg}\, X > \deg X - \mathrm{codim}\, X + 1$.

Reference

- J. McCullough and I. Peeva, Counterexamples to the Eisenbud-Goto regularity conjecture, J. Amer. Math. Soc. 31 (2018)
- J. McCullough and I. Peeva, The regularity conjecture for prime ideals in polynomial rings, EMS Survey Math. Sci. 7 (2020).

Counterexamples

Construction Method

(1) Take a bad ideal, that is, I is a homogeneous ideal of the standard polynomial ring S such that $\operatorname{reg} I \gg \deg S/I$, but I not prime.

(2) By using Rees-like algebra (or Rees algebra), take a homogeneous prime ideal P of the non-standard (weighted) polynomial ring T with $\operatorname{reg} P$ and $\deg T/P$ computable from $\operatorname{reg} I$ and $\deg S/I$.

(3) By step-by-step homogenization (or prime standardization), take a homogeneous prime ideal $P' = PT'$ of the standard polynomial ring T' with $\operatorname{reg} P' = \operatorname{reg} P$ and $\deg T'/P' = \deg T/P$.

Counterexamples

Proposition (cf. Bayer-Mumford)

Let I be a homogenous ideal of $k[x_0, \cdots, x_n]$. Then we have

(1) $\mathrm{char}\, k = 0$, $\mathrm{reg}\, I \leq (2\mathrm{maxdeg}(I))^{2^{n-1}}$

(2) $\mathrm{char}\, k > 0$, $\mathrm{reg}\, I \leq (2\mathrm{maxdeg}(I))^{n!}$

Example (Mayr-Meyer 1984)

There is an ideal I of $k[x_0, \cdots, x_n]$ with $\mathrm{maxdeg}(I) = 4$ and $\mathrm{reg}\, I \geq 2^{2^n} - 1$.

Example (Jee Koh 1998)

In the polynomial ring $k[x_1, \cdots, x_{22r-1}]$, there is an ideal I_r generated by 23 quadrics and one linear form such that $\mathrm{maxdeg}(Syz_1(I_r)) \geq 2^{2^{r-1}}$.

Rees Algebra

Definition

Let $I = (f_1, \cdots, f_r)$ be an ideal of the polynomial ring $S = k[x_1, \cdots, x_n]$.

The Rees algebra of I is defined as $R(I) = S[It](= \oplus_{d \geq 0} I^d) \subset S[t]$.
$\operatorname{Proj} R(I)$ is the blowing up of $\mathbb{A}^n$ along I. The defining ideal P is the kernel of $\varphi : S[y_1, \cdots, y_r] \to S[It]$ by $\varphi(y_i) = f_i t$.

P is, in general, difficult to compute.

Example (McCullough-Peeva)

Let $I = (u^6, v^6, u^2w^4 + v^2x^4 + uvwy^3 + uvxz^3)$ be an ideal of $S = k[u, v, w, x, y, z]$.
Let us take a defining prime ideal P of $T = S[w_1, w_2, w_3]$ of the Rees algebra $S[It]$.
By Bertini Theorem, we have a singular 3-fold X in $\mathbb{P}^5$ with $\deg X = 31$ and $\operatorname{reg} X \geq 38$ by computation with Macaulay2.

Rees-like Algebra

Definition

Let $I = (f_1, \cdots, f_r)$ be an ideal of the polynomial ring $S = k[x_1, \cdots, x_n]$.

The Rees-like algebra of I is defined as $\mathcal{RL}(I) = S[It, t^2] \subset S[t]$.
The defining ideal Q is the kernel of $\psi : T = S[y_1, \cdots, y_r, z] \to S[It, t^2]$ by $\psi(y_i) = f_i t$ and $\psi(z) = t^2$, where $\deg y_i = \deg f_i + 1$ and $\deg t = 2$.

NOT standard graded!

Example

Let $I = (x)$ be an ideal of $k[x]$. Then $\mathcal{RL}(I) = k[x, xt, t^2]$ and $P = (y^2 - x^2 z)$ in $k[x, y, z]$.

Rees-like Algebra

Theorem

(1) $\operatorname{reg} T/Q = \operatorname{reg} S/I + 2 + \sum_{i=1}^{r} \deg f_i$

(2) $\deg T/Q = 2\prod_{i=1}^{r}(\deg f_i + 1)$

(3) $\operatorname{ht} Q = r$

Sketch of Proof

The prime ideal Q of $T = k[x_1, \cdots, x_n, y_1, \cdots, y_r, z]$ is minimally generated by

$\{y_\alpha y_\beta - z f_\alpha f_\beta | 1 \le \alpha, \beta \le r\}$ and $\{\sum c_{ij} y_i | \sum c_{ij} f_i = 0\}$,

where the minimal free resolution of P as a graded S-module is

$$F_1 \xrightarrow{(c_{ij})} F_0 \xrightarrow{(f_i)} P \to 0.$$

Rees-like Algebra

Sketch of Proof

Since Q is homogeneous prime, z is a nonzerodivisor of T/Q.
Let $\bar{T} = T/(z)$ and $\bar{Q} = Q\bar{T}$.
Then the graded Betti numbers of T/Q and $\bar{T}/\bar{Q}$ is the same!

Now we have a homogeneous prime ideal $\bar{Q}$ in $\bar{T} = k[x_1, \cdots, x_n, y_1, \cdots, y_r]$. The prime ideal $\bar{Q}$ is generated by $M = (\{\sum_i c_{ij} y_i\})$ and $N = (\{y_i y_j\}) = (y_1, \cdots, y_r)^2$.

The minimal free resolution of $\bar{T}/\bar{Q}$ is constructed as the mapping cone of that of $(M+N)/N \to \bar{T}/N$, which is explicitely described.

In fact , the minimal $\bar{T}$-free resolution of $M+N/N(\cong M/M \cap N)$ comes from the minimal S-free resolution of $\mathrm{Syz}_1^S I$.
The minimal free resolution of $\bar{T}/\bar{Q}$ is the Eagon-Northcott resolution.

Homogenization

Definition and Proposition (Step-by-step homogenization)

Let $T = k[y_1, \cdots, y_p]$ be the polynomial ring with $\deg y_i > 1$ for $i \leq q$ and $\deg y_i = 1$ for $i > q$. Let $T' = k[y_1, \cdots, y_p, v_1, \cdots, v_q]$ be the standard polynomial ring. Let $\nu : T \to T'$ be a graded homomorphism by $\nu(y_i) = y_i v_i^{\deg y_i - 1}$ for $i \leq q$ and $\nu(y_i) = y_i$ for $i > q$.

Let P be a homogeneous prime ideal of T. Then PT' is a homogeneous prime ideal of T', and the graded Betti numbers of T/P and T'/P' is the same.

Remark

There is another homogenization preserving the graded Betti numbers. Mantero-McCullough-Miller use the Prime Standization by the Ananyan-Hochster theory (homogeneous prime sequence) to controll the singular locus.

Homogenization

Example (Affine Monomial Curve)

Let $P = (x^2 - y, xy - z)$ in $S = k[x, y, z]$, which is the kernel of $\varphi : S \to k[t]$ by $\varphi(x) = t$, $\varphi(y) = t^2$ and $\varphi(z) = t^3$.

Let us take a non-standard grading $\deg x = 1$, $\deg y = 2$ and $\deg z = 3$. Then it is graded, and $\operatorname{reg} P = 4$ since the minimal free resolution is:

$$0 \to S(-5) \to S(-2) \oplus S(-3) \to S \to S/P \to 0.$$

(1) Traditional homogenization gives $P' = (x^2 - yw, xy - zw, xz - y^2)$ in $S' = k[x, y, z, w]$, which is a twisted cubic curve, and $\operatorname{reg} P' = 2$.

(2) Step-by-step homogenization gives $Q = (x^2 - yu, wyu - zu^2)$ in $T = k[x, y, z, u, w]$, which is a complete intersection, and $\operatorname{reg} Q = 4$.

Counterexamples

Sketch of Proof (McCullough-Peeva Theorem)

From Jee Koh's example we have homogeneous prime ideals P_r of the standard polynomial ring R_r such that

- $\deg R_r/P_r \leq 4 \cdot 3^{22r-3} < 2^{50r}$
- $\mathrm{reg}\, P_r \geq \mathrm{maxdeg}(P_r) \geq 2^{2^{r-1}} + 1 > 2^{2^{r-1}}$,

which yields the assertion of the McCullough-Peeva theorem.

Horrocks Criterion

Theorem (Horrocks 1964)

Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^n$ of rank r.
Assume that $\mathcal{E}$ is ACM, that is, $\mathrm{H}^i_*(\mathcal{E}) = \oplus_{\ell\in\mathbb{Z}}\mathrm{H}^i(\mathbb{P}^n, \mathcal{E}(\ell)) = 0$ for $1 \leq i \leq n-1$.
Then $\mathcal{E}$ is isomorphic to a direct sum of line bundles.

Remark

There are several proofs for Horrocks Theorem.

- Horrocks' original proof
- Induction on the dimension of projective space (cf. Okonek-Schneider-Spindler)
- Auslander-Buchsbaum Theorem(cf. Matsumura)
- Use the Castelnuovo-Mumford regularity

Horrocks Criterion

Proof (cf. Okonek-Schneider-Spindler)

We will prove by induction on n. $n = 1$ is the Grothendieck Theorem.
For $n \geq 2$, let us take $\mathcal{F} = \oplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n}(a_i)$ from an isomorphism $\mathcal{E}|_H \cong \oplus_{i=1}^{r} \mathcal{O}_H(a_i)$. Then we have only to take a section of $\Gamma(\mathcal{F}^\vee \otimes \mathcal{E})$ by using the hypothesis of induction.

Proof (Auslander-Buchsbaum Theorem 1958)

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring.
Let M be a finitely generated R-module with $\operatorname{proj}\dim M < \infty$.
Then $\operatorname{depth} M + \operatorname{proj}\dim M = \operatorname{depth} R$.

Horrocks Criterion

Proof (Horrocks Criteiron using Castelnuovo-Mumford Regularity)

Let $\mathcal{E}$ be an ACM vector bundle on $\mathbb{P}^n$.
Assume that $\mathcal{E}$ is m-regular but not $(m-1)$-regular.
Then we have a surjective map $\varphi : \mathcal{O}_{\mathbb{P}^n}^{\oplus} \to \mathcal{E}(m)$.

Since $\mathcal{E}$ is ACM, we have $\mathrm{H}^n(\mathcal{E}(m-1-n)) \neq 0$, and $\mathrm{H}^0(\mathcal{E}^\vee(-m)) \neq 0$ by Serre duality.
Thus we have a nonzero map $\psi : \mathcal{E}(m) \to \mathcal{O}_{\mathbb{P}^n}$.

Since $\psi \circ \varphi$ is nonzero, it splits.
Hence $\mathcal{O}_{\mathbb{P}^n}$ is a direct summand of $\mathcal{E}(m)$.

Horrocks Correspondence

Theorem (Horrocks, Walter, Malaspina-Rao)

Let **VB** be the category of vector bundles on $\mathbb{P}^n$ modulo stable equivalence. Here vector bundles $\mathcal{E}$ and $\mathcal{F}$ on $\mathbb{P}^n$ are stable equivalent if there are direct sums of line bundles $\mathcal{L}$ and $\mathcal{M}$ such that $\mathcal{E} \oplus \mathcal{L} \cong \mathcal{F} \oplus \mathcal{M}$.

Let us write **FinL** for the full subcategory of $C^\bullet \in Ob(D^\flat(\mathrm{S}-\mathbf{Mod}))$ such that $\mathrm{H}^i(C^\bullet)$ is a finite over S for $0 < i < n$ and $\mathrm{H}^i(C^\bullet) = 0$ for all other i.

Then we have the following Horrocks correspondence:
A functor $\tau_{>0}\tau_{<n}\mathbb{R}\Gamma_* : \mathbf{VB} \to \mathbf{FinL}$ gives an equivalence of the categories.

Sketch of Proof (Walter, Malaspina-Rao)

Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^n = \mathrm{Proj}\, S = \mathrm{Proj}\, k[x_0, \cdots, x_n]$.
Let us put $E = \Gamma_* \mathcal{E}$.
A graded $S^\vee$-module $E^\vee$ is (negatively graded, but) $E^\vee$ is finitely generated with finite projective dimension.

Horrocks Correspondence

Sketch of Proof (Walter, Malaspina-Rao)

Note that $\operatorname{depth} E^\vee \geq 2$ since $E^{\vee\vee\vee} = E^\vee$.
By Auslander-Buchsbaum theorem, we have an exact sequence:

$$0 \to P^{n-1\vee} \to \cdots \to P^{0\vee} \to E^\vee \to 0,$$

where $P^{i\vee}$ is a dual of a graded free S-modules.
By taking dual, we have a complex of graded S-modules:

$$0 \to E \to P^0 \to \cdots \to P^{n-1} \to 0$$

Thus we have an exact sequence of sheaves on $\mathbb{P}^n$.

$$0 \to \mathcal{E} \to \mathcal{P}^0 \to \cdots \to \mathcal{P}^{n-1} \to 0$$

Horrocks Correspondence

Sketch of Proof (Walter, Malaspina-Rao)

A complex $P^\bullet : 0 \to P^0 \to \cdots P^{n-1} \to 0$ have $\mathrm{H}^i_*(\mathcal{E}) \cong \mathrm{H}^i(P^\bullet)$, $1 \leq i \leq n-1$, precisely, $\tau_{<n}\mathbb{R}\Gamma_*\mathcal{E} \cong P^\bullet$.
Then a complex $0 \to E \to P^0 \to \cdots \to P^{n-1} \to 0$ and the minimal free resolution of E, $0 \to P^{-n} \to \cdots \to P^{-1} \to E \to 0$, give a complex of graded S-mudules

$$P^\bullet : 0 \to P^{-n} \to \cdots \to P^0 \to \cdots \to P^{n-1} \to 0.$$

Here we remark that $\mathrm{H}^i(P^\bullet)$ has a finite length, especially $\mathrm{H}^i(P^\bullet) = 0$, $i \notin \{1, \cdots, n-1\}$.

Corollary

From the Horrocks correspondence, the vanishing of the intermediate cohomologies of a vector bundle $\mathcal{E}$ on $\mathbb{P}^n$, that is, $\tau_{>0}\tau_{<n}\mathbb{R}\Gamma_*(\mathcal{E}) = 0$ implies that $\mathcal{E}$ is isomorphic to a direct sum of line bundles, which is the original Horrocks theorem.

Buchsbaum Bundle

Definition

A vector bundle $\mathcal{E}$ on $\mathbb{P}^n$ is called a Buchsbaum bundle if $(x_0, \ldots, x_n)\mathrm{H}^i_*(\mathbb{P}^n, \mathcal{E}|_L) = 0$, $1 \leq i \leq r-1$ for any r-plane $L(\subseteq \mathbb{P}^n)$, $r = 1, \cdots, n$.

Definition and Proposition (Stückrad-Vogel, Schenzel)

Let $S = k[x_0, \cdots, x_n]$ be the polynomial ring over a field k with $\mathfrak{m} = (x_0, \cdots, x_n)$. A graded S-module M with $\dim M = d$ is called as a Buchsbaum module it the following equivalent conditions are satisfied.

(i) $\ell(M/\mathfrak{q}M) - e(\mathfrak{q}; M)$ does not depend on the choice of any homogeneous parameter ideal $\mathfrak{q} = (y_1, \cdots, y_d)$.

(ii) For any homogeneous system $y_1, \cdots, y_d$, $0 \leq i \leq d$ of parameters $\mathfrak{m}\mathrm{H}^j_{\mathfrak{m}}(M/(y_1, \cdots, y_i)M) = 0, \quad 0 \leq j \leq d-i-1$ holds.

(iii) $\tau_{<d}\mathbb{R}\Gamma_{\mathfrak{m}}(M)$ is isomorphic to a complex of k-vector spaces in $D^\flat(\mathrm{S}-\mathbf{Mod})$.

Buchsbaum Bundle

Theorem (Goto-Chang)

A Buchsbaum bundle $\mathcal{E}$ on $\mathbb{P}^n$ is isomorphic to a direct sum of sheaves of differential form, that is, $\mathcal{E} \cong \oplus \Omega^{k_i}_{\mathbb{P}^n}(\ell_i)$.

Remark

There are several proofs for the structure theorem of Buchsbaum bundles on $\mathbb{P}^n$..

- S. Goto, Maximal Buchsbaum modules over regular local rings and a structure theorem for generalized Cohen-Macaulay modules, ASPM 11(1987)
- M. C. Chang, Characterization of arithmetically Buchsbaum subschemes of codimension 2 in $\mathbb{P}^n$, J. Differential Geom. 31 (1990), 323–341.
- Horrocks Correspondence (Schenzel, Yoshino)
- Syzygy Theoretic Proof

Horrocks Correspondence

Question

Are there any criteria?

- Null-Correlation bundle on $\mathbb{P}^n$, n odd?
- Horrocks-Mumford bundle on $\mathbb{P}^4$?

Reference (Horrocks Correspondence)

- F. Malaspina and A. P. Rao, Horrocks correspondence on arithmetically Cohen-Macaulay varieties, Algebra Number Theory 9(2015).
- C. H. Walter, Pfaffian subschemes, J. Algebraic Geom. 5(1996).
- Y. Yoshino, Maximal Buchsbaum modules of finite projective dimension, J. Algebra 159(1993).
- F. Malaspina and C. Miyazaki, Cohomological property of vector bundles on biprojective spaces, Ric. mat. 67(2018).

Key Lemma of Goto-Chang's Proof

Lemma (Goto (3.5.2), Chang (1.3))

Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^n$ with $\mathrm{H}^1_*(\mathcal{E}) = 0$. Assume that there is an exact sequence $0 \to \mathcal{E} \to \mathcal{L} \to \mathcal{F} \to 0$, where $\mathcal{L}$ is a direct sum of line bundles not being any summand of $\mathcal{E}$, and $\mathcal{F} = \oplus_{p_j \geq 1} \Omega^{p_j}(k_j)$. Then we have $\mathcal{E} \cong \oplus_{p_j \geq 1} \Omega^{p_j+1}(k_j)$.

Goto's Proof

Sketch of Proof

Set $M = \Gamma_*(\mathcal{E})$ and $\mathbb{P}^n = \mathrm{Proj}\, S$, where $S = k[x_0, \cdots, x_n]$. We have only to consider the case M is not Cohen-Macaulay.

STEP I. Let us take a short exact sequence $0 \to N \to F \to M \to 0$, where F is graded free. Since N is Buchsbaum and $\mathrm{depth}\, N \geq \mathrm{depth}\, M + 1$, N is isomorphic to a direct sum of syzygy modules $E_i(k)$ by induction. By taking the dual sequence $0 \to M^* \to F^* \to N^* \xrightarrow{\partial} \mathrm{Ext}^1_S(M, S) \to 0$, we have short exact sequences $0 \to M^* \to F^* \to W \to 0$ and $0 \to W \to N^* \xrightarrow{\partial} \mathrm{Ext}^1_S(M, S) \to 0$.

STEP II. W is isomorphic to a direct sum of some copies of $E_j(\ell)$'s. Indeed, we see that $\partial(E_j) = 0$, $j = 1, \cdots, n$ and $\mathfrak{m}\mathrm{Ext}^1_S(M, S) = 0$ from the Buchsbaumness of M and the property of Koszul complex.

STEP III. Hence $\widetilde{M^*}$ is isomorphic to a direct sum of sheaves of differential p-forms with some twist by Lemma, and so is $\mathcal{E}$.

Chang's Proof

Sketch of Proof

STEP I. Let $\mathcal{E}$ be a Buchsbaum vector bundle on $\mathbb{P}^n$. If $\mathrm{H}^1_*(\mathcal{E}) \neq 0$, there is a short exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{L} \to 0$, where $\mathrm{H}^1_*(\mathcal{F})$ and $\mathcal{L}$ is a direct sum of line bundles.

STEP II. The minimal generator of $\Gamma_*(\mathcal{F}^\vee)$ give a short exact sequence $0 \to \mathcal{F} \to \mathcal{L} \to \mathcal{K} \to 0$, where $\mathcal{L}$ is a sum of line bundles. Then $\mathcal{K}$ is Buchsbaum with $i(\mathcal{K}) = i(\mathcal{E}) - 1$, where $i(\mathcal{E})$ is defined as the minimal i such that $\mathrm{H}^p_*(\mathcal{E}) = 0$ for $i+1 \leq p \leq n-1$. Thus we have $\mathcal{K}$ is isomorphic to a direct sum of $\Omega^{p_j}(k_j)$'s, and so is $\mathcal{F}$ by Lemma.

STEP III. A short exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{L} \to 0$ gives the assertion by using STEP II and the Buchsbaum property of $\mathcal{E}$.

Null-Correlation Bundle

Definition

Let n be an odd number. Let $\mathbb{P}^n = \operatorname{Proj} k[x_0, \cdots, x_n]$. From the Euler sequence, we see that $\Gamma(\Omega(2))$ is the kernel of $\Gamma(\mathcal{O}_{\mathbb{P}^n}(1))^{\oplus n+1} \to \Gamma(\mathcal{O}_{\mathbb{P}^n}(2))$.

Then $(x_1, -x_0, x_3, -x_2, \cdots, x_n, -x_{n-1}) \in \Gamma(\Omega(2))$ gives a map $\mathcal{O}_{\mathbb{P}^n} \to \Omega(2)$, which defines, by taking dual, a surjective morphism $\varphi : \mathcal{T}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n}(1)$.

A null-correlation bundle $\mathcal{N}$ is defined as $\operatorname{Ker} \varphi$, that is, gives a short exact sequence

$$0 \to \mathcal{N} \to \mathcal{T}_{\mathbb{P}^n}(-1)(\cong \Omega^{n-1}(n)) \to \mathcal{O}_{\mathbb{P}^n}(1) \to 0.$$

Remark

A null-correlation bundle $\mathcal{N}$ is quasi-Buchsbaum but not Buchsbaum.

In fact, the intermediate cohomologies appear only in $\mathrm{H}^1(\mathcal{N}(-1))(\cong k)$ and $\mathrm{H}^{n-1}(\mathcal{N}(-n))(\cong k)$.

Null-Correlation Bundle

Proposition

Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^n$ with n odd. Assume $\mathrm{H}^1_*(\mathcal{E}) \cong \mathrm{H}^{n-1}_*(\mathcal{E}) \cong k$ and $\mathrm{H}^i_*(\mathcal{E}) = 0$, $2 \le i \le n-2$. Then $\mathcal{E}$ is isomorphic to either a null-correlation bundle or a direct sum of a differential 1-form and $(n-1)$-form with some twist, modulo stable equivalence.

Remark

For a null-correlation bundle $\mathcal{E}$, which is quasi-Buchsbaum not Buchabaum, how about Goto-Chang's proof?

(1) In STEP II of Goto's proof, $\partial(E_i)$ is not necessarily zero because $\mathrm{Ext}^n_S(k, M) \to \mathrm{H}^n_{\mathfrak{m}}(M)$ is zero for an S-module M corresponding to a null-correlation bundle.

(2) In Chang's proof, $\mathcal{F}$ must be a direct sum of differential $(n-1)$-form with some twist, and then $\mathcal{E}$ is seen to be a null-correlation bundle.

Horrocks-Mumford bundle

Definition

A Horrocks-Mumford bundle $\mathcal{E}$ is defined by a monad from the following complex:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 5} \xrightarrow{\varphi} \Omega^2_{\mathbb{P}^4}(2)^{\oplus 2} \xrightarrow{\psi} \mathcal{O}_{\mathbb{P}^4}^{\oplus 5} \to 0$$

given by $\varphi(a_0, \cdots, a_4) = (a_0 e_2 \wedge e_3 + \cdots + a_4 e_1 \wedge e_2, a_0 e_1 \wedge e_4 + \cdots + a_4 e_0 \wedge e_3)$, and ψ given by dual of φ.

Remark

$$\begin{array}{ccccccccc}
 & & & & & & \mathcal{E} & & \\
 & & & & & & \downarrow & & \\
0 & \to & \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 5} & \xrightarrow{\varphi} & \Omega^2_{\mathbb{P}^4}(2)^{\oplus 2} & \to & \operatorname{Coker}\varphi & \to & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \to & \operatorname{Ker}\psi & \to & \Omega^2_{\mathbb{P}^4}(2)^{\oplus 2} & \xrightarrow{\psi} & \mathcal{O}_{\mathbb{P}^4}^{\oplus 5} & \to & 0 \\
 & & \downarrow & & & & & & \\
 & & \mathcal{E} & & & & & &
\end{array}$$

Horrocks-Mumford bundle

Remark

From the previous observation we have

- $\mathrm{H}^1_*(\mathcal{E}) \cong \mathrm{Coker}\,(\Omega^2_S(2)^{\oplus 2} \xrightarrow{\psi} S^{\oplus 5})$
- $\mathrm{H}^2_*(\mathcal{E}) \cong \mathrm{H}^2_*(\Omega^2_{\mathbb{P}^4}(2)^{\oplus 2}) \cong k^{\oplus 2}$
- $\mathrm{H}^3_*(\mathcal{E})$ is isomorphic to the dual of $\mathrm{H}^1_*(\mathcal{E})$.

Here is a cohomology table of $\mathrm{H}^i(\mathcal{E}(\ell))$.

	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
3	0	2	10	10	5	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	2	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	5	10	10	2	0

Question

Find criteria for Horrocks-Mumford bundle in terms of commutative algebra.

Syzygy Theoretic Method

Proposition

Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^n$ with $\mathrm{H}^p(\mathcal{E}) \neq 0$, where $1 \leq p \leq n-1$.
If a vector bundle $\mathcal{E}$ has the following condition:

(a) $\mathrm{H}^i(\mathcal{E}(p-i+1)) = 0$ for $1 \leq i \leq p$.

(b) $\mathrm{H}^i(\mathcal{E}(p-i-1)) = 0$ for $p \leq i \leq n-1$,

then $\Omega^p_{\mathbb{P}^n}$ is a direct summand of $\mathcal{E}$.

Sketch of Proof

By an exact sequence arising from the Koszul complex:

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}^{\oplus}_{\mathbb{P}^n}(1) \to \cdots \to \mathcal{O}^{\oplus}_{\mathbb{P}^n}(p) \to \Omega^{p\vee}_{\mathbb{P}^n} \to 0,$$

we have a surjective map $\varphi : \mathrm{H}^0(\mathcal{E} \otimes \Omega^{p\vee}_{\mathbb{P}^n}) \to \mathrm{H}^p(\mathcal{E})$ from the cohomological condition (a).

Syzygy Theoretic Method

Sketch of Proof

By an exact sequence arising from the Koszul complex:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-n-1) \to \mathcal{O}_{\mathbb{P}^n}^{\oplus}(-n) \to \cdots \to \mathcal{O}_{\mathbb{P}^n}^{\oplus}(-p-1) \to \Omega_{\mathbb{P}^n}^p \to 0,$$

we have a surjective map $\psi : \mathrm{H}^0(\mathcal{E}^\vee \otimes \Omega_{\mathbb{P}^n}^p) \to \mathrm{H}^p(\mathcal{E}^\vee(-n-1))$ from the cohomological condition (b).

$^\exists f \in \mathrm{H}^0(\mathcal{E} \otimes \Omega_{\mathbb{P}^n}^{p\vee})$ such that $\varphi(f) = s(\neq 0) \in \mathrm{H}^p(\mathcal{E})$.

$^\exists s^* \in \mathrm{H}^{n-p}(\mathcal{E}^\vee(-n-1))$ corresponding to $s \in \mathrm{H}^m(\mathcal{E})$.

$^\exists g \in \mathrm{H}^0(\mathcal{E}^\vee \otimes \Omega_{\mathbb{P}^n}^p)$ such that $\psi(g) = s^*(\neq 0) \in \mathrm{H}^{n-p}(\mathcal{E}^\vee(-n-1))$.

Now f and g are regarded as elements of $\mathrm{Hom}(\Omega_{\mathbb{P}^n}^p, \mathcal{E})$ and $\mathrm{Hom}(\mathcal{E}, \Omega_{\mathbb{P}^n}^p)$.

Syzygy Theoretic Method

Proof

From a commutative diagram:

$$\begin{array}{ccccc} f\otimes g & \in & \mathrm{H}^0(\mathcal{E}\otimes\Omega^{p\vee}_{\mathbb{P}^n})\otimes\mathrm{H}^0(\mathcal{E}^\vee\otimes\Omega^{p}_{\mathbb{P}^n}) & \to & \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}) \\ & & \downarrow & & \downarrow \\ s\otimes s^* & \in & \mathrm{H}^p(\mathcal{E})\otimes\mathrm{H}^{n-p}(\mathcal{E}^\vee(-n-1)) & \to & \mathrm{H}^n(\mathcal{O}_{\mathbb{P}^n}(-n-1)), \end{array}$$

a natural map $\mathrm{H}^0(\mathcal{E}\otimes\Omega^{p\vee}_{\mathbb{P}^n})\otimes\mathrm{H}^0(\mathcal{E}^\vee\otimes\Omega^{p}_{\mathbb{P}^n})\to\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n})$ yields that $g\circ f$ is an isomorphism, which implies $\Omega^p_{\mathbb{P}^n}$ is a direct summand of $\mathcal{E}$.

Multigraded Regularity, Syzygy Theoretic Method

Exercise

There are no vector bundles $\mathcal{E}$ on $X = \mathbb{P}^m \times \mathbb{P}^n$ such that $\mathrm{H}^i(\mathcal{E}(\ell_1, \ell_2)) = 0$ for all $\ell_1, \ell_2 \in \mathbb{Z}$ and $1 \leq i \leq m+n-1$.

Theorem (Costa-Miró Roig 2005,2008; Malaspina-Miyazaki 2018)

Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^m \times \mathbb{P}^n$ with $\mathrm{H}^{p+q}(\mathcal{E}) \neq 0$, where $1 \leq p \leq m-1$ and $1 \leq q \leq n-1$.
If a vector bundle $\mathcal{E}$ has the following condition:

(a) $\mathrm{H}^i(\mathcal{E}(a,b)) = 0$ for $1 \leq i \leq p+q$, $0 \leq a \leq p$, $0 \leq b \leq q$ with $i+a+b = p+q+1$.

(b) $\mathrm{H}^i(\mathcal{E}(a,b)) = 0$ for $p+q \leq i \leq m+n-1$, $p-m \leq a \leq 0$, $q-n \leq b \leq 0$ with $i+a+b = p+q-1$,

then $\Omega^p_{\mathbb{P}^m} \boxtimes \Omega^q_{\mathbb{P}^n}$ is a direct summand of $\mathcal{E}$.

Multigraded Regularity, Syzygy Theoretic Method

Example

Let $\mathcal{E}$ be an indecomposable vector bundle on $\mathbb{P}^2 \times \mathbb{P}^2$. Then the following conditions are equivalent:

(a) $\mathcal{E} \cong \Omega_{\mathbb{P}^2} \boxtimes \Omega_{\mathbb{P}^2}$.

(b) $\mathrm{H}^2(\mathcal{E}) \neq 0$ and $\mathrm{H}^1(\mathcal{E}(1,1)) = \mathrm{H}^2(\mathcal{E}(0,1)) = \mathrm{H}^2(\mathcal{E}(1,0)) = \mathrm{H}^2(\mathcal{E}(-1,0)) = \mathrm{H}^2(\mathcal{E}(0,-1)) = \mathrm{H}^3(\mathcal{E}(-1,-1)) = 0$.

Example

(1) $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{O}_{\mathbb{P}^1}(4)$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is not Buchsbaum (but quasi-Buchsbaum). In this case $\mathrm{H}^1(\mathcal{E}(-2)) \neq 0$ and $\mathrm{H}^2(\mathcal{E}(-4)) \neq 0$.

(2) $\mathcal{E} = \Omega_{\mathbb{P}^2} \boxtimes \Omega_{\mathbb{P}^2}(3)$ on $\mathbb{P}^2 \times \mathbb{P}^2$ is Buchsbaum, but $\mathrm{H}^1(\mathcal{E}) \neq 0$ and $\mathrm{H}^3(\mathcal{E}(-3)) \neq 0$.