Spectral Sequence Theory for Generalized Cohen-Macaulay Graded Modules

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1 Introduction

This paper is devoted to the spectral sequence theory of graded modules. The theory has been developed in [5] and [6] to study the Buchsbaum property of Segre products. In this paper, we generalize our previous results through the r-standard property. The r-standard property, defined in Section 2, has an important role to investigate the spectral sequence corresponding to the graded modules. Also, our paper is written for the self-contained introduction of the spectral sequence theory of graded modules, including quasi-homogeneous case. Further, we give some applications for the standard s.o.p. and the Buchsbaum property, renewing the viewpoint of some important theorems concerning the quasi-Buchsbaum property (cf. (3.7)) and some cohomological criteria (cf. Section 4).

In Section 2, we review and extend [5], Theorem 1.8 through the r-standard property. The point is the construction of a map $\varphi^q_{x_r\wedge\ldots\wedge x_1}(M)$ for an $(r-1)$-standard s.s.o.p. $x_1,\ldots,x_r$ for M in (2.3).

In Section 3, we investigate the correspondence between the map φ and the spectral sequence associated to M which is introduced in [5] and [6]. This section is the essence of our spectral sequence theory, not only giving a transparent proof of [5,(1.9)], but also yielding some corollaries with the thorough study of the property of the map φ. Also, we give another proof of [10,(3.6)].

In Section 4, we give some cohomological criteria for the r-standard property as an application of the spectral sequence theory. Through the spectral sequence, we study effectively the behavior of the local cohomology to generalize some cohomological criteria. For example, Proposition 4.1 is a generalization of [9,(3.1)] and [5,(2.6)] (See [2,(5.2)]). Proposition 4.2 and 4.3 is a generalization of [11,(3.4)], [5,(1.14)] and [13,(2.1)].

Throughout this paper, we follow the notation and terminology of [3]. We say that

R is a graded ring over a field k, if $R = \oplus_{n\geq 0} R_n$, $R_0 = k$ and R is finitely generated over k, but we do not assume that R as a k-algebra is generated by R_1. We always write m for the unique homogeneous maximal ideal. We say that a finitely generated R-module M is a generalized Cohen-Macaulay graded R-module (or FLC graded R-module) if $\ell_R\left(H^i_{\mathsf{m}}(M)\right) < \infty$ for $i \neq \dim\ M$. We say that a sequence $x_1, \ldots, x_n$ of homogeneous elements of R is a s.s.o.p. if the sequence is a part of homogeneous system of parameters.

2 r-Standard s.s.o.p.

Let R be a graded ring over a field k. Let m be the homogeneous maximal ideal of R. Let M be a generalized Cohen-Macaulay graded R-module with $\dim M = m+1 (\geq 1)$.

Definition 2.1 *Let $x_1, \ldots, x_n$ be a s.s.o.p. for M. Put $\mathsf{q} = (x_1, \ldots, x_n)$. We say $x_1, \ldots, x_n$ is r-standard, if, for any choice $x_{i_1}, \ldots, x_{i_\ell} (\ell \leq r-1)$,*

$$\mathsf{q} H^j_{\mathsf{m}}(M/(x_{i_1}, \ldots, x_{i_\ell})M) = 0$$

for $j + \ell \leq m$.

An ideal $J \subseteq \mathsf{m}$ is called r-standard if every s.s.o.p. for M contained in J is r-standard.

Remark 2.2 *A s.o.p. $x_1, \ldots, x_{m+1}$ for M is $(m+1)$-standard if and only if the s.o.p. $x_1, \ldots, x_{m+1}$ is standard (cf.* [11]*). The maximal ideal m is r-standard if and only if M is $(1, r)$-Buchsbaum (cf.* [2],[4],[6]*).*

Let $y_1, \ldots, y_n$ be a s.s.o.p. for M, with $\deg y_j = e_j \geq 1 (j = 1, \ldots, n)$. Under the assumption $y_1, \ldots, y_n$ is $(r-1)$-standard, we want to define graded R-homomorphisms

$$\varphi^q_{y_r \wedge \cdots \wedge y_1}(M) : H^q_{\mathsf{m}}(M)[-e_1 - \cdots - e_r] \to H^{q-r+1}_{\mathsf{m}}(M)$$

for $r - 1 \leq q \leq m$.

First, we define $\varphi^q_{y_1}(M) : H^q_{\mathsf{m}}(M)[-e_1] \to H^q_{\mathsf{m}}(M)$ by $\varphi^q_{y_1}(M)(u) = y_1 u$ for $u \in H^q_{\mathsf{m}}(M)$.

Next, we assume $n \geq 2$ and $y_1, \ldots, y_n$ is 1-standard. Let us consider the exact sequence

$$0 \to [0 : y_1]_M[-e_1] \to M[-e_1] \stackrel{\cdot y_1}{\to} M \to M/y_1 M \to 0.$$

Since M is generalized Cohen-Macaulay, $H^q_{\mathsf{m}}([0 : y_1]_M) = 0$ for $q \geq 1$. So we have the short exact sequence

$$0 \to H^{q-1}_{\mathsf{m}}(M) \to H^{q-1}_{\mathsf{m}}(M/y_1 M) \to H^q_{\mathsf{m}}(M)[-e_1] \to 0$$

for $1 \le q \le m$. Thus we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & H_{\mathfrak{m}}^{q-1}(M)[-\ell] & \to & H_{\mathfrak{m}}^{q-1}(M/y_1M)[-\ell] & \xrightarrow{f} & H_{\mathfrak{m}}^{q}(M)[-\ell - e_1] & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H_{\mathfrak{m}}^{q-1}(M) & \xrightarrow{g} & H_{\mathfrak{m}}^{q-1}(M/y_1M) & \to & H_{\mathfrak{m}}^{q}(M)[-e_1] & \to & 0, \end{array}$$

where $\ell = e_2$ and the vertical arrows are $\varphi_{y_2}^{q-1}(M)$, $\varphi_{y_2}^{q-1}(M/y_1M)$ and $\varphi_{y_2}^{q}(M)[-e_1]$ from left. Since $y_1, \ldots, y_n$ is 1-standard, $\varphi_{y_2}^{q-1}(M)$ and $\varphi_{y_2}^{q}(M)$ are zero maps. Thus we get a graded R-homomorhism

$$\phi : H_{\mathfrak{m}}^{q}(M)[-e_1 - e_2] \to H_{\mathfrak{m}}^{q-1}(M)$$

for $1 \le q \le m$ such that $g \circ \phi \circ f = \varphi_{y_2}^{q-1}(M/y_1M)$. We define $\varphi_{y_2 \wedge y_1}^{q}(M) = \phi$. Note that $\varphi_{y_2}^{q-1}(M/y_1M) = 0$ is equivalent to saying $\varphi_{y_2 \wedge y_1}^{q}(M) = 0$. Therefore, $y_2, \ldots, y_n$ is 1-standard for M/y_1M if and only if $\varphi_{y_j \wedge y_1}^{q}(M)$ is a zero map for $j = 2, \ldots, n$ and $q \le m - 1$. Hence $y_1, \ldots, y_n$ is a 2-standard s.s.o.p. for M if and only if $\varphi_{y_j \wedge y_i}^{q}(M)$ is a zero map for $i \ne j$ and $1 \le q \le m$.

Now assume $n \ge r \ge 3$ and $y_1, \ldots, y_n$ is $(r-1)$-standard. Similarly we have the short exact sequence

$$0 \to H_{\mathfrak{m}}^{q-1}(M) \to H_{\mathfrak{m}}^{q-1}(M/y_1M) \to H_{\mathfrak{m}}^{q}(M)[-e_1] \to 0$$

for $1 \le q \le m$. Since $y_2, \ldots, y_n$ is also $(r-2)$-standard s.s.o.p. for M/y_1M, $\varphi_{y_r \wedge \cdots \wedge y_2}^{q-1}(M/y_1M)$ is also defined. Thus we have the following diagram with exact rows for $r - 1 \le q \le m$

$$\begin{array}{ccccccccc} 0 & \to & H_{\mathfrak{m}}^{q-1}(M)[-\ell] & \to & H_{\mathfrak{m}}^{q-1}(M/y_1M)[-\ell] & \xrightarrow{f} & H_{\mathfrak{m}}^{q}(M)[-\ell - e_1] & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H_{\mathfrak{m}}^{q-r+1}(M) & \xrightarrow{g} & H_{\mathfrak{m}}^{q-r+1}(M/y_1M) & \to & H_{\mathfrak{m}}^{q-r+2}(M)[-e_1] & \to & 0, \end{array}$$

where $\ell = e_2 + \cdots + e_r$ and the vertical arrows are $\varphi_{y_r \wedge \cdots \wedge y_2}^{q-1}(M)$, $\varphi_{y_r \wedge \cdots \wedge y_2}^{q-1}(M/y_1M)$ and $\varphi_{y_r \wedge \cdots \wedge y_1}^{q}(M)[-e_1]$ from left. Our inductive construction of φ implies the commutativity of the above diagram (c.f. [5,(1.7.3)]). Thus we define a graded R-homomorphism

$$\varphi_{y_r \wedge \cdots \wedge y_1}^{q}(M) : H_{\mathfrak{m}}^{q}(M)[-e_1 \cdots - e_r] \to H_{\mathfrak{m}}^{q-r+1}(M)$$

for $r - 1 \le q \le m$ such that

$$g \circ \varphi_{y_r \wedge \cdots \wedge y_1}^{q}(M) \circ f = \varphi_{y_r \wedge \cdots \wedge y_2}^{q-1}(M/y_1M).$$

Similarly, the sequence $y_1, \ldots, y_n$ is r-standard for M if and only if $\varphi_{y_r \wedge \cdots \wedge y_1}^{q}(M)$ is a zero map for any choice $y_{i_1}, \ldots, y_{i_r}$ and $r - 1 \le q \le m$.

Hence we have the following.

Theorem 2.3 *Let R be a graded ring over a field k. Let m be the homogeneous maximal ideal of R. Let M be a generalized Cohen-Macaulay graded R-module with* $\dim M = m + 1 (\geq 1)$. *Let $y_1, \ldots, y_n$ be a s.s.o.p. for M. Suppose that $n \geq r$ and $y_1, \ldots, y_n$ is $(r-1)$-standard, then*

$$\varphi^q_{y_{i_r} \wedge \cdots \wedge y_{i_1}}(M) : H^q_{\mathsf{m}}(M)[-e_{i_1} \cdots - e_{i_r}] \to H^{q-r+1}_{\mathsf{m}}(M)$$

is well-defined for any choice $y_{i_1}, \ldots, y_{i_r}$ and $r - 1 \leq q \leq m$.

Furthermore, $y_1, \ldots, y_n$ is r-standard if and only if $\varphi^q_{y_{i_r} \wedge \cdots \wedge y_{i_1}}(M)$ is a zero map for any choice $y_{i_1}, \ldots, y_{i_r}$ and $r - 1 \leq q \leq m$.

3 Spectral Sequence Theory

Let R be a graded ring over a field k. Then we can write $R = P/I$, where $P = k[X_0, \ldots, X_N]$ is a polynomial ring, graded by $\deg(X_j) \geq 1$ for $0 \leq j \leq N$, and I is a homogeneous ideal of R. Let m be the homogeneous maximal ideal of R. Let M be a generalized Cohen-Macaulay graded R-module with $\dim M = m + 1 (\geq 1)$. Let q be a homogeneous ideal of R. Let $y_1, \ldots, y_n$ be homogeneous generators of q with $\deg y_j = d_j \geq 1 (j = 1, \ldots, n)$.

Let $\mathbf{P} = \mathrm{Proj}\ P$ and $X = \mathrm{Proj}\ R$. Let $\mathcal{F} = \widetilde{M}$ and $\mathcal{F}(\ell) = \widetilde{M(\ell)}$ on $\mathbf{P}$ for all integers ℓ . Notice that $\mathcal{F}(\ell)$ is not necessarily isomorphic to $\mathcal{F} \otimes \mathcal{O}_{\mathbf{P}}(\ell)$ in quasi-homogeneous cases. We often write $\mathcal{F}$ for its pull-back $\iota^* \mathcal{F}$ on X, where $\iota : X \to \mathbf{P}$ is a closed immersion. Then we have an isomorphism

$$H^i_*(X, \mathcal{F}) \cong H^{i+1}_{\mathsf{m}}(M)$$

for $i \geq 1$ and an exact sequence

$$0 \to H^0_{\mathsf{m}}(M) \to M \to \Gamma_*(X, \mathcal{F}) \to H^1_{\mathsf{m}}(M) \to 0,$$

where $\Gamma_*(X, \mathcal{F}) = \bigoplus_{\ell \in \mathbf{Z}} \Gamma(X, \mathcal{F}(\ell))$ and $H^i_*(X, \mathcal{F}) = \bigoplus_{\ell \in \mathbf{Z}} H^i(X, \mathcal{F}(\ell))$. (cf. [12].)

We will construct a spectral sequence corresponding the graded R-module M. Let $\mathcal{U} = \{U_\lambda\}$ be a finite affine open covering of X (or $\mathbf{P}$). Let $C^\bullet$ be the Čech complex $\bigoplus_{\ell \in \mathbf{Z}} C^\bullet(\mathcal{U}; \mathcal{F}(\ell))$. Then we put a complex $L^\bullet = (0 \to M \xrightarrow{\varepsilon} C^\bullet[-1])$, where $L^0 = M$, $L^i = C^{i-1}$ for $i \neq 0$ and ε is the natural map. Note that $H^i(L^\bullet)$ is isomorphic to $H^i_{\mathsf{m}}(M)$ as graded R-modules for every i. Let $K_\bullet$ be the Koszul complex $K_\bullet((y_1, \ldots, y_n); R)$. Then we consider the double complex $B^{\bullet\bullet} = \mathrm{Hom}_R(K_\bullet, L^\bullet)$. We write $B^{p,q} = \mathrm{Hom}_R(K_p, L^q)$ and we write its differentials as $d'^{p,q} : B^{p,q} \to B^{p+1,q}$ and $d''^{p,q} : B^{p,q} \to B^{p,q+1}$. When we emphasize M, we sometimes write $B^{\bullet\bullet}(M)$, $B^{p,q}(M)$, $d'^{p,q}(M)$ and $d''^{p,q}(M)$. Now let us take the first filtration $'F_t(B^{\bullet\bullet}) = \sum_{p \geq t} B^{p,q}$ and the second filtration $''F_t(B^{\bullet\bullet}) = \sum_{q \geq t} B^{p,q}$. Then the filtrations $'F_t$ and $''F_t$ give spectral sequences $\{'F^{p,q}_r\}$ and $\{''F^{p,q}_r\}$ respectively:

$$\left\{ \begin{array}{lll} 'F^{p,q}_1 = \mathrm{Ker}\ d''^{p,q} / \mathrm{Im}\ d''^{p,q-1} & \Rightarrow & \\ & & H^{p+q}(B^{\bullet\bullet}). \\ ''F^{p,q}_1 = \mathrm{Ker}\ d'^{p,q} / \mathrm{Im}\ d'^{p-1,q} & \Rightarrow & \end{array} \right.$$

Let $\{\mathbf{e}_1^*, \ldots, \mathbf{e}_n^*\}$ be the dual basis of $K_1((y_1, \ldots, y_n); R)$. Since

$$B^{p,q} \cong \begin{cases} \left(\bigoplus_{\ell \in \mathbf{Z}} C^{q-1}(\mathcal{U}; \mathcal{F}(\ell))\right) \otimes_R \bigwedge^p \left(\bigoplus_{i=1}^n R[d_i]\mathbf{e}_i^*\right) & q \neq 0 \\ M \otimes_R \bigwedge^p \left(\bigoplus_{i=1}^n R[d_i]\mathbf{e}_i^*\right) & q = 0, \end{cases}$$

we have

$${}'F_1^{p,q} \cong H^q_{\mathfrak{m}}(M) \otimes_R \bigwedge^p \left(\bigoplus_{i=1}^n R[d_i]\mathbf{e}_i^* \right).$$

If we assume the sequence $y_1, \ldots, y_n$ is a s.s.o.p. for M, then we have

$${}''F_1^{p,q} \cong \begin{cases} H^p((y_1, \ldots, y_n); M) & q = 0 \\ \left(\bigoplus_{\ell \in \mathbf{Z}} C^{q-1}\left(\mathcal{U}; \overline{\mathcal{F}}(\ell)\right)\right) \otimes_R \left(R\left[\sum_{i=1}^n d_i\right](\mathbf{e}_1^* \wedge \cdots \wedge \mathbf{e}_n^*)\right) & p = n, q \neq 0 \\ 0 & p \neq n, q \neq 0, \end{cases}$$

where $\overline{\mathcal{F}} = \mathcal{F}/(y_1, \ldots, y_n)\mathcal{F}$. Accordingly, we have

$${}''F_2^{p,q} \cong \begin{cases} H^p((y_1, \ldots, y_n); M) & p \neq n, q = 0 \\ H^q_{\mathfrak{m}}(M/(y_1, \ldots, y_n)M)[d_1 + \cdots + d_n] & p = n \\ 0 & p \neq n, q \neq 0. \end{cases}$$

Thus we have

$$H^{p+q}(B^{\bullet\bullet}) \cong \begin{cases} H^{p+q}((y_1, \ldots, y_n); M) & p + q < n \\ H^{p+q-n}_{\mathfrak{m}}(M/(y_1, \ldots, y_n)M)[d_1 + \cdots + d_n] & p + q \geq n, \end{cases}$$

if $y_1, \ldots, y_n$ is a s.s.o.p. for M. On the other hand, if we assume that $\mathfrak{q}$ is $\mathfrak{m}$-primary,

$$H^{p+q}(B^{\bullet\bullet}) \cong H^{p+q}((y_1, \ldots, y_n); M).$$

Now we simply write $F_r^{p,q}$ or $F_r^{p,q}(M)$ for ${}'F_r^{p,q}$. We see the spectral sequence $\{F_r^{p,q}\}$ converges to $H^{p+q}(B^{\bullet\bullet})$.

By the way, let us consider the double complex $C^{\bullet\bullet} = \mathrm{Hom}_R(K_\bullet, C^\bullet)$ with the first filtration and its spectral sequence $\{E_r^{p,q}\}$. Then we see

$$E_1^{p,q} \cong H^q_*(X, \mathcal{F}) \otimes_R \bigwedge^p \left(\bigoplus_{i=1}^n R[d_i]\mathbf{e}_i^* \right)$$

and $\{E_r^{p,q}\}$ converges to $H^{p+q}(C^{\bullet\bullet})$. In [6], we studied the spectral sequence $\{E_r^{p,q}\}$, but not much different from $\{F_r^{p,q}\}$.

Now let us characterize the r-standard property through the behavior of the spectral sequence $\{F_r^{p,q}\}$ (or $\{E_r^{p,q}\}$). Note that the spectral sequence does not depend on the choice of minimal generators $y_1, \ldots, y_n$ of $\mathfrak{q}$.

Theorem 3.1 *Let R be a graded ring over a field k. Let m be the homogeneous maximal ideal of R. Let M be a generalized Cohen-Macaulay graded R-module with* $\dim M = m + 1 (\geq 1)$. *Let r and n be integers with $r \leq n$. Let q be a homogeneous ideal which is generated by $y_1, \dots, y_n$ with $\deg y_j = d_j \geq 1 (j = 1, \dots, n)$. Assume that, for any $1 \leq i_1 < \cdots < i_r \leq n$, the sequence $y_{i_1}, \dots, y_{i_r}$ is an $(r-1)$-standard s.s.o.p. for M. Then we have*

(1) *There is an isomorphism*

$$F_r^{p,q} \cong H_{\mathsf{m}}^q(M) \otimes_R \bigwedge^p \left(\bigoplus_{i=1}^n R[d_i] \mathbf{e}_i^* \right) \quad \text{for } q \neq m+1.$$

(2) *The spectral sequence map $d_r^{p,q} : F_r^{p,q} \to F_r^{p+r,p-r+1}$ can be described through the isomorphism in (1) as*

$$\begin{aligned} & d_r^{p,q} \left(u \otimes \left(\mathbf{e}_{j_1}^* \wedge \cdots \wedge \mathbf{e}_{j_p}^* \right) \right) \\ & \quad = \sum_{1 \leq i_1 < \cdots < i_r \leq n} \varphi^q_{y_{i_r} \wedge \cdots \wedge y_{i_1}}(u) \otimes \left(\left(\mathbf{e}_{i_r}^* \wedge \cdots \wedge \mathbf{e}_{i_1}^* \right) \bigwedge \left(\mathbf{e}_{j_1}^* \wedge \cdots \wedge \mathbf{e}_{j_p}^* \right) \right) \end{aligned}$$

for $u \in H_{\mathsf{m}}^q(M)$.

Before the proof of (3.1), we state and prove some remarks and corollaries of Theorem 3.1.

Remark 3.2 *By the construction of spectral sequence, we can weaken the hypothesis of (3.1) as follows:*

Let s be an integer with $r \leq s \leq n$. Let q be a homogeneous ideal which is generated by $y_1, \dots, y_s$ with $\deg y_j = d_j \geq 1 (j = 1, \dots, s)$. Assume that, for any $1 \leq i_1 < \cdots < i_r \leq s$, the sequence $y_{i_1}, \dots, y_{i_r}$ is an $(r-1)$-standard s.s.o.p. for M. Let $y_j = 0$ with $\deg y_j = d_j$ for $s+1 \leq j \leq n$. We define $\varphi^q_{y_{i_r} \wedge \cdots \wedge y_{i_1}}(M) = 0$ if $y_{i_\ell} = 0$ for some i_ℓ. Then, in this case, we also have (3.1.1) and (3.1.2).

Corollary 3.3 *Under the assumptions of (3.1), the following conditions are equivalent.*

(a) *Every s.s.o.p. $y_{i_1}, \dots, y_{i_r} (1 \leq i_1 < \cdots < i_r \leq n)$ is r-standard.*

(b) q *is an r-standard ideal.*

(c) $d_r^{p,q} : F_r^{p,q} \to F_r^{p+r,q-r+1}$ *is a zero map for all p and $q (\neq m+1)$.*

(d) *For some fixed integer p with $0 \leq p \leq n-r$, $d_r^{p,q} : F_r^{p,q} \to F_r^{p+r,q-r+1}$ is a zero map for all $q (\neq m+1)$.*

Proof. The equivalence of (a), (c) and (d) follows immediately from (2.3) and (3.1). Clearly (b) implies (a). So we have only to prove the statement (b) under the assumptions (a), (c) and (d). Let $x_1, \ldots, x_r$ be homogeneous elements of $\mathfrak{q}$ with $\deg x_j = e_j$ such that the sequence $x_1, \ldots, x_r$ is a s.s.o.p. for M. We want to show that $x_1, \ldots, x_r$ is r-standard by the induction on r. The case r=1 is trivial. So we may assume that $x_1, \ldots, x_r$ is an $(r-1)$-standard s.s.o.p. from the hypothesis of induction. We put $x_i = 0$ for $i = r+1, \ldots, n$. Since $\mathfrak{q}$ is generated by $y_1, \ldots, y_n$, we can write

$$x_j = \sum_{i=1}^{n} a_{ji} y_i \text{ for } 1 \leq j \leq n,$$

where a_{ji} is a homogeneous element for $1 \leq i \leq n$ and $1 \leq j \leq n$ and $a_{ji} = 0$ for $1 \leq i \leq n$ and $r+1 \leq j \leq n$. Then we construct the spectral sequence $\{G_r^{p,q}, \overline{d}_r^{p,q}\}$ through the Koszul complex $K_\bullet((x_1, \ldots, x_n); R)$. We write the dual basis of $K_1((x_1, \ldots, x_n); R)$ as $\{\mathbf{f}_1^*, \ldots, \mathbf{f}_n^*\}$. Then we have

$$\psi : K^\bullet((y_1, \ldots, y_n); R) \to K^\bullet((x_1, \ldots, x_n); R) \text{ by } \psi(\mathbf{e}_i^*) = \sum_{j=1}^{n} a_{ij} \mathbf{f}_j^*$$

Thus we have $\psi_r^{p,q} : F_r^{p,q} \to G_r^{p,q}$ induced by ψ satisfying that $\overline{d}_r^{p,q} \circ \psi_r^{p,q} = \psi_r^{p+r,q-r+1} \circ d_r^{p,q}$. Through the isomorphism (3.1.1), there is a commutative diagram

$$\begin{array}{ccc}
H^q_{\mathfrak{m}}(M) & \xrightarrow{d_r^{0,q}} & H^{q-r+1}_{\mathfrak{m}}(M) \otimes_R \wedge^r \left(\sum_{i=1}^n R[d_i]\mathbf{e}_i^*\right) \\
\| & & \downarrow \psi_r^{r,q-r+1} \\
H^q_{\mathfrak{m}}(M) & \xrightarrow{\overline{d}_r^{0,q}} & H^{q-r+1}_{\mathfrak{m}}(M) \otimes_R \wedge^r \left(\sum_{i=1}^n R[e_i]\mathbf{f}_i^*\right)
\end{array} .$$

Thus we have

$$\begin{aligned}
\overline{d}_r^{0,q}(u) &= \psi_r^{r,q-r+1}(d_r^{0,q}(u)) \\
&= \psi_r^{r,q-r+1}\left(\sum_{1 \leq i_1 < \cdots < i_r \leq n} \varphi^q_{y_{i_r} \wedge \cdots \wedge y_{i_1}}(M)(u) \otimes \left(\mathbf{e}_{i_1}^* \wedge \cdots \wedge \mathbf{e}_{i_r}^*\right)\right) \\
&= \sum_{1 \leq i_1 < \cdots < i_r \leq n} \varphi^q_{y_{i_r} \wedge \cdots \wedge y_{i_1}}(M)(u) \otimes \left(\left(\sum_{j=1}^n a_{i_1 j} \mathbf{f}_j^*\right) \wedge \cdots \wedge \left(\sum_{j=1}^n a_{i_r j} \mathbf{f}_j^*\right)\right) \\
&= \sum_{1 \leq j_1 < \cdots < j_r \leq n} \sum_{1 \leq i_1 < \cdots < i_r \leq n} \begin{vmatrix} a_{i_1 j_1} & \cdots & a_{i_1 j_r} \\ \vdots & & \vdots \\ a_{i_r j_1} & \cdots & a_{i_r j_r} \end{vmatrix} \varphi^q_{y_{i_r} \wedge \cdots \wedge y_{i_1}}(M)(u) \otimes \left(\mathbf{f}_{j_1}^* \wedge \cdots \wedge \mathbf{f}_{j_r}^*\right)
\end{aligned}$$

for $u \in H^q_{\mathfrak{m}}(M)$. Hence we have

$$\varphi^q_{x_r \wedge \cdots \wedge x_1}(M) = \sum_{1 \leq i_1 < \cdots < i_r \leq n} \begin{vmatrix} a_{i_1 1} & \cdots & a_{i_r 1} \\ \vdots & & \vdots \\ a_{i_1 r} & \cdots & a_{i_r r} \end{vmatrix} \varphi^q_{y_{i_r} \wedge \cdots \wedge y_{i_1}}(M) . \tag{3.3.1}$$

Therefore, the statements (a), (c) and (d) implies (b) by (2.3) and (3.3.1).

Corollary 3.4 *Let R be a graded ring over a field k. Let M be a generalized Cohen-Macaulay graded R-module with* $\dim M = m + 1 (\geq 1)$. *Let $y_1, \ldots, y_r$ be an $(r-1)$-standard s.s.o.p. Then, $\varphi^q_{y_r \wedge \cdots \wedge y_1}(M)$ is skew-symmetric R-multilinear on $y_1, \ldots, y_r$ as follows:*

(1) *For $q \neq m+1$ and $1 \leq i < j \leq r$,*

$$\varphi^q_{y_r \wedge \cdots \wedge y_1}(M) = -\varphi^q_{y_r \wedge \cdots \wedge y_{j-1} \wedge y_i \wedge y_{j+1} \wedge \cdots \wedge y_{i-1} \wedge y_j \wedge y_{i+1} \wedge \cdots \wedge y_1}(M).$$

(2) *If x is a homogeneous element with* $\deg x = \deg y_1$ *such that $x, y_2, \ldots, y_r$ is an $(r-1)$-standard s.s.o.p. for M and $x + y_1, y_2, \ldots, y_r$ is a s.s.o.p., then $x + y_1, y_2, \ldots, y_r$ is $(r-1)$-standard and*

$$\varphi^q_{y_r \wedge \cdots \wedge y_2 \wedge (y_1 + x)}(M) = \varphi^q_{y_r \wedge \cdots \wedge y_2 \wedge y_1}(M) + \varphi^q_{y_r \wedge \cdots \wedge y_2 \wedge x}(M).$$

(3) *If z is a homogeneous element such that $zy_1, y_2, \ldots, y_r$ is a s.s.o.p. for M, then $zy_1, y_2, \ldots, y_r$ is $(r-1)$-standard and*

$$\varphi^q_{y_r \wedge \cdots \wedge y_2 \wedge zy_1}(M) = z \cdot \varphi^q_{y_r \wedge \cdots \wedge y_2 \wedge y_1}(M)$$

Proof. It follows immediately from (3.1), (3.3) and (3.3.1).

The following is an easy consequence of Corollary 3.4. We can show by the R-multilinearlity of the map φ. This gives another proof of [10,(3.6)].

Remark 3.5 *Let R be a graded ring over a field k. Let M be a generalized Cohen-Macaulay graded R-module. Let q be a 1-standard ideal for M. Let $x \in \mathsf{q}^2$ be a parameter for M. Then q is also a 1-standard ideal for M/xM.*

Now let us prove Theorem 3.1.

Proof of Theorem 3.1. We will prove by induction on r. Note that the hypothesis of induction is valid even for the results of (3.2), (3.3) and (3.4). The case $r = 1$ is trivial. Assume $r > 1$. The statement (1) follows immediately from the hypothesis of induction. So we have only to prove (2).

Put $\mathbf{e}^*_J = \mathbf{e}^*_{j_1} \wedge \cdots \wedge \mathbf{e}^*_{j_p}$ and $\bar{M} = M/y_{i_1}M$. Then we set

$$L^\bullet = \left(0 \to M \to \bigoplus_{\ell \in \mathbf{Z}} C^\bullet(\mathcal{U}; \widetilde{M}(\ell)) \right),$$

$$\bar{L}^\bullet = \left(0 \to \bar{M} \to \bigoplus_{\ell \in \mathbf{Z}} C^\bullet(\mathcal{U}; \widetilde{\bar{M}}(\ell))\right)$$

and

$$K_\bullet = K_\bullet((y_1, \ldots, y_n); R).$$

Let $B^{\bullet\bullet}(M)$ and $B^{\bullet\bullet}(\bar{M})$ be the double complexes $\mathrm{Hom}_R(K_\bullet, L^\bullet)$ and $\mathrm{Hom}_R(K_\bullet, \bar{L}^\bullet)$ respectively. Then we have an exact sequence

$$0 \to [0 : y_{i_1}]_M[-d_{i_1}] \to M[-d_{i_1}] \overset{\cdot y_{i_1}}{\to} M \to M/y_{i_1}M \to 0.$$

Thus we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & B^{p,q-1}(M) & \to & B^{p,q-1}(M) & \overset{\alpha}{\to} & B^{p,q-1}(\bar{M}) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B^{p,q}(M) & \overset{\beta}{\to} & B^{p,q}(M) & \to & B^{p,q}(\bar{M}) & \to & 0 \end{array}$$

for $1 \le q \le m$, where the vertical arrows are $d''^{p,q-1}$'s. On the other hand, by the proof of (2.3), we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & H^{q-1}_{\mathfrak{m}}(M)[-\ell] & \to & H^{q-1}_{\mathfrak{m}}(\bar{M})[-\ell] & \overset{f}{\to} & H^q_{\mathfrak{m}}(M)[-\ell - d_{i_1}] & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H^{q-r+1}_{\mathfrak{m}}(M) & \overset{g}{\to} & H^{q-r+1}_{\mathfrak{m}}(\bar{M}) & \to & H^{q-r+2}_{\mathfrak{m}}(M)[-d_{i_1}] & \to & 0 \end{array}$$

for $r-1 \le q \le m$, where $\ell = d_{i_2} + \cdots + d_{i_r}$ and the vertical arrows are $\varphi^{q-1}_{y_{i_r}\wedge\cdots\wedge y_{i_2}}(M)$, $\varphi^{q-1}_{y_{i_r}\wedge\cdots\wedge y_{i_2}}(\bar{M})$ and $\varphi^{q}_{y_{i_r}\wedge\cdots\wedge y_{i_1}}(M)[-d_{i_1}]$ from left. Also, the graded R-homomorphism

$$\varphi^q_{y_{i_r}\wedge\cdots\wedge y_{i_1}}(M) : H^q_{\mathfrak{m}}(M)[-d_{i_1} \cdots - d_{i_r}] \to H^{q-r+1}_{\mathfrak{m}}(M)$$

satisfies

$$g \circ \varphi^q_{y_{i_r}\wedge\cdots\wedge y_{i_1}}(M) \circ f = \varphi^{q-1}_{y_{i_r}\wedge\cdots\wedge y_{i_2}}(\bar{M})$$

for $r - 1 \le q \le m$.

Let u be an element of $H^q_{\mathfrak{m}}(M)$. Through the isomorphism $H^q_{\mathfrak{m}}(M) \cong H^q(L^\bullet)$, we take $\tilde{u} \in L^q$ such that $\tilde{u}(\mathrm{mod}\ I^q(L^\bullet)) = u$, where $I^q(L^\bullet) = \mathrm{Im}(L^{q-1} \to L^q)$. Then $\tilde{u} \otimes \mathbf{e}^*_J$ is an element of $B^{p,q}(M)$ and $u \otimes \mathbf{e}^*_J$ is an element of $F^{p,q}_r(M)$. What we have to do is to describe $\varphi_{y_{i_r}\wedge\cdots\wedge y_{i_1}}(M)(u)$ and $d^{p,q}_r(M)(u \otimes \mathbf{e}^*_J)$.

First, we take $v \in H^{q-1}_{\mathfrak{m}}(\bar{M})$ such that $f(v) = u$. On the other hand, we can take $\widetilde{w} \in L^{q-1}$ such that $\beta(\tilde{u} \otimes \mathbf{e}^*_J) = d''^{p,q-1}(M)(\widetilde{w} \otimes \mathbf{e}^*_J)$. We put $\alpha(\widetilde{w} \otimes \mathbf{e}^*_J) = \tilde{v} \otimes \mathbf{e}^*_J$ in $B^{p,q-1}(\bar{M})$, where $\tilde{v} \in \bar{L}^{q-1}$. Then we see $\tilde{v}(\mathrm{mod}\ I^{q-1}(L^\bullet)) = v$ in $H^{q-1}_{\mathfrak{m}}(\bar{M})$ by the construction of the map f.

Next, let us investigate $d^{p,q}_r(M)(u \otimes \mathbf{e}^*_J)$ through the double complex $B^{\bullet\bullet}$. Since $y_1, \ldots, y_n$ is 1-standard, there are elements $\widetilde{w}_\ell \in L^{q-1} (1 \le \ell \le n)$ satisfying

$$\begin{aligned} d'(\tilde{u} \otimes \mathbf{e}^*_J) &= \sum_{1 \le \ell \le n} (y_\ell \tilde{u}) \otimes (\mathbf{e}^*_\ell \wedge \mathbf{e}^*_J) \\ &= d''\left(\sum_{1 \le \ell \le n} \widetilde{w}_\ell \otimes (\mathbf{e}^*_\ell \wedge \mathbf{e}^*_J)\right) \end{aligned}$$

In particular, putting $\widetilde{v}_\ell = \widetilde{w}_\ell (\text{mod } y_{i_1} L^{q-1})$ in $\bar{L}^{q-1}$ for $1 \le \ell \le s$, we see that we may take $\widetilde{w} = \widetilde{w}_{i_1}$ and $\widetilde{v} = \widetilde{v}_{i_1}$ from the beginning. Now we have

$$d'(M)\left(\sum_{1\le \ell \le n} \widetilde{w}_\ell \otimes (\mathbf{e}^*_\ell \wedge \mathbf{e}^*_J)\right) = \sum_{1\le \ell < k \le n} (y_k \widetilde{w}_\ell - y_\ell \widetilde{w}_k) \otimes (\mathbf{e}^*_k \wedge \mathbf{e}^*_\ell \wedge \mathbf{e}^*_J).$$

The $\mathbf{e}^*_k \wedge \mathbf{e}^*_{i_1} \wedge \mathbf{e}^*_J$-component of $d'(M)\left(\sum_{1\le\ell\le n} \widetilde{w}_\ell \otimes (\mathbf{e}^*_\ell \wedge \mathbf{e}^*_J)\right)$ equals to $y_k \widetilde{w} - y_{i_1}\widetilde{w}_k$. Note that

$$y_k \widetilde{w} - y_{i_1} \widetilde{w}_k (\text{mod} y_{i_1} L^{q-1}) = y_k \widetilde{v}.$$

On the other hand, the $\mathbf{e}^*_k \wedge \mathbf{e}^*_{i_1} \wedge \mathbf{e}^*_J$-component of $d'(\bar{M})(\widetilde{v} \otimes (\mathbf{e}^*_{i_1} \wedge \mathbf{e}^*_J))$ equals to $y_k\widetilde{v}$. Thus we have the $(\mathbf{e}^*_{i_r} \wedge \cdots \wedge \mathbf{e}^*_{i_1} \wedge \mathbf{e}^*_J)$-component of $d^{p-1,q-1}_{r-1}(\bar{M})(v \otimes (\mathbf{e}_{i_1} \wedge \mathbf{e}^*_J))$ equals to the $(\mathbf{e}^*_{i_r} \wedge \cdots \wedge \mathbf{e}^*_{i_1} \wedge \mathbf{e}^*_J)$-component of $d^{p,q}_r(M)(u \otimes \mathbf{e}^*_J)$ modulo $y_{i_1} L$ from the construction of spectral sequence. By the hypothesis of induction, we see

$$\begin{aligned} & d^{p-1,q-1}_{r-1}(\bar{M})(v \otimes (\mathbf{e}_{i_1} \wedge \mathbf{e}^*_J)) \\ & \quad = \sum_{1 \le \ell_1 < \cdots < \ell_{r-1} \le n} \varphi^{q-1}_{y_{\ell_{r-1}} \wedge \cdots \wedge y_{\ell_1}}(\bar{M})(v) \otimes (\mathbf{e}^*_{\ell_{r-1}} \wedge \cdots \wedge \mathbf{e}^*_{\ell_1} \wedge \mathbf{e}^*_{i_1} \wedge \mathbf{e}^*_J) \end{aligned}$$

In particular, $\varphi^{q-1}_{y_{i_r} \wedge \cdots \wedge y_{i_2}}(\bar{M})(v)$ equals to the $(\mathbf{e}^*_{i_r} \wedge \cdots \wedge \mathbf{e}^*_{i_1} \wedge \mathbf{e}^*_J)$-component of $d^{p-1,q-1}_{r-1}(\bar{M})(v \otimes (\mathbf{e}_{i_1} \wedge \mathbf{e}^*_J))$. ¿From the injectivity of g, we have the $(\mathbf{e}^*_{i_r} \wedge \cdots \wedge \mathbf{e}^*_{i_1} \wedge \mathbf{e}^*_J)$-component of $d^{p,q}_r(M)(u \otimes \mathbf{e}^*_J)$ equals to $\varphi^q_{y_{i_r} \wedge \cdots \wedge y_{i_2}}(\bar{M})(v)$, and thereby equals to $\varphi^q_{y_{i_r} \wedge \cdots \wedge y_{i_1}}(M)(u)$. Hence the assertion is proved.

Remark 3.6 *Let r and n be integers with $r \le n$. Let $\mathfrak{q}$ be an $(r-1)$-standard ideal which is generated by $y_1, \ldots, y_n$ with* $\deg y_j = d_j \ge 1 (j = 1, \ldots, n)$. *Assume that for any $1 \le i_1 < \cdots < i_r \le n$, the sequence $y_{i_1}, \ldots, y_{i_r}$ is a s.s.o.p. for M. Let $x_1, \ldots, x_r$ be homogeneous elements of $\mathfrak{q}$. By (3.1), (3.2), (3.3) and (3.4), we define $\varphi^q_{x_r \wedge \cdots \wedge x_1}(M)$ through $d^{0,q}_r : F^{0,q}_r \to F^{r,q-r+1}_r$. ¿From (3.3.1), this definition does not depend on the choice of generators $y_1, \ldots, y_n$ of $\mathfrak{q}$. Further, $\varphi^q_{x_r \wedge \cdots \wedge x_1}(M)$ is skew-symmetric R-multilinear on $x_1, \ldots, x_r$.*

Remark 3.7 *By virtue of (3.1), we can see the r-standard property through the canonical dual. Let $M^\vee$ be the canonical dual module* $\mathrm{Ext}^t_R(M, K_R)$*, where K_R is the canonical module of the graded ring R and $t = \dim R - \dim M$. Assume that the sequence $y_1, \ldots, y_n$ is an r-standard s.s.o.p. for M. Then the sequence $y_1, \ldots, y_n$ also has the r-standard property for $M^\vee$. The proof is the same as in [6].*

4 Cohomological Criteria

Let R be a graded ring over a field k. Let $\mathfrak{m}$ be the homogeneous maximal ideal of R. Let M be a generalized Cohen-Macaulay graded R-module with $\dim M = m+1 \ge 1$.

Let $\mathfrak{q} = (x_1, \ldots, x_n)$ be a homogeneous ideal with $\deg x_j = e_j (j = 1, \ldots, n)$ such that every sequence $x_{i_1}, \ldots, x_{i_r} (1 \leq i_1 < \cdots < i_r \leq n)$ is a s.s.o.p. for M. Under the above conditions, we will investigate some cohomological criteria for the r-standard property.

Proposition 4.1 *Let us define*

$$\mathcal{S}(M) = \{(i, \ell) | [H^i_{\mathfrak{m}}(M)]_\ell \neq 0, 0 \leq i \leq m\}.$$

If S(M) satisfies the following conditions:

(a) *For any (j, ℓ_1) and (k, ℓ_2) with $j \geq k$ in $\mathcal{S}(M)$,*

$$\ell_2 - \ell_1 \neq \sum_{h=1}^{j-k+1} e_{i_h} \textit{ for every } 1 \leq i_1 < \cdots < i_{j-k+1} \leq n.$$

Then the ideal $\mathfrak{q}$ is r-standard.

Proof. By Theorem 2.3, we have only to show that, for any $s \leq r$,

$$\varphi^q_{y_{i_s} \wedge \cdots \wedge y_{i_1}}(M) : H^q_{\mathfrak{m}}(M)[-e_{i_1} \cdots - e_{i_s}] \to H^{q-s+1}_{\mathfrak{m}}(M)$$

is a zero map for every $1 \leq i_1 < \cdots < i_s \leq n$. If u is a non-zero homogeneous element of $[H^q_{\mathfrak{m}}(M)[\sum_{h=1}^s (-e_{i_h})]]_\ell$, then $\varphi^q_{y_{i_s} \wedge \cdots \wedge y_{i_1}}(M)(u) = 0$. In fact, u is an element of $H^q_{\mathfrak{m}}(M)$ with $\deg u = \ell - \sum_{h=1}^s e_{i_h}$, and $w = \varphi^q_{y_{i_s} \wedge \cdots \wedge y_{i_1}}(M)(u)$ is an element of $H^{q-s+1}_{\mathfrak{m}}(M)$ with $\deg w = \ell$. Put $j = q$, $k = q - s + 1$, $\ell_1 = \ell - \sum_{h=1}^s e_{i_h}$ and $\ell_2 = \ell$. Then we have $j \geq k$, $j - k + 1 = s$, $\ell_2 - \ell_1 = \sum_{h=1}^s (e_{i_h})$ and $(j, \ell_1) \in \mathcal{S}(M)$. Since $\mathcal{S}(M)$ satisfies the condition (a), we see (k, ℓ_2) is not in $\mathcal{S}(M)$. So we have $[H^{q-s+1}_{\mathfrak{m}}(M)]_\ell = 0$. Thus we have $w = 0$. Hence the assertion is proved.

Proposition 4.2 *Assume that $\mathfrak{q}$ is an $\mathfrak{m}$-primary ideal. Then the following conditions* (a)–(d) *are equivalent. If $\mathfrak{q}$ is r-standard, then the conditions* (a)–(d) *hold.*

(a) *The natural map*

$$H^i(\mathfrak{q}; M) \to H^i_{\mathfrak{m}}(M)$$

is surjective for $0 \leq i \leq r - 1$.

(b)

$$\ell_R\left(H^i(\mathfrak{q}; M)\right) = \sum_{j=0}^{i} \binom{n}{j} \ell_R\left(H^{i-j}_{\mathfrak{m}}(M)\right) \text{ for } 0 \leq i \leq r - 1.$$

(c)

$$\ell_R\left(H^{r-1}(\mathfrak{q}; M)\right) = \sum_{j=0}^{r-1} \binom{n}{j} \ell_R\left(H^{r-1-j}_{\mathfrak{m}}(M)\right).$$

(d) $\varphi^q_{y_{i_\ell}\wedge\cdots\wedge y_{i_1}}(M)$ *is a zero map for all* $1 \le i_1 < \cdots < i_\ell \le n$ *with* $\ell \le q \le r$.

Further, in case r=m+1, the converse is true.

Proof. If q is r-standard, then we have (d) by (2.3). Now we will show the equivalence (a)–(d). As we see in Section 3, there are isomorphisms

$$H^i(B^{\bullet\bullet}) \cong H^i(\mathsf{q}; M) \text{ and } F_1^{0,i}(M) \cong H^i_{\mathsf{m}}(M)$$

Thus $H^i(\mathsf{q}; M) \to H^i_{\mathsf{m}}(M)$ is surjective if and only if $d_\ell^{0,i} : F_\ell^{0,i} \to F_\ell^{\ell,i-\ell+1}$ is a zero map for $i \le m$ and $\ell \ge 1$. By (2.3), the statements (a) and (d) are equivalent. Next we assume (d). Then we see that $d_\ell^{j,i} : F_\ell^{j,i} \to F_\ell^{j+\ell,i-\ell+1}$ is a zero map for $i \le m$ and $\ell \ge 1$. So we have

$$F_\infty^{j,i} \cong H^i_{\mathsf{m}}(M) \otimes_R \bigwedge^i \left(\bigoplus_{k=1}^n R[e_k]\mathbf{e}_k^* \right) \text{ for } i \ne m+1.$$

Thus we have

$$\begin{aligned} \ell_R\left(H^i(\mathsf{q}; M)\right) &= \textstyle\sum_{j=0}^i \ell_R\left(F_\infty^{i-j,j}\right) \\ &= \textstyle\sum_{j=0}^i \begin{pmatrix} n \\ j \end{pmatrix} \ell_R\left(H^{i-j}_{\mathsf{m}}(M)\right) \quad \text{for } 0 \le i \le r-1. \end{aligned}$$

Thus we have (d) implies (c). Clearly (b) implies (c). Finally, if we assume (c), then $d_\ell^{j,r-1-j} : F_\ell^{j,r-1-j} \to F_\ell^{j+\ell,i-\ell}$ is a zero map for every j and $\ell(\ge 1)$. By (3.1), we have the statement (d).

In case $r = m+1$, similarly, the converse follows immediatelly from (3.1).

Proposition 4.3 *Assume that* $y_1, \ldots, y_n$ *is a s.s.o.p. for* M. *If* $y_1, \ldots, y_n$ *is r-standard, then the equivalent conditions* (a), (b), (c) *and* (d) *in (4.2) hold.*

Further, we assume n=r. If $y_1, \ldots, y_r$ *is r-standard, then we have*

$$\ell_R\left(H^i_{\mathsf{m}}(M/\mathsf{q}M)\right) = \sum_{j=0}^r \begin{pmatrix} r \\ j \end{pmatrix} \ell_R\left(H^{i+j}_{\mathsf{m}}(M)\right) \text{ for } 0 \le i \le m-r.$$

Conversely, if (a)–(d) *in (4.2) and the above equality hold, then* $y_1, \ldots, y_r$ *is r-standard.*

Proof. As we see in Section 3, there are isomorphisms

$$H^{p+q}(B^{\bullet\bullet}) \cong \begin{cases} H^{p+q}((y_1, \ldots, y_n); M) & p+q < n \\ \\ H^{p+q-n}_{\mathsf{m}}(M/(y_1, \ldots, y_n)M)[d_1 + \cdots + d_n] & p+q \ge n \end{cases}$$

and

$$'F_1^{p,q} \cong H^q_{\mathsf{m}}(M) \otimes_R \bigwedge^p \left(\bigoplus_{i=1}^n R[d_i]\mathbf{e}_i^* \right).$$

Hence, similarly as (4.2), the assertion follows immediatelly from (3.1).

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