Arithmetic and geometric degrees of graded modules

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1 Introduction

There are some basic measures such as dimension, degree and Castelnuovo-Mumford regularity to investigate the minimal free resolution of the graded modules and the defining ideals of projective schemes. The arithmetic and geometric degrees, which were introduced in Bayer-Mumford [1], and the idea of which can be also found in Hartshorne [8], involves the concept of length multiplicity which concerns lower-dimensional primary components, and enlarges the classical degree theory. In particular, the arithmetic degree is a basic measure gauging all primary components including the embedded

*Partially supported by Grant-in-Aid for Scientific Research (no. 08740040), Ministry of Education, Science, Sports and Culture, Japan

components.

The purpose of this paper is to provide a down-to-earth introduction to some degree theory of graded modules and projective schemes. Recently we have already got a nice lecture note by Vasconcelos [14] on the recent developement of this topic, including a survey of his and his colleagues' series of papers [4, 15, 16]. We try to present this survey paper from rather different viewpoint, focusing on the behaviour of the arithmetic and geometric degrees under flat deformation and hyperplane section, describing a Bertini-type theorem related to this topic, and studying upper bounds of the degrees. We think highly of explaining clearly and simply, sometimes by avoiding a general statement, particularly to help a novice reader to have a good understanding.

The authors would like to thank the organizers of the Hanoi Conference and the people helping this conference for giving us a chance to present this paper, which is an enlarged lecture note based on the first author's talk there.

Let S be the polynomial ring $k[x_0, \cdots, x_n]$ over a field k. Let $\mathfrak{m}$ be the irrelevant ideal of S. Let M be a finitely generated graded S-module. We define the length multiplicity $\mathrm{mult}_P(M)$ of M at a prime ideal P for the length of S_P-module $\Gamma_P(M_P)$. The r-th arithmetic (resp. geometric) degree $\text{arith-deg}_r(M)$ (resp. $\text{geom-deg}_r(M)$) of M is defined as the sum of $\mathrm{mult}_M(P) \cdot \deg(M)$ running through all the associated (resp. minimal) primes P of M with $\dim(S/P) = r + 1$ for an integer r with $r \geq -1$ as in [1]. We remark here that $\text{arith-deg}_u(M) = \text{geom-deg}_u(M) = \deg(M)$, where $u = \dim(M) - 1$.

In Section 2, we give the definition of the degrees for graded modules and study the basic property. The facts (2.3) and (2.5) play an important role in this paper. To analylize the embedded primary ideals of the graded S-module M, it is useful to correpond the primary ideals of M with the minimal ideals of $\mathrm{Ext}^{\bullet}_S(M, S)$, which is stated in (2.3). Also the formula in (2.5) for the caluculation of the arithmetic degree gives not only a cohomological method but also computational one doing without computing the associated primes, while primary decomposition itself is relatively difficult for computer systems. In fact, by virtue of the first equality in (2.5), we can compute the arithmetic degrees of various examples, say, by using a script `ext(-,R)` of Macaulay [2].

In Section 3, we study the behaviour of the degrees under flat deformation. The upper and lower semi-continuous property is one of the basic theme for the study of an invariant of a flat family of graded modules or coherent sheaves. Firstly we give a proof of the upper semi-continuous property of the arithmetic degree under flat deformation. This property was essentially shown in Hartshorne [8], and later Sturmfels-Trung-Vogel give an elementary proof of $\text{arith-deg}_r(S/in(I)) \geq \text{arith-deg}_r(S/I)$ in [13], where $in(I)$ is the initial ideal for any term order. In (3.1), we prove, in an elementary way, this property in the general case. Next we give a proof of the lower semi-continuous property of the geometric degree under flat deformation. The result is maybe a folk-theorem for the specialist. Also, Sturmfels et al. again give an elementary proof of $\text{geom-deg}_r(S/in(I)) \leq \text{geom-deg}_r(S/I)$ in [13]. However, as far as the authors know, there are no references about this property in the general case. We give a proof of this property in (3.2).

In Section 4, we study the behaviour of the arithmetic degree after hypersurface sections. As for the classical degree theory, it is well-known that $\deg(M/fM) = \deg(f) \cdot \deg(M)$ for a homogeneous polynomial f which is a non-zero-divisor for the S-module M. Of course, it is also clear to have an equality, for all r, $\text{geom-deg}_{r-1}(M/fM) = \deg(f) \cdot \text{geom-deg}_r(M)$ if f does not belong to any associated prime of the S-module M. However, it is harder to control the behaviour of arithmetic degree under hypersurface sections. Although the equality $\text{arith-deg}_{r-1}(M/fM) = \deg(f) \cdot \text{arith-deg}_r(M)$ holds for a general element f when k an infinite field and $r > 0$, there are examples such that $\text{arith-deg}_{r-1}(M/fM) > \deg(f) \cdot \text{arith-deg}_r(M)$ for some $r > 0$, even in the case I is a prime ideal, $M = S/I$ and $f \notin I$, see, e.g., (4.4), (4.5) and (4.6). In this section, we study the difference $\text{arith-deg}_{r-1}(M/fM) - \deg(f) \cdot \text{arith-deg}_r(M)$ under the assumption that f is a non-zero-divisor for the S-module M. This assumption is rather stronger than that of Miyazaki-Vogel [10] or Miyazaki-Vogel-Yanagawa [11], but is taken for the clearness and the simplicity of the proof. Also, the proof of a Bezout-type theorem in (4.1) given here is somewhat different from those of the above papers. Roughly speaking, we describe the arithmetic degree arith-$\deg_r(M)$ of the S-module M in terms of Hilbert polynomial of $\mathrm{Ext}_S^{n-r}(M,S)$ based on an idea of [6] and reduce the classical degree case by using the notion of S-module $M_{\geq r}$ defined in Section 2. We note that this result yields an effective criterion whether the equality arith-$\deg_{r-1}(M/fM) = \deg(f) \cdot$ arith-$\deg_r(M)$ holds.

In Section 5, we study the behaviour of the associated primes under hypersurface sections in order to obtain a Bertini-type theorem for the associated primes including the embedded primary components. In fact, we are motivated by the result in Section 4 to study the relationship between $\mathrm{Ass}_S(M)$ and $\mathrm{Ass}_S(M/fM)$ in terms of $\mathrm{Ext}_S^{n-r}(M,S)$. The theorem of this section, see (5.1), gives a description of an obstruction for the equality

$$\mathrm{Ass}_S(M/fM) \setminus \{\mathsf{m}\} = \bigcup_{P \in \mathrm{Ass}_S(M)} \mathrm{Min}(S/P + (f)),$$

This type of theorem was firstly obtained by Flenner, see, e.g., [3] through the viewpoint of his local Bertini's theorem. Recently a precise description of the obstruction was given in [11]. The statement in (5.1) is a special case of [11, (3.1)], but the proof illustrates an essential point of the original one.

In Section 6, we study upper bounds on the arithmetic and geometric degrees of a homogeneous ideal I of S. First we describe an upper bound on the arithmetic degree in terms of the Castelnuovo-Mumford regularity in (6.3). This result was obtained in [1] and also obtained in an improved version in [10], see (6.2). Next we describe an upper bound on the geometric degree in terms of the maximal degree of the minimal generator of I, firstly obtained by Masser-Wüstholz [9] in a slightly general statement. However the proof in their paper takes more than 4 pages and later a vivid sketch of the proof was illustrated in [1]. We give a self-contained and short proof of this result following an idea of [9].

Acknowledgement. The first author regrets to report that Professor Wolfgang Vogel passed away on 2 October 1996 during writing this paper. My

interest in this topic have begun through a co-operative research with him while I had been staying at Massey University in New Zealand from April 1994 to March 1995. He will be in my memory.

2 Preliminalies

Let S be the polynomial ring $k[x_0, \cdots, x_n]$ over a field k. Let m is the irrelevant ideal of S. Let M be a finitely generated graded S-module. For a homogeneous prime ideal P of S we define the length-multiplicity of M at P as the length of S_P-module $\Gamma_P(M_P)$ and denote it by $\mathrm{mult}_M(P)$. It is easy to see that $\mathrm{mult}_M(P) \neq 0$ if and only if P is an associated prime of M.

Definition 2.1 ([1, 14]). Let r be an integer with $r \geq -1$. We define the r-th arithmetic degree of M as

$$\text{arith-deg}_r(M) = \sum_{P \in \mathrm{Ass}_S(M) \text{ with } \dim(S/P) = r+1} \mathrm{mult}_M(P) \cdot \deg(S/P)$$

and the r-th geometric degree of M as

$$\text{geom-deg}_r(M) = \sum_{P \in \mathrm{Min}_S(M) \text{ with } \dim(S/P) = r+1} \mathrm{mult}_M(P) \cdot \deg(S/P)$$

Here $\mathrm{Ass}_S(M)$ and $\mathrm{Min}_S(M)$ denote the set of the associated primes of S-module M and that of the minimal primes of S-module M respectively.

Let $0 = \bigcap_\lambda (N_\lambda)$ be a minimal primary decomposition in the S-module M. Note that $\mathrm{Ass}_S(M/N_\lambda)$ consists of one prime ideal P_λ of S for each λ. We define

$$M_r = \bigcap_{\dim(S/P_\lambda) \geq r+1} N_\lambda$$

and

$$M_{\geq r} = M/M_r$$

for each integer $r \geq -1$ (cf. [8, 13]), which do not depend on the choice of the primary decomposition.

Remark 2.2.

$$\mathrm{Ass}(M_r) = \{P \in \mathrm{Ass}M \,|\, \dim S/P \leq r\}$$

$$\mathrm{Ass}(M_{\geq r}) = \{P \in \mathrm{Ass}(M) \,|\, \dim(S/P) \geq r+1\} = \mathrm{Ass}(M) \setminus \mathrm{Ass}(M_r).$$

Proposition 2.3 (See, e.g., [6, (1.1)]). *Let r be an integer with $r \geq -1$. Then we have $\dim(\mathrm{Ext}_S^{n-r}(M, S)) \leq r+1$. Furthermore, let P be a homogeneous prime ideal of S with $\dim(S/P) = r+1$. Then $P \in \mathrm{Ass}_S(M)$ if and only if P is a minimal prime of $\mathrm{Ext}_S^{n-r}(M, S)$.*

Proof. It follows from the local duality of S_P-module. The details are left to the readers.

Definition 2.4. *Let M be an S-module with $\dim M \leq r+1$. We define*

$$e_r(M) = \begin{cases} \deg(M) & \text{if } \dim M = r+1 \\ 0 & \text{otherwise.} \end{cases}$$

The arithmetic degree of M is described by the Hilbert polynomials of M_{r+1} or $\mathrm{Ext}_S^{n-r}(M, S)$ as follows. Note that both M_{r+1} and $\mathrm{Ext}_S^{n-r}(M, S)$ have Krull-dimension at most $r+1$.

Proposition 2.5 (See, e.g., [11, 13, 14, 15]) *Under the above condition, we have*

$$\text{arith-deg}_r(M) = e_r(\mathrm{Ext}_S^{n-r}(M, S)) = e_r(M_{r+1}).$$

Proof. The first equality follows from the local duality and (2.3). Also, $\text{arith-deg}_r(M) = e_r(M_{r+1})$ follows from (2.2).

Example 2.6 (See [1, pp.23-24]). Let S be the polynomial ring $k[x, y, z]$ over a field k. Let I be a homogeneous ideal of S generated by x^2 and xy. Then we have a minimal primary decomposition $I = (x) \cap (x^2, y)$. Since $\dim(S/I) = 2$, we see that $\text{arith-deg}_1(S/I) = \deg(S/I) = 1$. On the other hand, we see that $(S/I)_{\geq 1} = S/xS$. Thus we have

$$\text{arith-deg}_0(S/I) = e_0((x)/(x^2, y)) = 1.$$

3 Degrees in flat families

In this section we investigate the behaviour of the arithmetic and geometric degrees with respect to flat families. We study the upper semi-continuous property of the arithmetic degree and the lower semi-continuous property of the geometric degree under flat deformation.

Theorem 3.1. *Let R be a discrete valuation ring and t be a uniformizing parameter of R. Let K be the quotient field of R and $k = R/tR$. Let S be the polynomial ring $R[x_0, \cdots, x_n]$ over R. Let M be a finitely generated graded S-module which is flat over R. Then we have*

$$\text{arith-deg}_r(M \otimes_R k) \geq \text{arith-deg}_r(M \otimes_R K).$$

Proof. Let $\mathbf{F}_\bullet$ be a free resolution of the graded S-module M. Since M is flat over R, $\mathbf{F}_\bullet \otimes_R k$ is a free resolution of the graded $S \otimes_R k$-module $M \otimes_R k$. Thus we see that

$$\text{Ext}^i_S(M, S) = \text{H}^i(\text{Hom}_S(\mathbf{F}_\bullet, S))$$

and

$$\text{Ext}^i_{S\otimes_R k}(M \otimes_R k, S \otimes_R k) = \text{H}^i(\text{Hom}_{S\otimes_R k}(\mathbf{F}_\bullet \otimes_R k, S \otimes_R k))$$

for all i. By the short exact sequence of the complexes of graded S-modules

$$0 \to \text{Hom}_S(\mathbf{F}_\bullet, S) \xrightarrow{\cdot t} \text{Hom}_S(\mathbf{F}_\bullet, S) \to \text{Hom}_{S\otimes_R k}(\mathbf{F}_\bullet \otimes_R k, S \otimes_R k) \to 0,$$

we have an exact sequence

$$\mathrm{Ext}_S^{n-r}(M,S) \xrightarrow{\cdot t} \mathrm{Ext}_S^{n-r}(M,S) \to \mathrm{Ext}_{S\otimes_R k}^{n-r}(M\otimes_R k, S\otimes_R k).$$

Therefore we have

$$\begin{aligned} & e_r(\mathrm{Ext}_{S\otimes_R K}^{n-r}(M\otimes_R K, S\otimes_R K)) = e_r(\mathrm{Ext}_S^{n-r}(M,S)\otimes_R K) \\ \leq\ & e_r(\mathrm{Ext}_S^{n-r}(M,S)\otimes_R k) \leq e_r(\mathrm{Ext}_{S\otimes_R k}^{n-r}(M\otimes_R k, S\otimes_R k)). \end{aligned}$$

Hence the assertion is proved by (2.5).

Theorem 3.2. *Let R be a discrete valuation ring and t be a uniformizing parameter of R. Let K be the quotient field of R and $k = R/tR$. Let S be the polynomial ring $R[x_0, \cdots, x_n]$ over R. Let M be a finitely generated graded S-module which is flat over R. Then we have*

$$\text{geom-deg}_r(M\otimes_R k) \leq \text{geom-deg}_r(M\otimes_R K).$$

Proof. By the flatness of M over R there is one-to-one correspondence between $\mathrm{Ass}_S(M)$ and $\mathrm{Ass}_{S\otimes_R K}(M\otimes_R K)$. In particular, any irreducible component of $\mathrm{V}(\mathrm{Ann}(M))$ in $\mathbf{P}_R^n$ has a non-empty intersection with generic fibre $\mathbf{P}_K^n$. Let $0 = \bigcap_\lambda(N_\lambda)$ be a minimal primary decomposition in M with $\mathrm{Ass}(M/N_\lambda) = \{P_\lambda\}$. Let N be the intersection of N_λ such that the corresponding associated prime P_λ is a minimal prime of M with $\dim(S/P_\lambda) = r+1$. Let $\bar{M} = M/N$. Note that $\bar{M}$ is flat over R and $\dim \bar{M} = r+1$. Then we have

$$\text{geom-deg}_r(M\otimes_R K) = \deg(\bar{M}\otimes_R K).$$

On the other hand, any r-dimensional irreducible component Z of a closed subset $\mathrm{V}(\mathrm{Ann}(M\otimes_R k))$ in $\mathbf{P}_k^n$ is an irreducible component of the closed fibre $W\otimes_R k$ of some r-dimensional irreducible component W of $\mathrm{V}(\mathrm{Ann}(M))$ in $\mathbf{P}_R^n$. Thus we have

$$\deg(\bar{M}\otimes_R K) = \deg(\bar{M}\otimes_R k) \leq \text{geom-deg}_r(M\otimes_R k).$$

Hence the assertion is proved.

Let Y be a connected scheme of finite type over an algebraically closed field k. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_Y$ which is flat over Y. For $y \in Y$ we define functions $\text{arith-deg}_r(y, \mathcal{F})$ and $\text{geom-deg}_r(y, \mathcal{F})$ as the r-th arithmetic degree $\text{arith-deg}_r(\mathcal{F} \otimes_Y k(y))$ and the r-th geometric degree $\text{geom-deg}_r(\mathcal{F} \otimes_Y k(y))$ of the pull-back $\mathcal{F} \otimes_Y k(y)$ on $\mathbf{P}^n_{k(y)}$, where $k(y)$ is the corresponding residue field. Then we have the following.

Corollary 3.3. *Under the above condition, the function* $\text{arith-deg}_r(y, \mathcal{F})$ *is upper semi-continuous and the function* $\text{geom-deg}_r(y, \mathcal{F})$ *is lower semi-continuous.*

Proof. It immediately follows from (3.1) and (3.2).

Example 3.4 ([13, (2.4)]). The arithmetic degree is not necessarily constant under flat deformation. In fact, let I be a prime ideal of the monomial curve given parametrically by $(s^7 : s^5t^2 : s^2t^5 : t^7)$ in $\mathbf{P}^3_k = \text{Proj}(S)$, where S is the polynomial ring $k[x_0, x_1, x_2, x_3]$. In particular, $\text{arith-deg}_0(S/I) = 0$. It is known that the ideal I has the universal Gröbner basis:

$$\{x_0^5x_3^2 - x_1^7, x_0^4x_2x_3 - x_1^6, x_0^3x_2^2 - x_1^5, x_0^2x_3^5 - x_2^7, x_0x_1x_3^4 - x_2^6,$$
$$x_0x_3 - x_1x_2, x_0^2x_2^3 - x_1^4x_3, x_0x_2^4 - x_1^3x_3^2, x_1^2x_3^3 - x_2^5\},$$

having 14 distinct initial ideals $in(I)$ and all the ideals $in(I)$ fail to be square-free. For example, if we consider the reverse lexicographic order, then

$$\begin{aligned} in(I) &= (x_0^5x_3^2, x_0^4x_2x_3, x_0^3x_2^2, x_0^2x_3^5, x_0x_1x_3^4, x_0x_3, x_1^4x_3, x_1^3x_3^2, x_1^2x_3^3) \\ &= (x_0, x_1^2) \cap (x_0^3, x_3) \cap (x_2^2, x_3) \cap (x_0, x_1^3, x_3^3) \cap (x_0, x_1^4, x_3^2). \end{aligned}$$

Thus we see that $\text{arith-deg}_0(S/in(I)) > 0$.

Example 3.5. The geometric degree is not necessarily constant under flat deformation. In fact, let I be an ideal $(z(z - ty), xz)$ of the polynomial ring $S = k[t, x, y, z]$. Then we see that S/I is flat over $k[t]$. Also we easily have that $\text{geom-deg}_0(S/I)_t = 1$ for $t \neq 0$ and $\text{geom-deg}_0(S/I)_0 = 0$.

4 Arithmetic degree and hypersurface section

In this section we consider the arithmetic degree under hypersurface section and describe a Bezout-type theorem for the arithmetic degree.

Theorem 4.1 ([10, 11]). *Let S be the polynomial ring $k[x_0, \cdots, x_n]$ over a field k. Let M be a finitely generated graded S-module. Let r be a non-negative integer. Let f be a homogeneous element of S with* $\deg(f) = \tau \geq 1$. *Assume that f is a non-zero-divisor for the S-module M. Then*

$$\begin{aligned} &\text{arith-deg}_{r-1}(M/fM) - \tau \cdot \text{arith-deg}_r(M) \\ = \ &\text{arith-deg}_{r-1}(M_{\geq r+1}/fM_{\geq r+1}). \end{aligned}$$

Proof . Let us consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & M_{r+1}(-\tau) & \to & M(-\tau) & \to & M_{\geq r+1}(-\tau) & \to & 0 \\ & & \downarrow \cdot f & & \downarrow \cdot f & & \downarrow \cdot f & & \\ 0 & \to & M_{r+1} & \to & M & \to & M_{\geq r+1} & \to & 0 \end{array}$$

Since each row is exact and the vertical map $M_{\geq r+1}(-\tau) \xrightarrow{\cdot f} M_{\geq r+1}$ is injective, we have the following short exact sequence:

$$0 \to M_{r+1}/fM_{r+1} \to M/fM \to M_{\geq r+1}/fM_{\geq r+1} \to 0$$

from snake lemma. Since $\dim(M_{r+1}/fM_{r+1}) \leq r$, we see that

$$\text{Ext}_S^{n-r}(M_{r+1}/fM_{r+1}, S) = 0.$$

Thus we have the exact sequence:

$$\begin{aligned} 0 \to \text{Ext}_S^{n-r+1}(M_{\geq r+1}/fM_{\geq r+1}, S) \to \text{Ext}_S^{n-r+1}(M/fM, S) \to \\ \text{Ext}_S^{n-r+1}(M_{r+1}/fM_{r+1}, S) \to \text{Ext}_S^{n-r+2}(M_{\geq r+1}/fM_{\geq r+1}, S) \end{aligned}$$

Thus we have

$$\begin{aligned}
&\text{arith-deg}_{r-1}(M/fM) = e_{r-1}(\mathrm{Ext}_S^{n-r+1}(M/fM,S)) \\
&= \; e_{r-1}(\mathrm{Ext}_S^{n-r+1}(M_{\geq r+1}/fM_{\geq r+1},S)) + e_{r-1}(\mathrm{Ext}_S^{n-r+1}(M_{r+1}/fM_{r+1},S)) \\
&= \; \text{arith-deg}_{r-1}(M_{\geq r+1}/fM_{\geq r+1}) + \text{arith-deg}_{r-1}(M_{r+1}/fM_{r+1}).
\end{aligned}$$

Since $\dim(M_{r+1}/fM_{r+1}) \leq r$, we have

$$\begin{aligned}
&\text{arith-deg}_{r-1}(M_{r+1}/fM_{r+1}) = e_{r-1}(M_{r+1}/fM_{r+1}) \\
&= \; \tau \cdot e_r(M_{r+1}) = \tau \cdot \text{arith-deg}_r(M).
\end{aligned}$$

by the classical Bezout's Theorem and (2.5). Therefore we have

$$\begin{aligned}
&\text{arith-deg}_{r-1}(M/fM) \\
&= \; \text{arith-deg}_{r-1}(M_{\geq r+1}/fM_{\geq r+1}) + \tau \cdot \text{arith-deg}_r(M).
\end{aligned}$$

Hence the assertion is proved.

By using a Bertini-type theorem, which is proved in the next section, we have the following.

Corollary 4.2 (cf. [1, 10, 11]). *Let r be τ positive integers. Let S be the polynomial ring $k[x_0, \cdots, x_n]$ over an infinite field k. Let M be a finitely generated graded S-module. Let f be a generic homogeneous element of S with* $\deg(f) = \tau$. *Then*

$$\text{arith-deg}_{r-1}(M/fM) = \tau \cdot \text{arith-deg}_r(M).$$

Proof. We may assume that $\text{depth}(M) \geq 1$, and by using (4.1) we have only to show that $\text{arith-deg}_{r-1}(M_{\geq r+1}/fM_{\geq r+1}) = 0$. Since k is infinite, we can take a generic polynomial f such that

$$f \notin \bigcup_{r \geq 1} \{P \in \mathrm{Min}_S(\mathrm{Ext}_S^{n-r}(M,S)) \mid \dim(S/P) = r\}.$$

By (5.1), any associated prime P of $\mathrm{Ass}_S(M_{\geq r+1}/fM_{\geq r+1})$ is a minimal prime of $P' + (f)$ for some associated prime P' of $M_{\geq r+1}$. Thus we see that $\dim(S/P) \geq r+1$. Hence the assertion is proved.

Remark 4.3. We can prove (4.2) directly. In fact, we may assume that $\mathrm{depth}(M) > 0$ and take enough general f such that f does not belong to any associated prime P of $\mathrm{Ext}_S^{n-r}(M, S)$ with $\dim(S/P) = r, r+1$. By the short exact sequence:

$$0 \to M(-\tau) \xrightarrow{\cdot f} M \to M/fM \to 0,$$

we have an exact sequnece:

$$\mathrm{Ext}_S^{n-r}(M, S) \xrightarrow{\varphi} \mathrm{Ext}_S^{n-r}(M, S)(\tau) \xrightarrow{\psi} \mathrm{Ext}_S^{n-r+1}(M/fM, S),$$

such that $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\psi)$ have Krull-dimension $\leq r-2$. Thus the assertion follows by (2.5).

The equality $\text{arith-deg}_{r-1}(M/fM) = \deg(f) \cdot \text{arith-deg}_r(M)$ is frequently violated as the following examples.

Example 4.4. Let S be the polynomial ring over a field k. Let us take a homogeneous prime ideal I of S with $\dim(S/I) \geq 2$ and $\mathrm{depth}(S/I) = 1$. Then we see that, for a homogeneous polynomial f, $\text{arith-deg}_0(S/I) = 0$ and $\text{arith-deg}_{-1}(S/(I + (f))) > 0$.

Example 4.5 ([11, (2.9.(i))]). Let $S = k[x_0, x_1, x_2, x_3, x_4]$ be the polynomial ring over a field k. Let us take an ideal $I = (x_1x_4 - x_2x_3, x_0x_1x_2 - x_0x_2^2 + x_1^2x_3, x_0x_2x_3 - x_0x_2x_4 + x_1x_3^2, x_0x_3x_4 - x_0x_4^2 + x_3^3)$ of S. (See, e.g., [12, (V.5.2)].) The surface $X = \mathrm{Proj}(S/I)$ in $\mathbf{P}_k^4$ is called as Hartshorne surface. It is known that S/I is isomorphic to a subring

$$k[s^3, s^2t, stu, su(u-s), u^2(u-s)] \subset k[s, t, u].$$

It is easy to see that $\dim(S/I) = 3$, $\text{arith-deg}_2(S/I) = \deg(S/I) = 4$ and $\text{arith-deg}_i(S/I) = 0$ for $i = -1, 0, 1$. A calculation gives $\mathrm{Ann}_S(\mathrm{Ext}_S^3(S/I, S))$

$= (x_1, x_2, x_3, x_4)$. In particular, X is not locally Cohen-Macaulay at $P = (x_1, x_2, x_3, x_4)$. If taking a homogeneous polynomial $f \in P \setminus I$, we see that $\text{arith-deg}_0(S/(I + (f))) > 0$. Thus we have

$$\text{arith-deg}_0(S/(I + (f))) > \deg(f) \cdot \text{arith-deg}_1(S/I).$$

Example 4.6 ([10, (5.1)]). Let $S = k[x_0, x_1, x_2, x_3]$ be the polynomial ring over a field k. Let m be the irrelevant ideal of S. Let us take an ideal $Q = (x_0x_3 - x_1x_2, x_0^2, x_1^2, x_0x_1)$ of S. Note that Q is a primary ideal belonging to (x_0, x_1). We set $I = Q \cap (x_0^2, x_1, x_2)$. In other words, I is the defining ideal of a double line in the smooth quadric surface with an embedded point. A computer software, say Macaulay [2], shows that $\dim(\text{Ext}_S^3(S/I, S)) = 1$ and $\deg(\text{Ext}_S^3(S/I, S)) = 1$. Hence we have $\text{arith-deg}_0(S/I) = 1$. Also we have that $\mathsf{m} \in \text{Ass}(\text{Ext}_S^3(S/I, S))$ by computing $\text{Ext}_S^4(\text{Ext}_S^3(S/I, S), S))$. Thus we see

$$\text{arith-deg}_{-1}(S/(I + (f))) > \deg(f) \cdot \text{arith-deg}_0(S/I) = \deg(f) > 0,$$

for any $f \in \mathsf{m} \setminus (x_0, x_1, x_2)$ by (4.1).

Analyzing the classical degree theory (see, e.g., [7, (3.5)]), one might be attempted to ask the following question:

Assume that $\text{arith-deg}_{r-1}(S/(I+(f))) \geq \deg(f) \cdot \text{arith-deg}_r(S/I)$ *for some integer* r *and* $\deg(f) \geq 2$*. Then is there* f *not belonging to any associated prime* P *of* I *with* $\dim(S/P) \geq r + 1$*?*

However this question has a negative answer:

Example 4.7. Let S be the polynomial ring $k[x, y, z]$ and t be an integer with $t \geq 2$. We set $I = (xz, xy^t) = (x) \cap (y^t, z)$ of S. We take $f = xz \in I$ and $r = 1$. Then we see that $\text{arith-deg}_0(S/(I + (xz))) = \text{arith-deg}_0(S/I) = t$ and $\deg(xz) \cdot \text{arith-deg}_1(S/I) = 2$.

5 Bertini-type theorem for primary components

A study of the arithmetic degree involves Bertini-type results, see, e.g., [1, 10, 11]. In this section we describe a Bertini-type theorem for primary components including embedded primes.

Theorem 5.1 ([11, (3.1)]). *Let S be the polynomial ring $k[x_0, \cdots, x_n]$ over a field k. Let m be the irrelevant ideal of S. Let M be a finitely generated graded S-module with* $\operatorname{depth}(M) \geq 1$*. Let f be a homogeneous element of S with* $\deg(f) = \tau \geq 1$ *satisfying that f is a non-zero-divisor for M. If*

$$f \notin \bigcup_{r \geq 1} \{P \in \operatorname{Min}_S(\operatorname{Ext}_S^{n-r}(M, S)) \mid \dim(S/P) = r\},$$

then

$$\operatorname{Ass}_S(M/fM) \setminus \{\mathsf{m}\} \subseteq \bigcup_{P' \in \operatorname{Ass}_S(M)} \operatorname{Min}_S(S/P' + (f)).$$

Proof. Let P be an associated prime of M/fM with $\dim(S/P) = r > 0$. From the short exact sequence:

$$0 \to M \xrightarrow{\cdot f} M \to M/fM \to 0,$$

we have the following exact sequence of S-modules:

$$\operatorname{Ext}_S^{n-r}(M, S)_P \to \operatorname{Ext}_S^{n-r+1}(M/fM, S)_P \to \operatorname{Ext}_S^{n-r+1}(M, S)_P.$$

On the other hand, since $f \in P$, we see that $P \notin \operatorname{Ass}(M)$ by the assumption. In other words, $\operatorname{Ext}_S^{n-r+1}(M, S)_P = 0$ by (2.3). Further we know that $\operatorname{Ext}_S^{n-r+1}(M/fM, S)_P \neq 0$ by (2.3). Therefore we have $\operatorname{Ext}_S^{n-r}(M, S)_P \neq 0$. By the assumption, P is not a minimal prime of $\operatorname{Ext}_S^{n-r}(M, S)$. Hence there exists a homogeneous prime P' such that $P' \subset P$ and $\operatorname{Ext}_S^{n-r}(M, S)_{P'} \neq 0$. Since $\dim(\operatorname{Ext}_S^{n-r}(M, S)) \leq r + 1$, we see that P' is a minimal prime of M with $\dim(S/P') = r + 1$. In particular, f does not belong to P'. Therefore

P is a minimal prime of $P' + (f)$. Hence the assertion is proved.

Remark 5.2. The converse of (5.1) is also true. Further, an extended result is shown in [11, (3.1)].

The following examples illustrate the converse.

Example 5.3. Let $S = k[x_0, x_1, x_2, x_3, x_4]$ be the polynomial ring over a field k. Let us take an ideal $I = (x_1, x_2) \cap (x_3, x_4)$ of S. We see that $\mathrm{Ass}_S(\mathrm{Ext}^3_S(S/I, S)) = \{P\}$, where $P = \{(x_1, x_2, x_3, x_4)\}$. For any non-zero-divisor $f \in P$ for S/I, we see that

$$P \in \mathrm{Ass}_S(S/I + (f))$$

and

$$P \notin \bigcup_{Q \in \mathrm{Ass}_S(S/I)} \mathrm{Min}_S(S/Q + (f)).$$

From a geometric point of view, the ideal I is the defining ideal of two planes meeting at the closed point p, corresponding to the prime ideal P, in $\mathbf{P}^4$. Let F be a hypersurface of $\mathbf{P}^4$ which intersects each primary component of X transversally. If F contains p, then p is an embedded point of $X \cap F$. Note that the local ring of X at p has depth 1, since X is not connected in codimension 1 at p.

Example 5.4. Let us consider the example in (4.5). Then $\mathrm{Ext}^3_S(S/I, S)$ has a minimal prime $P = (x_1, x_2, x_3, x_4)$ and $\dim(S/P) = 1$. Then we see that, for any homogeneous element $f \in P \setminus I$, $P \in \mathrm{Ass}_S(S/I + (f))$. On the other hand, since the ideal I is a prime ideal with $\dim(S/I) = 3$, we see that $P \notin \mathrm{Min}_S(S/I + (f))$.

Remark 5.5. Although we state the results of this section in homogeneous case, the proof works similarly for the statement over a Cohen–Macaulay local ring S admitting the canonical module ω_S if we replace $\mathrm{Ext}^{n-r}_S(M, S)$ by $\mathrm{Ext}^{n-r}_S(M, \omega_S)$.

6 Some bounds on degrees

In this section we investigate upper bounds on the arithmetic and geometirc degrees according to [1].

First we study an upper bound on the arithmetic degree in term of the Catselnuovo-Mumford regularity. Let S be the polynomial ring $k[x_0, \cdots, x_n]$ over a field k. Let m be the irrelevant ideal of S. Let M be a finitely generated graded S-module. Let m be an integer. Then the S-module M is said to be m-regular if

$$\left[\mathrm{H}^i_{\mathsf{m}}(M)\right]_j = 0,$$

for all i and j with $i + j \geq m$, where $[N]_j$ denotes the j-th graded part of a graded S-module N. The Castelnuovo-Mumford regularity of M is the smallest integer m such that M is m'-regular for all $m' > m$ and is denoted by $\mathrm{reg}(M)$. Equivalent definitions of the Castelnuovo-Mumford regularity are given in terms of the minimal free resolution of the graded S-module M. (See [1, (3.2)] or [5].)

The following is a well-known result which describes an important property of the regularity.

Lemma 6.1. Let S be the polynomial ring $k[x_0, \cdots, x_n]$ over a field k. Let M be a graded S-module. We denote the Hilbert funtion and the Hilbert polynomial of the S-module M by $\mathrm{H}(M, \ell)$ and $\mathrm{P}(M, \ell)$ respectively. Then we have

$$\mathrm{H}(M, \ell) = \mathrm{P}(M, \ell)$$

for all $\ell \geq \mathrm{reg}(M) - \mathrm{depth}(M) + 1$.

Proof. Let us take the minimal free resolution of the S-module M

$$0 \to F_s \to \cdots \to F_1 \to F_0 \to M \to 0,$$

where $s = n + 1 - \operatorname{depth}(M)$ and $F_i = \oplus_j S(-\alpha_{ij})$ for some α_{ij}. Then we have

$$\mathrm{H}(M, \ell) = \sum_{i,j} (-1)^i \dim_k [S(-\alpha_{ij})]_\ell$$

for all ℓ. Thus we see that $\mathrm{H}(M, \ell)$ is a polynomial for all $\ell \geq \max_{i,j}(\alpha_{ij} - n)$, because $\dim_k [S]_\ell = (\ell + n) \cdots (\ell + 1)/n!$ for all $\ell \geq -n$. On the other hand, $\ell \geq \operatorname{reg}(M) - \operatorname{depth}(M) + 1 \geq (\alpha_{ij} - i) - (n + 1 - i) + 1 = \alpha_{ij} - n$ for all i and j. Hence the assertion is proved.

Theorem 6.2 ([10, (3.1)]). Let S be the polynomial ring $k[x_0, \cdots, x_n]$ over a field k. Let M be a graded S-module. Then we have, for any integer $r \geq 0$,

$$\text{arith-deg}_r(M) \leq \Delta^r \mathrm{P}(M, \ell)$$

for all integers $\ell \geq \operatorname{reg}(M)$, where $\Delta^r \mathrm{P}(M, \ell)$ is defined inductively as $\Delta \mathrm{P}(M, \ell) = \mathrm{P}(M, \ell) - \mathrm{P}(M, \ell - 1)$ and $\Delta^r \mathrm{P}(M, \ell) = \Delta^{r-1} \mathrm{P}(M, \ell) - \Delta^{r-1} \mathrm{P}(M, \ell - 1)$.

Proof. We may assume that $\operatorname{depth}(M) > 0$. First we prove the case $r = 0$. By (2.5), we see that

$$\text{arith-deg}_0(M) = \mathrm{P}(M_1, \ell).$$

Note that $\dim(M_1) \leq 1$. From the short exact sequence:

$$0 \to M_1 \to M \to M_{\geq 1} \to 0,$$

we see that

$$\mathrm{H}^1_{\mathsf{m}}(M_1) \to \mathrm{H}^1_{\mathsf{m}}(M)$$

is injective, where m is the irrelevant ideal of S. Hence we have $\operatorname{reg}(M_1) \leq \operatorname{reg}(M)$, and by (6.1) $\mathrm{H}(M, \ell) = \mathrm{P}(M, \ell)$ and $\mathrm{H}(M_1, \ell) = \mathrm{P}(M_1, \ell)$ for $\ell \geq \operatorname{reg}(M)$. Also we see that

$$\mathrm{H}(M_1, \ell) = \mathrm{H}(M, \ell) - \mathrm{H}(M_{\geq 1}, \ell),$$

for all ℓ. Thus we have

$$\text{arith-deg}_0(M) = \mathrm{P}(M, \ell) - \mathrm{P}(M_{\geq 1}, \ell) \geq \mathrm{P}(M, \ell)$$

for $\ell \geq \operatorname{reg}(M)$.

Now let us assume that $r > 0$. For general hyperplanes $h_1, \cdots, h_r$, we have

$$\begin{aligned} \text{arith-deg}_r(M) &= \text{arith-deg}_0(M/(h_1, \cdots, h_r)M) \\ &= \text{arith-deg}_0((M/(h_1, \cdots, h_r)M)_{\geq 0}) \end{aligned}$$

by (4.2). On the other hand, we see that

$$\begin{aligned} \Delta^r \mathrm{P}(M, \ell) &= \mathrm{P}(M/(h_1, \cdots, h_r)M, \ell) \\ &= \mathrm{P}((M/(h_1, \cdots, h_r)M)_{\geq 0}, \ell) \end{aligned}$$

for all ℓ. Since $\operatorname{reg}(M) \geq \operatorname{reg}((M/(h_1, \cdots, h_r)M)_{\geq 0})$, we have

$$\text{arith-deg}_r(M) \leq \Delta^r \mathrm{P}(M, \ell)$$

for $\ell \geq \operatorname{reg}(M)$. Hence the assertion is proved.

Corollary 6.3 ([1, (3.6)]). Let S be the polynomial ring $k[x_0, \cdots, x_n]$ over a field k, and let I be a homogeneous ideal of S. Let r be a non-negative integer. Let $\operatorname{reg}(I) = m$. Then we have, for any integer $r \geq 0$,

$$\text{arith-deg}_r(S/I) \leq \Delta^r \mathrm{P}(S, \ell)$$

for all integers $\ell \geq m - 1$. In particular,

$$\text{arith-deg}_r(S/I) \leq m^{n-r}.$$

Proof. First we show $\Delta^r \mathrm{P}(I, \ell) \geq 0$ for all $\ell \geq m - 1$ by induction, which is left to the readers, or see [10, (3.5)]. Since $\operatorname{reg}(S/I) = m - 1$, we have

$$\begin{aligned} \text{arith-deg}_r(S/I) &\leq \Delta^r \mathrm{P}(S/I, \ell) \\ &\leq \Delta^r \mathrm{P}(S, \ell) \end{aligned}$$

for all integers $\ell \geq m - 1$ by (6.2). The second assertion follows immediately from the first one.

Next we study an upper bound on the geometric degree of homogeneous ideals in term of the degree of the minimal generators of the ideal, according to [9]. Let S be the polynomial ring $k[x_0, \cdots, x_n]$ over a field k. Let I be an homogeous ideal of S. For a minimal generater $f_1, \cdots, f_t$ of the ideal I of S, we set $d_1 = \deg(f_1), \cdots, d_t = \deg(f_t)$ and assume $d_1 \geq \cdots \geq d_t$. The number of the generaters t and the sequence $d_1, \cdots, d_t$ does not depend on the choice of minimal generators of I. Then we define $\mathrm{d}(I) = d_1$ and $\mu(I) = t$. It is known that $\mathrm{reg}(I) \geq \mathrm{d}(I)$.

Theorem 6.4 ([1, (3.5)], [9, Theorem II]). Let S be the polynomial ring $k[x_0, \cdots, x_n]$ over a field k. Let I be an homogeous ideal of S. Then we have

$$\text{geom-deg}_r(S/I) \leq \mathrm{d}(I)^{n-r}$$

for all integer r.

Proof. We may assume that $r \geq 0$ and k is algebraically closed. Further we may assume that the ideal I has at least one isolated prime P with $\dim(S/P) = r + 1$. Then we note that $\mu(I) \geq n - r$. Let $P_1, \cdots, P_m$ be the isolated primes of I of height $n - r$. Then, for each $i = 1, \cdots, m$, we can take a homogeneous prime ideal Q_i with $\dim(S/Q_i) = 1$ which contains P_i and does not contain any other minimal prime of I. For a homogeneous ideal J of S, we define the contracted extension J^* as

$$J^* = JS_{Q_1} \cap \cdots \cap JS_{Q_m} \cap S.$$

Note that the set of the minimal primes of I^* is $\{P_1, \cdots, P_m\}$. Let h be a linear from of S such that h is not contained in any of $Q_1, \cdots, Q_m$. Let $f_1, \cdots, f_{\mu(I)}$ be a generator of I with degree $d_1, \cdots, d_{\mu(I)}$. We set that $\mathcal{S}$ is a k-linear subspace of the homogeneous polynomials of S of degree $\mathrm{d}(I)$ generated by $h^{a_1} f_1, \cdots, h^{a_{\mu(I)}} f_{\mu(I)}$, where $a_1 = \mathrm{d}(I) - d_1, \cdots, a_{\mu(I)} = \mathrm{d}(I) - d_{\mu(I)}$.

Now we inductively construct, for $i = 1, \cdots, n-r$, an ideal I_i generated by elements $g_1, \cdots, g_i$ of $\mathcal{S}$ such that the contracted extention $(I_i)^*$ is unmixed of height i.

In case $i = 1$, we take a non-zero element g_1 in $\mathcal{S}$. Suppose that $I_{i-1} = (g_1, \cdots, g_{i-1})$ such that the contracted extention $(I_{i-1})^*$ is unmixed of height $i-1$. In order to construct I_i we show that, for any minimal prime P of $(I_{i-1})^*$, not all of $f_1, \cdots, f_{\mu(I)}$ lie in P. In fact, if not, then $I^* \subseteq P$. It contradicts that $\mathrm{ht}(P) = i-1$ and $\mathrm{Min}_S(S/I^*) = \{P_1, \cdots, P_m\}$. Thus we can take an element g_i in $\mathcal{S}$ such that g_i is not contained in any minimal prime of the ideal $(I_{i-1})^*$, because h is not contained in any minimal prime P of $(I_{i-1})^*$ and k is an infinite field.

Now we have the ideal I_{n-r} generated by elements $g_1, \cdots, g_{n-r}$ of $\mathcal{S}$ such that $(I_{n-r})^*$ is unmixed of height $n-r$. Thus we see that

$$\text{geom-deg}_r(S/I) = \deg(S/I^*) \leq \deg(S/(I_{n-r})^*) \leq \mathrm{d}(I)^{n-r}.$$

Hence the assertion is proved.

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