Some properties of weighted operator means due to Pálfia and Petz

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This is a joint work with Doctor Yoichi Udagawa and Professor Masahiro Yanagida
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2 Weighted operator means

3 Adjoint, orthogonal and dual of weighted operator means

4 A characterization of operator interpolational means

5 Supplements

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Some properties of weighted operator means
Operator monotone functions

- $\mathcal{H}$: Hilbert space.
- $B(\mathcal{H})$: the set of all bounded linear operators on $\mathcal{H}$.
- $B(\mathcal{H})_{sa}$, $B(\mathcal{H})_+$: the sets of all self-adjoint and positive definite operators, respectively.
- Let $A, B \in B(\mathcal{H})_{sa}$. Then $A \leq B$ is defined by
  $$\langle Ax, x \rangle \leq \langle Bx, x \rangle \quad \text{for all } x \in \mathcal{H}.$$
Operator monotone functions

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- Let $A, B \in B(\mathcal{H})_{sa}$. Then $A \leq B$ is defined by
  \[ \langle Ax, x \rangle \leq \langle Bx, x \rangle \quad \text{for all } x \in \mathcal{H}. \]

Operator monotone function

A real-valued function $f$ defined on a real interval $I$ is operator monotone if

\[ A \leq B \implies f(A) \leq f(B) \]

for $A, B \in B(\mathcal{H})_{sa}$ whose spectral are included in $I$. 

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Operator monotone functions

A real-valued function $f$ defined on a real interval $I$ is operator monotone if

$$A \preceq B \implies f(A) \preceq f(B)$$

for $A, B \in B(\mathcal{H})_{sa}$ whose spectral are included in $I$.

Examples.

- $f(t) = t^\alpha$ for $\alpha \in [0, 1]$ (Loewner-Heinz inequality),
- $f(t) = \log t,$
- $f(t) = \left( \frac{1 + t^r}{2} \right)^{\frac{1}{r}}$ for $r \in [-1, 1] \setminus \{0\}$. 
Operator mean

Operator mean (Kubo-Ando, 1980)

For \( A, B \in B(\mathcal{H})_+ \), a binary operation \( \mathcal{M}(A, B) \in B(\mathcal{H})_+ \) is called an **operator mean** if the following conditions are satisfied,

(i) \( \mathcal{M}(A, B) \leq \mathcal{M}(C, D) \) if \( A \leq C \) and \( B \leq D \),

(ii) \( C\mathcal{M}(A, B)C \leq \mathcal{M}(CAC, CBC) \) for \( C \in B(\mathcal{H})_{sa} \),

(iii) \( \mathcal{M}(A_n, B_n) \downarrow \mathcal{M}(A, B) \) if \( A_n \downarrow A \) and \( B_n \downarrow B \), strongly,

(iv) \( I \sigma I = I \).
Kubo-Ando theory

Theorem A (Kubo-Ando, 1980)

For each operator mean $\mathcal{M}(\cdot, \cdot)$, $\exists_1$ positive operator monotone function $f$ such that $f(1) = 1$ and

$$f(t)I = \mathcal{M}(l, tl),$$
Kubo-Ando theory

Theorem A (Kubo-Ando, 1980)

- For each operator mean $\mathcal{M}(\cdot, \cdot)$, there exists a positive operator monotone function $f$ such that $f(1) = 1$ and

$$f(t)1 = \mathcal{M}(1, tl),$$

- for $A, B \in B(\mathcal{H})_+$, the following formula holds:

$$\mathcal{M}(A, B) = A^{\frac{1}{2}} f\left(A^{\frac{-1}{2}} BA^{\frac{-1}{2}} \right) A^{\frac{1}{2}}.$$
Kubo-Ando theory

Theorem A (Kubo-Ando, 1980)

- For each operator mean \( \mathcal{M}(\cdot, \cdot) \), \( \exists_1 \) positive operator monotone function \( f \) such that \( f(1) = 1 \) and
  \[
  f(t)I = \mathcal{M}(I, tI),
  \]
- for \( A, B \in B(\mathcal{H})_+ \), the following formula holds:
  \[
  \mathcal{M}(A, B) = A^{\frac{1}{2}} f\left(A^{\frac{-1}{2}} BA^{\frac{-1}{2}} \right) A^{\frac{1}{2}}.
  \]

Let \( \gamma \) be the set of all pairs of an operator mean and its representing function, i.e., we considered \( (\mathcal{M}(\cdot, \cdot), f(x)) \in \gamma \) in Theorem A.
Examples of operator means

- **Trivial means:** \( f(x) = 1, \ f(x) = x \)

  \[ \mathcal{M}(A, B) = A \ (\text{left}), \quad \mathcal{M}(A, B) = B \ (\text{right}). \]

- **Weighted geometric mean:** \( f(t; x) = x^t \)

  \[ \mathcal{G}(t; A, B) = A^{\frac{1}{2}} (A^{-\frac{1}{2}} BA^{-\frac{1}{2}})^t A^{\frac{1}{2}} \quad \text{for} \ t \in [0, 1]. \]

- **Weighted power mean:**

  for \( s \in [-1, 1] \setminus \{0\} \), \( f_s(t; x) = [(1 - t) + tx^s]^\frac{1}{s} \)

  \[ P_s(t; A, B) = A^{\frac{1}{2}} [(1-t)I + (A^{-\frac{1}{2}} BA^{-\frac{1}{2}})^s]^\frac{1}{s} A^{\frac{1}{2}} \quad \text{for} \ t \in [0, 1]. \]

  *We will not treat trivial means without explanations.*
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Some properties of weighted operator means
**Definition 1** (**\( t \)-Weighted operator mean).**

Let \((\mathcal{M}(\cdot, \cdot), f(x)) \in \gamma\). Then

\[
\mathcal{M}(\cdot, \cdot) \colon \text{\( t \)-weighted operator mean} \overset{\text{def}}{\leftrightarrow} f'(1) = t.
\]
Definition 1 (t-Weighted operator mean).

Let \((\mathcal{M}(\cdot, \cdot), f(x)) \in \gamma\). Then

\[\mathcal{M}(\cdot, \cdot) : \text{t-weighted operator mean } \overset{\text{def}}{\iff} f'(1) = t.\]

Theorem B (Pálfia-Petz, 2014)

Let \((\mathcal{M}(\cdot, \cdot), f(x)) \in \gamma\). Then \(f'(1) \in [0, 1]\).
Generalized weighted mean process

Definition 2 (Pálfia-Petz, 2014)

Let $(\mathcal{M}(\cdot, \cdot), f(x)) \in \gamma$. Let $A, B \in B(\mathcal{H})_+$ and $t \in [0, 1]$. Define $a_n, b_n \in [0, 1]$ and $A_n, B_n \in B(\mathcal{H})_+$ by the following procedure for all $n = 0, 1, 2, \ldots$:

\[(0)\] Initial conditions:
\[a_0 = 0 \text{ and } b_0 = 1, \ A_0 = A \text{ and } B_0 = B,\]

\[(1)\] $a_n = t \quad \iff \quad \begin{cases} a_{n+1} := a_n & \text{and } b_{n+1} := a_n, \\ A_{n+1} := A_n & \text{and } B_{n+1} := A_n, \end{cases}$

\[(2)\] $b_n = t \quad \iff \quad \begin{cases} a_{n+1} := b_n & \text{and } b_{n+1} := b_n, \\ A_{n+1} := B_n & \text{and } B_{n+1} := B_n, \end{cases}$
Definition 2 (Pálfia-Petz, 2014)

Let $(\mathcal{M}(\cdot, \cdot), f(x)) \in \gamma$. Let $A, B \in B(\mathcal{H})_+$ and $t \in [0, 1]$. Define $a_n, b_n \in [0, 1]$ and $A_n, B_n \in B(\mathcal{H})_+$ by the following procedure for all $n = 0, 1, 2, \ldots$:

\begin{align*}
(3) \quad & (1 - f'(1))a_n + f'(1)b_n \leq t \implies \\
& \begin{cases} 
    a_{n+1} := (1 - f'(1))a_n + f'(1)b_n & \text{and } b_{n+1} := b_n, \\
    A_{n+1} := \mathcal{M}(A_n, B_n) & \text{and } B_{n+1} := B_n,
\end{cases}
\end{align*}

\begin{align*}
(4) \quad & (1 - f'(1))a_n + f'(1)b_n > t \implies \\
& \begin{cases} 
    a_{n+1} := a_n & \text{and } b_{n+1} := (1 - f'(1))a_n + f'(1)b_n, \\
    A_{n+1} := A_n & \text{and } B_{n+1} := \mathcal{M}(A_n, B_n).
\end{cases}
\end{align*}
Example. The case $f'(1) = \frac{1}{2}$ and $t = \frac{1}{8}$

\[ a_0 = 0 \quad b_0 = 1 \]

\[ A_0 = A \quad B_0 = B \]
Example. The case $f'(1) = \frac{1}{2}$ and $t = \frac{1}{8}$

$a_0 = 0 \quad b_0 = 1$

$(1 - f'(1))a_0 + f'(1)b_0 = \frac{1}{2} > \frac{1}{8} = t$

$A_0 = A \quad B_0 = B$
Example. The case \( f'(1) = \frac{1}{2} \) and \( t = \frac{1}{8} \)

\[
\begin{align*}
a_1 &= a_0 \\

b_1 &= (1 - f'(1))a_0 + f'(1)b_0 = \frac{1}{2} > \frac{1}{8} = t \\

b_0 &= 1
\end{align*}
\]

\[
A_0 = A \quad B_0 = B
\]
Example. The case $f'(1) = \frac{1}{2}$ and $t = \frac{1}{8}$

$a_1 = a_0$  \hspace{1cm}  $b_1$  \hspace{1cm}  $b_0 = 1$

\[(1 - f'(1))a_0 + f'(1)b_0 = \frac{1}{2} > \frac{1}{8} = t\]

\[b_1 = (1 - f'(1))a_0 + f'(1)b_0 = \frac{1}{2}\]

$A_1 = A_0$  \hspace{1cm}  \hspace{1cm}  $B_0 = B$

\[B_1 = M(A, B)\]
Example. The case \( f'(1) = \frac{1}{2} \) and \( t = \frac{1}{8} \)

\[
(1 - f'(1))a_1 + f'(1)b_1 = \frac{1}{4} > \frac{1}{8} = t
\]

\[
B_1 = \mathcal{M}(A, B)
\]

\[
b_0 = 1
\]
Example. The case \( f'(1) = \frac{1}{2} \) and \( t = \frac{1}{8} \)

\[
\begin{align*}
 a_2 &= a_1 \\
 b_2 &= b_1 \\
 b_0 &= 1 \\
\end{align*}
\]

\[
(1 - f'(1))a_1 + f'(1)b_1 = \frac{1}{4} > \frac{1}{8} = t
\]

\[
b_2 = (1 - f'(1))a_1 + f'(1)b_1 = \frac{1}{4}
\]

\[
A_1 \\ B_0 = B
\]

\[
B_1 = M(A, B)
\]
Example. The case \( f'(1) = \frac{1}{2} \) and \( t = \frac{1}{8} \)

\[
\begin{align*}
 a_2 &= a_1 \\
 b_2 &= b_1 \\
 b_0 &= 1
\end{align*}
\]

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(1 - f'(1))a_1 + f'(1)b_1 = \frac{1}{4} > \frac{1}{8} = t
\]

\[
b_2 = (1 - f'(1))a_1 + f'(1)b_1 = \frac{1}{4}
\]

\[
A_2 = A_1 \\
B_0 = B
\]

\[
B_2 = \mathcal{M}(A_1, B_1)
\]

\[
B_1 = \mathcal{M}(A, B)
\]
Example. The case \( f'(1) = \frac{1}{2} \) and \( t = \frac{1}{8} \)

\[
(1 - f'(1))a_2 + f'(1)b_2 = \frac{1}{8} \leq \frac{1}{8} = t
\]

\( A_2 \)

\( B_0 = B \)

\( B_2 = \mathcal{M}(A_1, B_1) \)

\( B_1 = \mathcal{M}(A, B) \)
Example. The case \( f'(1) = \frac{1}{2} \) and \( t = \frac{1}{8} \)

\[
(1 - f'(1))a_2 + f'(1)b_2 = \frac{1}{8} \leq \frac{1}{8} = t
\]

\[
a_3 = (1 - f'(1))a_2 + f'(1)b_2 = \frac{1}{8}
\]

\[
A_2 \quad B_0 = B
\]

\[
B_2 = \mathcal{M}(A_1, B_1)
\]

\[
B_1 = \mathcal{M}(A, B)
\]
Example. The case \( f'(1) = \frac{1}{2} \) and \( t = \frac{1}{8} \)

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(1 - f'(1))a_2 + f'(1)b_2 = \frac{1}{8} \leq \frac{1}{8} = t
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\]

\[
A_2 \quad B_0 = B
\]

\[
A_3 = \mathcal{M}(A_2, B_2) \quad B_3 = B_2
\]

\[
B_1 = \mathcal{M}(A, B)
\]
Example. The case $f'(1) = \frac{1}{2}$ and $t = \frac{1}{8}$.

\[ a_3 = \frac{1}{8} = t \]

\[ A_2 \]
\[ A_3 = \mathcal{M}(A_2, B_2) \]

\[ B_3 \]

\[ B_1 = \mathcal{M}(A, B) \]

\[ b_0 = 1 \]
Example. The case $f'(1) = \frac{1}{2}$ and $t = \frac{1}{8}$

\[ a_2 \quad b_3 \quad b_1 \quad b_0 = 1 \]

\[ a_4 = b_4 = a_3 \quad a_3 = \frac{1}{8} = t \]

\[ A_2 \quad A_3 = \mathcal{M}(A_2, B_2) \quad B_0 = B \]

\[ B_1 = \mathcal{M}(A, B) \]
Example. The case $f'(1) = \frac{1}{2}$ and $t = \frac{1}{8}$

\[ a_2 \quad b_3 \quad b_1 \quad b_0 = 1 \]

\[ a_4 = b_4 = a_3 \quad a_3 = \frac{1}{8} = t \]

\[ A_2 \quad A_4 = B_4 = A_3 \quad B_0 = B \]

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Example. The case $f'(1) = \frac{1}{2}$ and $t = \frac{1}{8}$

\[a_2 \quad b_3 \quad b_1 \quad b_0 = 1\]

\[a_4 = b_4 = a_3 \quad a_3 = \frac{1}{8} = t\]

\[A_2 = A \quad A_4 = B_4 = A_3 \quad B_0 = B\]

\[B_1 = \mathcal{M}(A, B)\]
The sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ given in Definition 2 converge to the common limit point, i.e.,

$$\lim_{n \to \infty} A_n = \lim_{n \to \infty} B_n = \exists M_t \in B(\mathcal{H})_+$$

in the Thompson metric.
Weighted operator mean

Theorem B (Pálfia-Petz, 2014).

The sequences \( \{A_n\}_{n=0}^\infty \) and \( \{B_n\}_{n=0}^\infty \) given in Definition 2 converge to the common limit point, i.e.,

\[
\lim_{n \to \infty} A_n = \lim_{n \to \infty} B_n = \exists M_t \in B(\mathcal{H})_+
\]

in the Thompson metric.

Definition 3 (Pálfia-Petz, 2014).

The common limit point \( M_t \) of \( \{A_n\}_{n=0}^\infty \), \( \{B_n\}_{n=0}^\infty \) in Theorem B will be denoted by \( M(t; A, B) \).
Basic properties

Proposition C (Pálfia-Petz, 2014).

Let \((\mathcal{M}(\cdot,\cdot), f(x)) \in \gamma\), and let \(\mathcal{N}(\cdot,\cdot)\) be an operator mean. \(\mathcal{M}(t; A, B)\) for \(A, B \in B(\mathcal{H})_+\) and \(t \in [0,1]\) fulfills the following properties:

1. \(\mathcal{M}(t; A, B)\) is an operator mean,

2. if \(\mathcal{N}(A, B) \leq \mathcal{M}(A, B)\), then \(\mathcal{N}(t; A, B) \leq \mathcal{M}(t; A, B)\),

3. \(\mathcal{M}(f'(1); A, B) = \mathcal{M}(A, B)\),

4. \(\mathcal{M}(t; A, B)\) is continuous in \(t\).
Remark

Let \((M(\cdot, \cdot), f(x)) \in \gamma\).

- For \(t \in [0, 1]\), we write \((M(t; \cdot, \cdot), f(t; x)) \in \gamma\) as a pair of a corresponding weighted operator mean \(M(t; \cdot, \cdot)\) to \(M(\cdot, \cdot)\) and its representing function.
Remark

Let $(\mathcal{M}(\cdot, \cdot), f(x)) \in \gamma$.

- For $t \in [0, 1]$, we write $(\mathcal{M}(t; \cdot, \cdot), f(t; x)) \in \gamma$ as a pair of a corresponding weighted operator mean $\mathcal{M}(t; \cdot, \cdot)$ to $\mathcal{M}(\cdot, \cdot)$ and its representing function.

- Since

$$\frac{\partial}{\partial x} f(t; x) \bigg|_{x=1} = t,$$

$\mathcal{M}(t; \cdot, \cdot)$ is a $t$-weighted operator mean.
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Some properties of weighted operator means
Definition 4.

Let \((\mathcal{M}(\cdot, \cdot), f(x)) \in \gamma\). The operator mean characterized by

- \(f(x^{-1})^{-1}\) is called the **adjoint** of \(\mathcal{M}(\cdot, \cdot)\),
- \(xf(x^{-1})\) is called the **orthogonal** of \(\mathcal{M}(\cdot, \cdot)\),
- \(xf(x)^{-1}\) is called the **dual** of \(\mathcal{M}(\cdot, \cdot)\).
**Adjoint, orthogonal and dual of operator means**

**Definition 4.**

Let \((M(\cdot, \cdot), f(x)) \in \gamma\). The operator mean characterized by
- \(f(x^{-1})^{-1}\) is called the **adjoint** of \(M(\cdot, \cdot)\),
- \(xf(x^{-1})\) is called the **orthogonal** of \(M(\cdot, \cdot)\),
- \(xf(x)^{-1}\) is called the **dual** of \(M(\cdot, \cdot)\).

Especially,

\[
M : \text{symmetric} \iff M(A, B) = M(B, A)
\]

for all \(A, B \in B(H)_+\). Also \(f(x) = xf(x^{-1})\) and \(f'(1) = \frac{1}{2}\) hold.
Adjoint of weighted operator means

Proposition 1.

Let \((\mathcal{M}(\cdot, \cdot), f(x)) \in \gamma\). For \(s \in [-1, 1] \setminus \{0\}\), put
\[g_s(x) := f(x^s)^{\frac{1}{s}}\], and let \((\mathcal{M}_s(\cdot, \cdot), g_s(x)) \in \gamma\).

Then for each \(t \in [0, 1]\),
\[g_s(t; x) = f(t; x^s)^{\frac{1}{s}}\]
Adjoint of weighted operator means

Proposition 1.

Let \((\mathcal{M}(\cdot, \cdot), f(x)) \in \gamma\). For \(s \in [-1, 1] \setminus \{0\}\), put
\[
g_s(x) := f(x^s)^{\frac{1}{s}},
\]
and let \((\mathcal{N}_s(\cdot, \cdot), g_s(x)) \in \gamma\).

Then for each \(t \in [0, 1]\),
\[
g_s(t; x) = f(t; x^s)^{\frac{1}{s}}
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Adjoint of weighted operator means

Proposition 1.

Let \((\mathcal{M}(\cdot, \cdot), f(x)) \in \gamma\). For \(s \in [-1, 1] \setminus \{0\}\), put \(g_s(x) := f(x^s)^{\frac{1}{s}}\), and let \((\mathcal{N}_s(\cdot, \cdot), g_s(x)) \in \gamma\).

Then for each \(t \in [0, 1]\),

\[ g_s(t; x) = f(t; x^s)^{\frac{1}{s}} \]

The case \(s = -1\), we have an adjoint of \(f(t; x)\)
Outline of the proof

Let \{a_n\}, \{b_n\}, \{a'_n\}, \{b'_n\} \subset [0, 1], A, B \in B(\mathcal{H})_+ and \{A_n\}, \{B_n\}, \{A'_n\}, \{B'_n\} \subset B(\mathcal{H})_+ such that

\[ a_0 = a'_0 = 0, \quad b_0 = b'_0 = 1 \quad \text{and} \quad A_0 = A'_0 = A, \quad B_0 = B'_0 = B \]
Outline of the proof

Let \( \{a_n\}, \{b_n\}, \{a'_n\}, \{b'_n\} \subset [0, 1], \ A, B \in B(\mathcal{H})_+ \) and \( \{A_n\}, \{B_n\}, \{A'_n\}, \{B'_n\} \subset B(\mathcal{H})_+ \) such that

\[
a_0 = a'_0 = 0, \ b_0 = b'_0 = 1 \quad \text{and} \quad A_0 = A'_0 = A, \ B_0 = B'_0 = B
\]

- \( (1 - f'(1))a_n + f'(1)b_n \leq t \implies \)
  \[
  \begin{cases}
  a_{n+1} := (1 - f'(1))a_n + f'(1)b_n & \text{and} \ b_{n+1} := b_n, \\
  A_{n+1} := M(A_n, B_n)
  \end{cases}
  \]

- \( (1 - g'_s(1))a'_n + g'_s(1)b'_n \leq t \implies \)
  \[
  \begin{cases}
  a'_{n+1} := (1 - g'_s(1))a'_n + g'_s(1)b_n & \text{and} \ b'_{n+1} := b'_n, \\
  A'_{n+1} := M_s(A'_n, B'_n)
  \end{cases}
  \]
Outline of the proof

- \((1 - f'(1)) a_n + f'(1) b_n \leq t \implies \)
  \[
  \begin{cases}
  a_{n+1} := (1 - f'(1)) a_n + f'(1) b_n & \text{and} \ b_{n+1} := b_n, \\
  A_{n+1} := \mathcal{M}(A_n, B_n) & \text{and} \ B_{n+1} := B_n,
  \end{cases}
  \]

- \((1 - g'(1)) a'_n + g'(1) b'_n \leq t \implies \)
  \[
  \begin{cases}
  a'_{n+1} := (1 - g'(1)) a'_n + g'(1) b'_n & \text{and} \ b'_{n+1} := b'_n, \\
  A'_{n+1} := \mathcal{N}_s(A'_n, B'_n) & \text{and} \ B'_{n+1} := B'_n,
  \end{cases}
  \]

★ For each \(n = 0, 1, \ldots, A_n, A'_n, B_n, B'_n\) are operator means.
Outline of the proof

\[ (1 - f'(1))a_n + f'(1)b_n \leq t \implies \]
\[
\begin{align*}
    a_{n+1} &:= (1 - f'(1))a_n + f'(1)b_n \\
    A_{n+1} &:= \mathcal{M}(A_n, B_n)
\end{align*}
\]
and \( b_{n+1} := b_n \), \( B_{n+1} := B_n \),

\[ (1 - g'_s(1))a'_n + g'_s(1)b'_n \leq t \implies \]
\[
\begin{align*}
    a'_{n+1} &:= (1 - g'_s(1))a'_n + g'_s(1)b_n \\
    A'_{n+1} &:= \mathcal{N}_s(A'_n, B'_n)
\end{align*}
\]
and \( b'_{n+1} := b'_n \), \( B'_{n+1} := B'_n \),

\[ \star \text{ For each } n = 0, 1, \ldots, A_n, A'_n, B_n, B'_n \text{ are operator means.} \]

\[ \star \text{ Let } f_{L,n}^{(t)}(x), g_{L,n}^{(t)}(x) f_{R,n}^{(t)}(x), g_{R,n}^{(t)}(x) \text{ be representing functions of } A_n, A'_n, B_n, B'_n, \text{ respectively.} \]
Outline of the proof

• \((1 - f'(1))a_n + f'(1)b_n \leq t \Rightarrow \)
  \[
  \begin{align*}
  a_{n+1} &:= (1 - f'(1))a_n + f'(1)b_n \\
  A_{n+1} &:= \mathcal{M}(A_n, B_n)
  \end{align*}
  \]
  and \(b_{n+1} := b_n\), and \(B_{n+1} := B_n\),

• \((1 - g'_s(1))a'_n + g'_s(1)b'_n \leq t \Rightarrow \)
  \[
  \begin{align*}
  a'_{n+1} &:= (1 - g'_s(1))a'_n + g'_s(1)b_n \\
  A'_{n+1} &:= \mathcal{M}_s(A'_n, B'_n)
  \end{align*}
  \]
  and \(b'_{n+1} := b'_n\), and \(B'_{n+1} := B'_n\),

★ Since \(f'(1) = g'_s(1)\), we have \(a_n = a'_n\), \(b_n = b'_n\), and

\[
  g^{(t)}_{L,n}(x) = f^{(t)}_{L,n}(x^s)^{\frac{1}{s}} \quad \text{and} \quad g^{(t)}_{R,n}(x) = f^{(t)}_{R,n}(x^s)^{\frac{1}{s}}.
\]

for all \(n = 0, 1, 2, \ldots\).
Outline of the proof

★ Since $f'(1) = g'_s(1)$, we have $a_n = a'_n$, $b_n = b'_n$, and

$$g_L^{(t)}(x) = f_L^{(t)}(x^s)^{\frac{1}{s}} \text{ and } g_R^{(t)}(x) = f_R^{(t)}(x^s)^{\frac{1}{s}}.$$  

for all $n = 0, 1, 2, \ldots$.

★ Hence

$$g_s(t; x) = \lim_{n \to \infty} g_L^{(t)}(x) = \lim_{n \to \infty} f_L^{(t)}(x^s)^{\frac{1}{s}} = f(t; x^s)^{\frac{1}{s}}.$$
Orthogonal and dual of operator means

Proposition 2.

Let \((\mathcal{M}(\cdot, \cdot), f(x)) \in \gamma\).

1. Put \(g(x) := xf(x^{-1})\) and let \((\mathcal{M}(\cdot, \cdot), g(x)) \in \gamma\). Then

\[ g(t; x) = xf(1 - t; x^{-1}). \]
Orthogonal and dual of operator means

Proposition 2.

Let \((\mathcal{M}(\cdot, \cdot), f(x)) \in \gamma\).

(2) Put \(h(x) := xf(x)^{-1}\) and let \((\mathfrak{I}(\cdot, \cdot), h(x)) \in \gamma\). Then

\[ h(t; x) = xf(1 - t; x)^{-1} \cdot \]

\[
\begin{align*}
 f(x) & \xrightarrow{h} g_s(x) := xf(x)^{-1} \\
 1 - t \downarrow & \quad t \downarrow \\
 f(1 - t; x) & \xrightarrow{h} g_s(t; x) = xf(1 - t; x)^{-1}
\end{align*}
\]
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2 Weighted operator means

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4 A characterization of operator interpolational means

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Some properties of weighted operator means
Operator interpolational mean

Definition 5 (J. I. Fujii, 2012)

Let $\mathcal{M} : [0, 1] \times (0, \infty)^2 \rightarrow (0, \infty)$. Then

$\mathcal{M}$: interpolational mean $\overset{def}{\iff}$

$$
\mathcal{M}((1 - t)\alpha + t\beta; a, b) = \mathcal{M}(t; \mathcal{M}(\alpha; a, b), \mathcal{M}(\beta; a, b))
$$

for all $\alpha, \beta, t \in [0, 1]$ and $a, b \in (0, \infty)$. 

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Some properties of weighted operator means
Operator interpolational mean

**Definition 5 (J. I. Fujii, 2012)**

Let $\mathcal{M}(\cdot; \cdot, \cdot)$ be a weighted operator mean. Then $\mathcal{M}$: **operator interpolational mean** $\overset{\text{def}}{\iff}$

$\mathcal{M}((1 - t)\alpha + t\beta; A, B) = \mathcal{M}(t; \mathcal{M}(\alpha; A, B), \mathcal{M}(\beta; A, B))$

for all $\alpha, \beta, t \in [0, 1]$ and $A, B \in B(\mathcal{H})_+$. 

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Some properties of weighted operator means
Characterizations of operator interpolational means


Let $\mathcal{M}(\cdot, \cdot)$ be a symmetric operator mean. $\mathcal{M}_F(\cdot; \cdot, \cdot)$ is operator interpolational if and only if

$$\mathcal{M}(\mathcal{M}(aI, bI), \mathcal{M}(cI, dI)) = \mathcal{M}(\mathcal{M}(aI, cI), \mathcal{M}(bI, dI))$$

holds for all positive numbers $a$, $b$, $c$ and $d$. 

Remark. $\mathcal{M}_F(t; \cdot, \cdot)$ is a $t$-weighted operator mean defined by J. I. Fujii, 2012. (Definition will be introduced in the later.)
Characterizations of operator interpolational means


Let $\mathcal{M}(\cdot, \cdot)$ be a symmetric operator mean. $\mathcal{M}_F(\cdot; \cdot, \cdot)$ is operator interpolational if and only if

$$\mathcal{M}(\mathcal{M}(aI, bI), \mathcal{M}(cI, dI)) = \mathcal{M}(\mathcal{M}(aI, cI), \mathcal{M}(bI, dI))$$

holds for all positive numbers $a, b, c$ and $d$.

Remark. $\mathcal{M}_F(t; \cdot, \cdot)$ is a $t$-weighted operator mean defined by J. I. Fujii, 2012. (Definition will be introduced in the later.)

Let \( (\mathcal{M}(\cdot, \cdot), f(x)) \in \gamma \), and \( \mathcal{M}(\cdot, \cdot) \) is symmetric. If \( \mathcal{M}_F(\cdot; \cdot, \cdot) \) is an operator interpolational mean, then

\[
f(\mathcal{M}(xI, yI)) = \mathcal{M}(f(x)I, f(y)I),
\]

or equivalently

\[
\mathcal{M}(xI, yI) = f^{-1}(\mathcal{M}(f(x)I, f(y)I))
\]

for all positive numbers \( x \) and \( y \).
A characterization of interpolational means

Theorem 3.

Let \( M : [0, 1] \times (0, \infty)^2 \to (0, \infty) \). Assume

(i) \( M(0; a, b) = a, M(1; a, b) = b \) and \( M(t; a, a) = a \) for all \( t \in [0, 1] \) and \( a, b \in (0, \infty) \),

(ii) for a fixed \( t \in (0, 1) \), if \( M(t; a, b) = a \) or \( b \), then \( a = b \).
A characterization of interpolational means

Theorem 3.

Let $\mathcal{M} : [0, 1] \times (0, \infty)^2 \to (0, \infty)$. Assume

(i) $\mathcal{M}(0; a, b) = a$, $\mathcal{M}(1; a, b) = b$ and $\mathcal{M}(t; a, a) = a$ for all $t \in [0, 1]$ and $a, b \in (0, \infty)$,

(ii) for a fixed $t \in (0, 1)$, if $\mathcal{M}(t; a, b) = a$ or $b$, then $a = b$.

Then the following conditions are equivalent:

1. $\mathcal{M}$: interpolational mean,
2. $\exists f; \mathcal{M}(t; a, b) = f^{-1}[(1 - t)f(a) + tf(b)]$
   for all $t \in [0, 1]$ and $a, b \in (0, \infty)$. 
Outline of the proof

(1) $\mathcal{M}$: interpolational mean,

(2) $\exists f; \mathcal{M}(t; a, b) = f^{-1}[(1 - t)f(a) + tf(b)]$

for all $t \in [0, 1]$ and $a, b \in (0, \infty)$.

Proof of (2) $\Rightarrow$ (1) is easy. To prove (1) $\Rightarrow$ (2), we prove
Outline of the proof

(1) $\mathcal{M}$: interpolational mean,

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for all $t \in [0, 1]$ and $a, b \in (0, \infty)$.

Proof of (2) $\implies$ (1) is easy. To prove (1) $\implies$ (2), we prove

$\bullet$ $M_{a,b}(t) := \mathcal{M}(t; a, b)$ is one-to-one and onto for
$t \in [0, 1]$,
Outline of the proof

(1) \( M \): interpolational mean,

(2) \( \exists f; M(t; a, b) = f^{-1}[(1 - t)f(a) + tf(b)] \)
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Proof of (2) \( \implies (1) \) is easy. To prove (1) \( \implies (2) \), we prove

- \( M_{a,b}(t) := M(t; a, b) \) is one-to-one and onto for \( t \in [0, 1] \),

- \( \exists f_{a,b}; M_{a,b}(t) = f_{a,b}^{-1}[(1 - t)f_{a,b}(a) + tf_{a,b}(b)] \)
  for all \( t \in [0, 1] \),
Outline of the proof

(1) \( \mathcal{M} \): interpolational mean,

(2) \( \exists f; \mathcal{M}(t; a, b) = f^{-1}[(1 - t)f(a) + tf(b)] \) for all \( t \in [0, 1] \) and \( a, b \in (0, \infty) \).

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- \( M_{a,b}(t) := \mathcal{M}(t; a, b) \) is one-to-one and onto for \( t \in [0, 1] \),

- \( \exists f_{a,b}; M_{a,b}(t) = f_{a,b}^{-1}[(1 - t)f_{a,b}(a) + tf_{a,b}(b)] \) for all \( t \in [0, 1] \),

- \( f_{a,b} \) is independent of \( a \) and \( b \) without affine transforms.
Corollary 4.

Let $\mathcal{M} : [0, 1] \times (0, \infty)^2 \rightarrow (0, \infty)$. Assume

$$[(1 - t)a^{-1} + tb^{-1}]^{-1} \leq \mathcal{M}(t; a, b) \leq (1 - t)a + tb$$

for all $t \in [0, 1]$ and $a, b \in (0, \infty)$.

Then the following conditions are equivalent:

(1) $\mathcal{M}$: interpolational mean,
(2) $\exists f; \mathcal{M}(t; a, b) = f^{-1}[(1 - t)f(a) + tf(b)]$

for all $t \in [0, 1], a, b \in (0, \infty)$. 
A characterization of operator interpolational means

Theorem 5.

Let \((M(t; \cdot, \cdot), f(t; x)) \in \gamma\). Assume

\[
[(1 - t)A^{-1} + tB^{-1}]^{-1} \leq M(t, A, B) \leq (1 - t)A + tB
\]

for all \(t \in [0, 1]\) and \(A, B \in B(\mathcal{H})_+\).
A characterization of operator interpolational means

Theorem 5.

Let \((M(t; \cdot, \cdot), f(t; x)) \in \gamma\). Assume

\[
[(1 - t)A^{-1} + tB^{-1}]^{-1} \leq M(t, A, B) \leq (1 - t)A + tB
\]

for all \(t \in [0, 1]\) and \(A, B \in B(\mathcal{H})_+\).

Then the following conditions are equivalent:

1. \(M\) is an operator interpolational mean,
2. \(\exists r \in [-1, 1]; f(t; x) = [(1 - t) + tx^r]^{\frac{1}{r}}\) for all \(t \in [0, 1]\) and \(x > 0\).

In the case \(r = 0\), we consider \(f(t; x) = x^t\).
Outline of the proof

(1) \( \mathcal{M} \) is an operator interpolational mean,

(2) \( \exists r \in [-1, 1]; f(t; x) = [(1 - t) + tx^r]^\frac{1}{r} \)

for all \( t \in [0, 1] \) and \( x > 0 \).

Proof of (2) \( \Rightarrow \) (1) is easy. To prove (1) \( \Rightarrow \) (2), we prove...
Outline of the proof

(1) $\mathcal{M}$ is an operator interpolational mean,

(2) $\exists r \in [-1, 1]; f(t; x) = [(1 - t) + tx']^{\frac{1}{r}}$

for all $t \in [0, 1]$ and $x > 0$.

Proof of (2) $\implies$ (1) is easy. To prove (1) $\implies$ (2), we prove

since $\mathcal{M}$ is an interpolational mean, $\exists f$;

$\mathcal{M}(t; a_l, b_l) = f^{-1}[(1 - t)f(a) + tf(b)]l,$
Outline of the proof

(1) $\mathcal{M}$ is an operator interpolational mean,

(2) $\exists r \in [-1, 1]; f(t; x) = [(1 - t) + tx^r]^\frac{1}{r}$

for all $t \in [0, 1]$ and $x > 0$.

Proof of (2) $\implies$ (1) is easy. To prove (1) $\implies$ (2), we prove

- since $\mathcal{M}$ is an interpolational mean, $\exists f$;

$$\mathcal{M}(t; al, bl) = f^{-1}[(1 - t)f(a) + tf(b)]l,$$

- since $\mathcal{M}(t; \alpha al, \alpha bl) = \alpha \mathcal{M}(t; al, bl)$ for $\alpha > 0, \exists r \in \mathbb{R}$;

$$\mathcal{M}(t; al, bl) = [(1 - t)a^r + tb^r]^\frac{1}{r}l,$$

by Hardy-Littlewood-Pólya (Inequalities, Theorem 84),

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Outline of the proof

(1) $M$ is an operator interpolational mean,

(2) $\exists r \in [-1, 1]; f(t; x) = [(1 - t) + tx']^{\frac{1}{r}}$

for all $t \in [0, 1]$ and $x > 0$.

Proof of (2) $\implies$ (1) is easy. To prove (1) $\implies$ (2), we prove

- since $M$ is an interpolational mean, $\exists f$;

\[ M(t; aI, bI) = f^{-1}[(1 - t)f(a) + tf(b)]I, \]

- since $M(t; \alpha aI, \alpha bI) = \alpha M(t; aI, bI)$ for $\alpha > 0$, $\exists r \in \mathbb{R}$;

\[ M(t; aI, bI) = [(1 - t)a^r + tb^r]^{\frac{1}{r}}I, \]

by Hardy-Littlewood-Pólya (Inequalities, Theorem 84),

- $\exists r \in [-1, 1]; f(t; x) = M(t; 1, x) = [(1 - t) + tx']^{\frac{1}{r}}$. 

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Some properties of weighted operator means
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## Definition 6 (J. I. Fujii, 2012)

Let $\mathcal{M}(\cdot, \cdot)$ be a symmetric operator mean. Then the correspondence **weighted operator mean** $\mathcal{M}_F(t; A, B)$ to $\mathcal{M}(A, B)$ is defined by the following procedure:

1. **Initial conditions:** $\mathcal{M}_F(0; A, B) := A$, $\mathcal{M}_F(1; A, B) := B$

and

$$\mathcal{M}_F \left( \frac{1}{2}; A, B \right) := \mathcal{M}(A, B),$$
Weighted operator mean

**Definition 6 (J. I. Fujii, 2012)**

Let $\mathcal{M}(\cdot, \cdot)$ be a symmetric operator mean. Then the correspondence weighted operator mean $\mathcal{M}_F(t; A, B)$ to $\mathcal{M}(A, B)$ is defined by the following procedure:

(2) Define $\mathcal{M}_F\left(\frac{2k+1}{2^{n+1}}; A, B\right)$ for $n, k \in \mathbb{N}$ with $2k + 1 < 2^{n+1}$ by the following inductive relation

\[
\mathcal{M}_F\left(\frac{2k+1}{2^{n+1}}; A, B\right) = \mathcal{M}\left(\mathcal{M}_F\left(\frac{k}{2^n}; A, B\right), \mathcal{M}_F\left(\frac{k+1}{2^n}; A, B\right)\right)
\]

\[
= \mathcal{M}\left(\mathcal{M}_F\left(\frac{k+1}{2^n}; A, B\right), \mathcal{M}_F\left(\frac{k}{2^n}; A, B\right)\right).
\]
Weighted operator mean (J. I. Fujii, 2012)

\[ m_F(0; A, B) = A \]

\[ B = m_F(1; A, B) \]

\[ m_F \left( \frac{1}{2}; A, B \right) = \mathcal{M}(A, B) \]
Weighted operator mean (J. I. Fujii, 2012)

\[ m_F(0; A, B) = A \]

\[ m_F \left( \frac{1}{4}; A, B \right) \]

\[ m_F \left( \frac{3}{4}; A, B \right) \]

\[ B = m_F(1; A, B) \]

\[ m_F \left( \frac{1}{2}; A, B \right) \]
Weighted operator mean (J. I. Fujii, 2012)
Weighted operator mean (J. I. Fujii, 2012)
Two weighted operator means

Proposition 6.

The weighted operator means by Pálfia-Patz and J. I. Fujii coincide with each other if an operator mean is symmetric.
Examples

Let \((\mathcal{M}(\cdot, \cdot), f(x)) \in \gamma\) with \(f'(1) = \frac{1}{2}\).
By \(\mathcal{M}(a, b) = af(a^{-1}b)\), we have
Examples

Let \((\mathcal{M}(\cdot, \cdot), f(x)) \in \gamma\) with \(f'(1) = \frac{1}{2}\).
By \(\mathcal{M}(a, b) = af(a^{-1}b)\), we have

\[(1) \quad f \left( \frac{1}{2}; x \right) = f(x),\]
Examples

Let \((\mathcal{M}(\cdot, \cdot), f(x)) \in \gamma\) with \(f'(1) = \frac{1}{2}\).

By \(\mathcal{M}(a, b) = af(a^{-1}b)\), we have

(1) \(f \left( \frac{1}{2}; x \right) = f(x)\),

(2) \(f \left( \frac{1}{4}; x \right) = \mathcal{M} \left( \frac{1}{2}; f(0; x), f \left( \frac{1}{2}; x \right) \right) = \mathcal{M} \left( \frac{1}{2}; 1, f \left( \frac{1}{2}; x \right) \right) = f(f(x))\),
Examples

Let \((\mathcal{M}(\cdot, \cdot), f(x)) \in \gamma\) with \(f'(1) = \frac{1}{2}\).

By \(\mathcal{M}(a, b) = af(a^{-1}b)\), we have

1. \(f \left( \frac{1}{2}; x \right) = f(x)\),
2. \(f \left( \frac{1}{4}; x \right) = \mathcal{M} \left( \frac{1}{2}; f(0; x), f \left( \frac{1}{2}; x \right) \right) = \mathcal{M} \left( \frac{1}{2}; 1, f \left( \frac{1}{2}; x \right) \right) = f(f(x))\),
3. \(f \left( \frac{3}{4}; x \right) = \mathcal{M} \left( \frac{1}{2}; f \left( \frac{1}{2}; x \right), f(1; x) \right) = \mathcal{M} \left( \frac{1}{2}; f \left( \frac{1}{2}; x \right), x \right) = f(x)f(xf(x)^{-1})\).
Examples

If $\mathcal{M}$ is an operator interpolational mean, then

$$
\mathcal{M} \left( \frac{1}{2}; f \left( \frac{1}{4}; x \right), f \left( \frac{3}{4}; x \right) \right) = \mathcal{M} \left( \frac{1}{2}; 1, x \right),
$$

holds for all $x > 0$. 

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Some properties of weighted operator means
Examples

If $M$ is an operator interpolational mean, then

$$M\left(\frac{1}{2}; f\left(\frac{1}{4}; x\right), f\left(\frac{3}{4}; x\right)\right) = M\left(\frac{1}{2}; 1, x\right),$$

i.e.,

$$f\left(\frac{1}{4}; x\right) f\left(f\left(\frac{3}{4}; x\right) f\left(\frac{1}{4}; x\right)^{-1}\right) = f(x)$$

holds for all $x > 0$. 

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Examples (Logarithmic mean)

Let $(\mathcal{M}(\cdot,\cdot), f(x)) \in \gamma$ such that $f(x) = \frac{x-1}{\log x}$, $f(1) = 1$. ($\mathcal{M}$ is called the logarithmic mean and $f'(1) = \frac{1}{2}$.)
Let \((\mathcal{M}(\cdot, \cdot), f(x)) \in \gamma\) such that \(f(x) = \frac{x-1}{\log x}, f(1) = 1\). (\(\mathcal{M}\) is called the logarithmic mean and \(f'(1) = \frac{1}{2}\).) Then
\[
\mathcal{M} \left( \frac{1}{2}; 1, 10 \right) = f \left( \frac{1}{2}, 10 \right) = f(10) = 3.90865,
\]
Examples (Logarithmic mean)

Let \((\mathcal{M}(\cdot, \cdot), f(x)) \in \gamma\) such that \(f(x) = \frac{x-1}{\log x}\), \(f(1) = 1\). (\(\mathcal{M}\) is called the logarithmic mean and \(f'(1) = \frac{1}{2}\).) Then

\[
\mathcal{M} \left( \frac{1}{2}; 1, 10 \right) = f \left( \frac{1}{2}, 10 \right) = f(10) = 3.90865,
\]

\[
\mathcal{M} \left( \frac{1}{2}; f \left( \frac{1}{4}; 10 \right), f \left( \frac{3}{4}; 10 \right) \right) = 3.9141.
\]
Examples (Logarithmic mean)

Let \((M(\cdot, \cdot), f(x)) \in \gamma\) such that \(f(x) = \frac{x-1}{\log x}, f(1) = 1.\) (\(M\) is called the logarithmic mean and \(f'(1) = \frac{1}{2}.\)) Then

\[
M \left( \frac{1}{2}; 1, 10 \right) = f \left( \frac{1}{2}, 10 \right) = f(10) = 3.90865, \\
M \left( \frac{1}{2}; f \left( \frac{1}{4}; 10 \right), f \left( \frac{3}{4}; 10 \right) \right) = 3.9141.
\]

Hence the logarithmic mean is not an interpolational mean.
Examples (Storalsky mean)

Let \((M(\cdot, \cdot), g(x)) \in \gamma\) such that \(g(x) = \frac{1}{e} x^{\frac{x}{x-1}}\), \(g(1) = 1\). (\(M\) is called the Storalsky mean and \(g'(1) = \frac{1}{2}\).)
Examples (Storalsky mean)

Let \((\mathcal{M}(\cdot, \cdot), g(x)) \in \gamma\) such that \(g(x) = \frac{1}{x} e^x \cdot x^{-1}\), \(g(1) = 1\). \((\mathcal{M}\) is called the Storalsky mean and \(g'(1) = \frac{1}{2}\).\) Then

\[
\mathcal{M}\left(\frac{1}{2}; 1, 10\right) = g\left(\frac{1}{2}, 10\right) = g(10) = 4.75135,
\]
Examples (Storalsky mean)

Let $(M(\cdot, \cdot), g(x)) \in \gamma$ such that $g(x) = \frac{1}{e} x^{\frac{x}{x-1}}$, $g(1) = 1$. ($M$ is called the Storalsky mean and $g'(1) = \frac{1}{2}$.) Then

$$M\left(\frac{1}{2}; 1, 10\right) = g\left(\frac{1}{2}, 10\right) = g(10) = 4.75135,$$

$$M\left(\frac{1}{2}; g\left(\frac{1}{4}; 10\right), g\left(\frac{3}{4}; 10\right)\right) = 4.7465.$$
Examples (Storalsky mean)

Let \((\mathcal{M}(\cdot, \cdot), g(x)) \in \gamma\) such that \(g(x) = \frac{1}{e} x^{\frac{x}{x-1}}\), \(g(1) = 1\). (\(\mathcal{M}\) is called the Storalsky mean and \(g'(1) = \frac{1}{2}\).) Then

\[
\mathcal{M}\left(\frac{1}{2}; 1, 10\right) = g\left(\frac{1}{2}, 10\right) = g(10) = 4.75135,
\]

\[
\mathcal{M}\left(\frac{1}{2}; g\left(\frac{1}{4}; 10\right), g\left(\frac{3}{4}; 10\right)\right) = 4.7465.
\]

Hence the Storalsky mean is not an interpolational mean.
Let

\[ \varphi(t; x) = \sqrt{x} \frac{t\sqrt{x} + (1 - t)}{t + (1 - t)\sqrt{x}}. \]
Barbour path (Kubo-Nakamura-Ohno-Wada, 2011)

Let

\[ \varphi(t; x) = \sqrt{x} \frac{t \sqrt{x} + (1 - t)}{t + (1 - t) \sqrt{x}}. \]

Then for each \( t \in [0, 1] \),

- \( \varphi(t; x) \) is an operator monotone function with \( \varphi(t; 1) = 1 \).
Barbour path (Kubo-Nakamura-Ohno-Wada, 2011)

Let

$$\varphi(t; x) = \sqrt{x} \frac{t\sqrt{x} + (1 - t)}{t + (1 - t)\sqrt{x}}.$$ 

Then for each $t \in [0, 1]$,

- $\varphi(t; x)$ is an operator monotone function with $\varphi(t; 1) = 1$,

- $\varphi(0; x) = 1$, $\varphi(\frac{1}{2}; x) = \sqrt{x}$ and $\varphi(1; x) = x$,
Let

\[ \varphi(t; x) = \sqrt{x} \frac{t \sqrt{x} + (1 - t)}{t + (1 - t) \sqrt{x}}. \]

Then for each \( t \in [0, 1], \)

- \( \varphi(t; x) \) is an operator monotone function with \( \varphi(t; 1) = 1, \)
- \( \varphi(0; x) = 1, \varphi(\frac{1}{2}; x) = \sqrt{x} \) and \( \varphi(1; x) = x, \)
- \( \frac{\partial}{\partial x} \varphi(t; x) \bigg|_{x=1} = t. \)

Hence \( (\mathcal{M}_\varphi(t; \cdot, \cdot), \varphi(t; x)) \in \gamma \) and \( \mathcal{M}_\varphi(t; \cdot, \cdot) \) is a \( t \)-weighted operator mean.
Barbour path (Kubo-Nakamura-Ohno-Wada, 2011)

By easy calculation,

\[ M_\varphi \left( \frac{1}{2}; 1, x \right) = \varphi \left( \frac{1}{2}, x \right) = \sqrt{x}, \]
\[ M_\varphi \left( \frac{1}{2}; \varphi \left( \frac{1}{4}; x \right), \varphi \left( \frac{3}{4}; x \right) \right) = \sqrt{x}. \]

Hence

\[ M_\varphi \left( \frac{1}{2}; 1, x \right) = M_\varphi \left( \frac{1}{2}; \varphi \left( \frac{1}{4}; x \right), \varphi \left( \frac{3}{4}; x \right) \right) \]
Barbour path (Kubo-Nakamura-Ohno-Wada, 2011)

However

\[
M_\varphi \left( \frac{3}{4}; 1, x \right) = \varphi \left( \frac{3}{4}, x \right) = \sqrt{x} \frac{3\sqrt{x} + 1}{3 + \sqrt{x}},
\]
Barbour path (Kubo-Nakamura-Ohno-Wada, 2011)

However

\[
M_\phi \left( \frac{3}{4}; 1, x \right) = \phi \left( \frac{3}{4}, x \right) = \sqrt{x} \frac{3\sqrt{x} + 1}{3 + \sqrt{x}},
\]

\[
M_\phi \left( \frac{1}{2}; \varphi \left( \frac{1}{2}; x \right), \varphi \left( 1; x \right) \right) = \varphi \left( \frac{1}{2}; x \right) \varphi \left( \frac{1}{2}; \varphi \left( 1; x \right) \varphi \left( \frac{1}{2}; x \right)^{-1} \right) = x^{\frac{3}{4}},
\]

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Barbour path (Kubo-Nakamura-Ohno-Wada, 2011)

However

\[ M_\varphi \left( \frac{3}{4}; 1, x \right) = \varphi \left( \frac{3}{4}, x \right) = \sqrt{x} \frac{3\sqrt{x} + 1}{3 + \sqrt{x}}, \]

\[ M_\varphi \left( \frac{1}{2}; \varphi \left( \frac{1}{2}; x \right), \varphi \left( 1; x \right) \right) = \varphi \left( \frac{1}{2}; x \right) \varphi \left( \frac{1}{2}; \varphi \left( 1; x \right) \varphi \left( \frac{1}{2}; x \right)^{-1} \right) = x^{\frac{3}{4}}, \]

i.e.,

\[ M \left( \frac{1}{2}; f \left( \frac{1}{2}; x \right), f \left( 1; x \right) \right) \neq M \left( \frac{3}{4}; 1, x \right) \]

Hence \( M_\varphi(t, \cdot, \cdot) \) is not an interpolational mean.
Conclusion

- It is known two definitions of weighted operator means. (But they are the same.)
- Adjoint, orthogonal and dual of weighted operator means are obtained.
- Characterizations of interpolational and operator interpolational means are different.
- Operator interpolationl means are just only the power means.
References


Thanks for your attention!