On a generalization of the Aluthge transform in the viewpoint of the operator means

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**Introduction (Aluthge transform)**

- $B(H)$: $C^*$-algebra of all bounded linear operators on a Hilbert space

**Definition 1 (Aluthge transform).**
Let $T = U|T|$ be the polar decomposition. Then the Aluthge transform $\tilde{T}$ of $T$ is defined as follows.

$$\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$$

- $\sigma(T) = \sigma(\tilde{T})$
- If $T$ is semi-hyponormal (i.e., $|T^*| \leq |T|$), then $\tilde{T}$ is hyponormal (i.e., $|\tilde{T}^*|^2 \leq |\tilde{T}|^2$).

- Aluthge, Integral Equations Operator Theory, **13** (1990), 307-315.
**Basic properties**

- $\tilde{T}$ has an invariant subspace iff $T$ does so.
- If $T$ is a $n \times n$ matrix, then iteration of the Aluthge transform converges to a normal matrix $N$ such that $\sigma(N) = \sigma(T)$.
- $\lim_{n \to \infty} \|\tilde{T}^{(n)}\| = r(T)$, where $\tilde{T}^{(n)}$ means $n$-th iterated of the Aluthge transform.
- $\cos \sigma(T) = \bigcap_{n \in \mathbb{N}} W(\tilde{T}^{(n)})$.

- Jung, Ko, Pearcy, IEOT, **37** (2000), 437-448.
- Ando, Linear and Multilinear Algebra, **52** (2004), 281-292.
**Definition 2 (mean transform).**

Let $A, B \in B(\mathcal{H})$. Define linear mappings $B(\mathcal{H}) \to B(\mathcal{H})$ defined as

$$
\mathbb{L}_A X := AX, \quad \mathbb{R}_B X := XB \quad (X \in B(\mathcal{H})).
$$

**Remark**

- $\mathbb{L}_A$ and $\mathbb{R}_B$ commute with each other, i.e.,
  $$
  \mathbb{L}_A \mathbb{R}_B X = \mathbb{L}_A XB = AXB = \mathbb{R}_B AX = \mathbb{R}_B \mathbb{L}_A X.
  $$
- If $A$ and $B$ are positive semi-definite (or positive invertible), then
  $$
  (\mathbb{L}_A)^\alpha = \mathbb{L}_A^\alpha, \quad (\mathbb{R}_B)^\alpha = \mathbb{R}_B^\alpha \quad (\alpha > 0 \text{ or } \alpha \in \mathbb{R}).
  $$
- Geometric mean
  $$
  (\mathbb{R}_B \#_\lambda \mathbb{L}_A) X = A^\lambda XB^{1-\lambda} \quad \text{for } \lambda \in [0,1].
  $$
  Especially, $\left(\mathbb{R}_{|T|} \#_\lambda \mathbb{L}_{|T|}\right) U = |T|^\lambda U |T|^{1-\lambda}$ is the $\lambda$–Aluthge transform.

Definition 3 (Operator mean).
Let $\sigma: \mathcal{P}^2 \to \mathcal{P}$. If $\sigma$ satisfies the following conditions, then $\sigma$ is called an operator mean.
1. $\sigma(A, B) \leq \sigma(C, D)$ if $A \leq C$ and $B \leq D$,
2. $X^*\sigma(A, B)X \leq \sigma(X^*AX, X^*BX)$ for all bounded linear operator $X$,
3. $\sigma$ is upper semi-continuous on $\mathcal{P}^2$,
4. $\sigma(I, I) = I$.

Introduction

◆ ℳ: The set of all operator monotone functions on \((0, \infty)\).

**Theorem A (representing function).**
Let \(\sigma\) be an operator mean. Then \(\exists f \in \mathcal{M}\) such that \(f(1) = 1\) and
\[
\sigma(A, B) = \frac{1}{A^2} f\left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^2
\]
for all \(A, B \in \mathcal{P}\).

**Examples.** Let \(\lambda \in [0,1]\).

- Arithmetic mean: \(f(x) = 1 - \lambda + \lambda x\),
- Geometric mean: \(f(x) = x^\lambda\),
- Harmonic mean: \(f(x) = [1 - \lambda + \lambda x^{-1}]^{-1}\),
- Power mean: \(f_r(x) = [1 - \lambda + \lambda x^r]^\frac{1}{r}\) \((-1 \leq r \leq 1)\).

Definition 4 (Extension of the Aluthge transform).
Let \( T = U|T| \) be the polar decomposition. For an operator mean \( \sigma \), the extension of the Aluthge transform \( \Delta_\sigma(T) \) of \( T \) is defined as follows.

\[
\Delta_\sigma(T) = \sigma(\mathbb{R}|T|, \mathbb{L}|T|)U
\]

Examples.
- Arithmetic mean case. \( \Delta_A(T) = (1 - \lambda)U|T| + \lambda|T|U \)
  (mean transform, S.H. Lee-W.Y. Lee-Yoon, 2014.)
- Geometric mean case. \( \Delta_G(T) = |T|^\lambda U|T|^{1-\lambda} \)
  (\( \lambda \) – Aluthge transform, Furuta, 1996.)

Definition 4 (Extension of the Aluthge transform).
Let \( T = U|T| \) be the polar decomposition. For an operator mean \( \sigma \), the extension of the Aluthge transform \( \Delta_\sigma(T) \) of \( T \) is defined as follows.
\[
\Delta_\sigma(T) = \sigma(\mathbb{R}|T|, \mathbb{L}|T|)U
\]

Matrix case
Let \( T = U|T| \) be the polar decomposition and
\[
U = (u_{ij}), \quad |T| = V^* \text{diag}(s_1, \ldots, s_n)V, \quad (V: \text{unitary}).
\]
Then
\[
\Delta_\sigma(T) = V^*\{VUV^* \circ [\sigma(s_i, s_j)]\}V.
\]
Proposition 1.
Let $T \in B(\mathcal{H})$ be invertible and $T = U|T|$ be the polar decomposition, and let $H_\lambda$ be the $\lambda$-weighted harmonic mean for $\lambda \in [0,1]$. Then

$$\Delta_{H_\lambda}(T) = \int_0^\infty e^{-\lambda t|T|^{-1}} U e^{-(1-\lambda)t|T|^{-1}} dt.$$ 

The harmonic mean case for a **unilateral shift** is firstly considered by S.H. Lee, 2016.

Proof

Theorem A (Bhatia, Matrix Analysis, Theorem VII.2.3). Let $A$ and $B$ be operators whose spectra are contained in the open right half-plane and the open left-plane, respectively. Then the solution of the equation $AX - XB = Y$ can be expressed as

$$X = \int_0^\infty e^{-tA}Ye^{tB} \, dt.$$  

Proof of Proposition 1. Let $X = \Delta_{H\lambda}(T)$. Then we have

$$[(1 - \lambda)\mathbb{R}_{|T|^{-1}} + \lambda\mathbb{L}_{|T|^{-1}}]^{-1}U = X$$

It is equivalent to

$$U = [(\lambda\mathbb{L}_{|T|^{-1}} + (1 - \lambda)\mathbb{R}_{|T|^{-1}}]X$$

$$= \lambda|T|^{-1}X + (1 - \lambda)X|T|^{-1}$$

$$= (\lambda|T|^{-1})X - X(-(1 - \lambda)|T|^{-1})$$

Hence

$$\Delta_{H\lambda}(T) = \int_0^\infty e^{-\lambda t|T|^{-1}}Ue^{-(1-\lambda)t|T|^{-1}} \, dt.$$
Other means cases

Theorem 2.
Let $T \in B(\mathcal{H})$ be invertible and $T = U|T|$ be the polar decomposition and let $\sigma$ be an operator mean. Then there exists a probability measure $d\mu(\lambda)$ on $[0, 1]$, s.t.,

$$\Delta_\sigma(T) = \int_0^1 \int_0^\infty e^{-\lambda t|T|^{-1}} U e^{-(1-\lambda)t|T|^{-1}} \, dt \, d\mu(\lambda).$$

Corollary 3. $\text{tr}(\Delta_\sigma(T)) = \text{tr}(T)$.

- Spectral of $T$ and $\Delta_\sigma(T)$ do not coincide, generally.
Theorem 2.
Let $T \in B(\mathcal{H})$ be invertible and $T = U|T|$ be the polar decomposition and let $\sigma$ be an operator mean. Then there exists a probability measure $d\mu(\lambda)$ on $[0, 1]$, s.t.,

$$\Delta_{\sigma}(T) = \int_0^1 \int_0^\infty e^{-\lambda t|T|^{-1}} U e^{-(1-\lambda)t|T|^{-1}} \, dt \, d\mu(\lambda).$$

Proof. Every representing function of operator mean can be given by

$$f(x) = \int_0^1 [1 - \lambda + \lambda x^{-1}]^{-1} d\mu(\lambda)$$

for a probability measure $d\mu(\lambda)$. Hence we have

$$\Delta_{\sigma}(T) = \int_0^1 [(1 - \lambda)\mathbb{R}|T|^{-1} + \lambda \mathbb{L}|T|^{-1}]^{-1} d\mu(\lambda)U$$

$$= \int_0^1 \Delta_{H_{\lambda}}(T') \, d\mu(\lambda) = \int_0^1 \int_0^\infty e^{-\lambda t|T|^{-1}} U e^{-(1-\lambda)t|T|^{-1}} \, dt \, d\mu(\lambda).$$
Theorem 3.
Let $T \in B(H)$ and let $\sigma$ be an operator mean. Then
\[
\Delta_\sigma(T) = T \iff |T|U = U|T| \quad \text{(i.e., $T$ is quasinormal)}.
\]

Proof. $(\iff)$ is easy. We shall show $(\Rightarrow)$.
(i) Assume $T$ is invertible. Let $f$ be a representing function of $\sigma$. Then
\[
T = \Delta_\sigma(T) = \mathbb{R}_{|T|}f(\mathbb{R}_{|T|^{-1}}L_{|T|})U = f(\mathbb{R}_{|T|^{-1}}L_{|T|})T
\]
Hence
\[
f(\mathbb{R}_{|T|^{-1}}L_{|T|})^nT = T \quad \text{for } n \in \mathbb{Z}
\]
Since $f$ is an operator monotone function on $(0, \infty)$, then $f$ is analytic and there exists an inverse function $f^{-1}$. Its Laurent expansion at $x = 0$ can be obtain as follows:
\[
f^{-1}(x) = \sum_{n=-\infty}^{\infty} a_n x^n
\]
**Theorem 3.**
Let $T \in B(\mathcal{H})$ and let $\sigma$ be an operator mean. Then
\[ \Delta_\sigma(T) = T \iff |T|U = U|T| \] (i.e., $T$ is quasinormal).

Hence
\[
\mathbb{R}|T|^{-1}\|T\|T = f^{-1}\left(f\left(\mathbb{R}|T|^{-1}\|T\|\right)\right)T
\]
\[
= \sum_{n=-\infty}^{\infty} a_n f\left(\mathbb{R}|T|^{-1}\|T\|\right)^n T
\]
\[
= \sum_{n=-\infty}^{\infty} a_n T = T.
\]

That is, $T = \mathbb{R}|T|^{-1}\|T\|T = |T|T|T|^{-1}$, i.e., $|T|U = U|T|$. 

\[ f^{-1}(1) = \sum_{n=-\infty}^{\infty} a_n = 1 \]
Theorem 3.
Let $T \in B(\mathcal{H})$ and let $\sigma$ be an operator mean. Then
\[ \Delta_\sigma(T) = T \iff |T|U = U|T| \] (i.e., $T$ is quasinormal).

Proof.
(ii) Assume $T$ is non-invertible. For $\epsilon > 0$, let
\[ |T|_\epsilon := |T| + \epsilon I > 0, \quad T_\epsilon = U|T|_\epsilon \quad \text{and} \quad X_\epsilon := \mathbb{R}|T|_\epsilon^{-1} \mathbb{L}|T|_\epsilon. \]
Assume
\[ \mathbb{R}|T|_\epsilon f \left( \mathbb{R}|T|_\epsilon^{-1} \mathbb{L}|T|_\epsilon \right) U = \Delta_\sigma(T_\epsilon) \to T \quad \text{(as } \epsilon \to 0) \]
i.e., $f(X_\epsilon)U|T|_\epsilon \to T$. Then
\[ f(X_\epsilon)T = f(X_\epsilon)U|T|_\epsilon |T||T|_\epsilon^{-1} \to TU^*U = T. \]
Therefore

\[ f(X_\varepsilon)^nT \to T \text{ for } n \in \mathbb{Z} \]

By the same argument of (i) we have \( \mathbb{R}|T|_\varepsilon^{-1}\mathbb{L}|T|_\varepsilon T = X_\varepsilon T \to T \).

Since

\[
\mathbb{R}|T|_\varepsilon^{-1}\mathbb{L}|T|_\varepsilon T = (|T| + \varepsilon I)T(|T| + \varepsilon I)^{-1}
\]

\[
= (|T| + \varepsilon I)U|T|(|T| + \varepsilon I)^{-1} \to |T|UP = |T|U,
\]

We have \( |T|U = U|T| \).
Iteration (finite dimensional case)

**Theorem 4.** Let $T$ be an invertible matrix and $A$ be an non-weighted arithmetic mean. Then the sequence $\{\Delta^n_A(T)\}$ of $n$-th iterated mean transform converges to a normal matrix.

**Proof.** Let $T = U|T|$ be the polar decomposition of $T$, where $U$ is unitary. Then the mean transform of $T$ is obtained as

$$T_1 := \Delta_A(T) = \frac{|T|U + U|T|}{2}.$$ 

Here $|T_1|^2$ is given by

$$T_1^*T_1 = \frac{1}{4} (U^*|T| + |T|U^*)(|T|U + U|T|)$$

$$= \frac{1}{4} (U^*|T|U + |T|)U^*U(U^*|T|U + |T|) = \frac{1}{4} (|T| + U^*|T|U)^2.$$ 

Hence we have the polar decomposition of $T_1$.

$$T_1 = U_1|T_1| = U \cdot \frac{1}{2} (|T| + U^*|T|U).$$
Iteration (finite dimensional case)

**Theorem 4.**
Let $T$ be an invertible matrix and $A$ be an non-weighted arithmetic mean. Then the sequence $\{\Delta^n_A(T)\}$ of $n$-th iterated mean transform converges to a normal matrix.

For $T_1 = U_1 |T_1| = U \cdot \frac{1}{2} (|T| + U^* |T|U)$, we have

$$T_2 := \Delta_A(T_1) = \frac{1}{2} (|T_1|U_1 + U_1 |T_1|)$$

$$= \frac{1}{2} U (|T_1| + U^* |T_1|U)$$

$$= U \cdot \frac{1}{4} (|T| + 2U^* |T|U + (U^*)^2 |T|U^2).$$

By induction, we have the **polar decomposition of $T_n := \Delta_A(T_{n-1}) = \Delta^n_A(T)$** as follows:

$$T_n = U \cdot \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} (U^*)^k |T|U^k$$
Let \( U = V^*DV \), with unitary \( V \) and \( D = \text{diag}(e^{\theta_1i}, ..., e^{\theta_mi}) \), \((i = \sqrt{-1})\). Then

\[
T = U|T| = V^*DV|T| = V^*(D \cdot V|T|V^*)V = V^*(D \cdot P)V,
\]

where \( P := V|T|V^* > 0 \). Then

\[
T_n = U \cdot \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{k} (U^*)^k|T|U^k = V^* \left\{ D \cdot \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (D^*)^kPD^k \right\} V.
\]

Here \((D^*)^kPD^k = P \circ \left[ e^{k(\theta_j-\theta_i)i} \right] \) and

\[
\frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (D^*)^kPD^k = \left[ \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} e^{k(\theta_j-\theta_i)i} \right] \circ P = \left[ \left( \frac{1 + e^{(\theta_j-\theta_i)i}}{2} \right)^n \right] \circ P
\]
Iteration (finite dimensional case)

**Theorem 4.**
Let $T$ be an invertible matrix and $A$ be an non-weighted arithmetic mean. Then the sequence $\{\Delta^n_A(T)\}$ of $n$-th iterated mean transform converges to a normal matrix.

\[
\frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (D^*)^k PD^k = \left[ \left( \frac{1 + e^{(\theta_j - \theta_i)i}}{2} \right)^n \right] \circ P
\]

We notice that

\[
\left| \frac{1 + e^{(\theta_j - \theta_i)i}}{2} \right| < 1 \quad \text{(if } \theta_j \neq \theta_i + 2n\pi),
\]

\[
\left| \frac{1 + e^{(\theta_j - \theta_i)i}}{2} \right| = 1 \quad \text{(if } \theta_j = \theta_i + 2n\pi).
\]

Hence

\[
\lim_{n \to \infty} \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (D^*)^j PD^j = \exists P_0 \geq 0.
\]
Theorem 4.
Let $T$ be an invertible matrix and $A$ be an non-weighted arithmetic mean. Then the sequence $\{\Delta^n_A(T)\}$ of $n$-th iterated mean transform converges to a normal matrix.

$$\lim_{n \to \infty} \frac{1}{2^n} \sum_{k=1}^{n} \begin{pmatrix} n \end{pmatrix} (D^*)^j P D^j = \exists P_0 \geq 0.$$ 

Therefor

$$\lim_{n \to \infty} T_n = \lim_{n \to \infty} V^* \left\{ D \cdot \frac{1}{2^n} \sum_{k=0}^{n} \begin{pmatrix} n \end{pmatrix} (D^*)^k P D^k \right\} V = V^* D V \cdot V^* P_0 V \equiv N.$$ 

Since $\Delta_A(N) = N$, $N$ is a normal matrix.

(Quasinormal matrix is a normal matrix.)
Remarks.

- There exists a weighted shift operator s.t., \( \{ \Delta_G^n(T) \} \) does not converge, where \( G \) means a non-weighted geometric mean.

- There exists a weighted shift operator s.t., \( \{ \Delta_A^n(T) \} \) does not converge, where \( A \) means a non-weighted geometric mean by modification of the geometric mean case.

Prolem.

Let \( T \) be a matrix, and \( G \) be an non-weighted geometric mean. Then can you give another proof that \( \{ \Delta_G^n(T) \} \) converge to a normal matrix by modification of the arithmetic mean case?

Proposition 5.
Let $T$ be an invertible semi-hyponormal operator and $A$ be an non-weighted arithmetic mean. Then the sequence $\{\Delta^n_A(T)\}$ of $n$-th iterated mean transform converges to a quasinormal operator, strongly.

Proof. Let $T = U|T|$ be the polar decomposition of $T$, where $U$ is unitary. Then the polar decomposition of $T_1$ as follows:
\[
\Delta_A(T) = U_1|\Delta_A(T)| = U \cdot \frac{1}{2}(|T| + U^*|T|U), \text{ and}
\]
\[
|\Delta_A(T)^*| = U|\Delta_A(T)|U^* = \frac{1}{2}(|T| + |T^*|).
\]
Hence we have
\[
|\Delta_A(T)^*| \leq |T| \leq |\Delta_A(T)| \leq |\Delta^2_A(T)| \leq \cdots \leq |\Delta^n_A(T)| \leq \cdots,
\]
i.e., $\Delta_A(T)$ is a semi-hyponormal operator.
Proposition 5.
Let $T$ be an invertible semi-hypnormal operator and $A$ be an non-weighted arithmetic mean. Then the sequence $\{\Delta^n_A(T)\}$ of $n$-th iterated mean transform converges to a quasinormal operator, strongly.

\[|\Delta_A(T)^*| \leq |T| \leq |\Delta_A(T)| \leq |\Delta^2_A(T)| \leq \cdots \leq |\Delta_n^A(T)| \leq \cdots.\]

Since

\[\|\Delta_A(T)\| = \left\|\frac{1}{2}(|T|U + U|T|)\right\| \leq \|T\|,

We have $\|\Delta_n^A(T)\| \leq \|T\|$ for all $n = 1, 2, \ldots$ Hence $\Delta_n^A(T) = U|\Delta_n^A(T)|$ converges to an operator $T_0$. Since $\Delta_A(T_0) = T_0$ holds, $T_0$ is quasinormal.
Problem.
Let $T$ be an invertible semi-hypnormal operator and $G$ be an non-weighted geometric mean. Then does the sequence $\{\Delta_G^n(T)\}$ of $n$-th iterated mean transform converge to a quasinormal operator?
Thanks for your attention!