On numerical range of a generalization of the Aluthge transform

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Introduction (Aluthge transform)

- $B(\mathcal{H})$: $C^*$-algebra of all bounded linear operators on a Hilbert space

**Definition 1 (Aluthge transform).**

Let $T = U|T|$ be the polar decomposition. Then the Aluthge transform $\tilde{T}$ of $T$ is defined as follows.

$$\tilde{T} = |T|^{1/2}U|T|^{1/2}$$

- $\sigma(T) = \sigma(\tilde{T})$
- If $T$ is semi-hyponormal (i.e., $|T^*| \leq |T|$), then $\tilde{T}$ is hyponormal (i.e., $|\tilde{T}^*|^2 \leq |\tilde{T}|^2$).

**Aluthge, Integral Equations Operator Theory, 13 (1990), 307-315.**
Basic properties

- $\tilde{T}$ has an invariant subspace iff $T$ does so.
- If $T$ is a $n \times n$ matrix, then iteration of the Aluthge transform converges to a normal matrix $N$ such that $\sigma(N) = \sigma(T)$.
- $\lim_{n \to \infty} \|\tilde{T}^{(n)}\| = r(T),$
  where $\tilde{T}^{(n)}$ means $n$-th iterated of the Aluthge transform.
- $cos(T) = \cap_{n \in \mathbb{N}} W(\tilde{T}^{(n)})$.

Definition 2 (left and right multiplication).
Let $A, B \in B(H)$. Define linear mappings $B(H) \to B(H)$ defined as
\[
\mathbb{L}_A X := AX, \quad \mathbb{R}_B X := XB \quad (X \in B(H)).
\]

Remark

- $\mathbb{L}_A$ and $\mathbb{R}_B$ commute with each other, i.e.,
  \[
  \mathbb{L}_A \mathbb{R}_B X = \mathbb{L}_A XB = AXB = \mathbb{R}_B AX = \mathbb{R}_B \mathbb{L}_A X.
  \]
- If $A$ and $B$ are positive semi-definite (or positive invertible), then
  \[
  (\mathbb{L}_A)^\alpha = \mathbb{L}_A^\alpha, \quad (\mathbb{R}_B)^\alpha = \mathbb{R}_B^\alpha \quad (\alpha > 0 \text{ or } \alpha \in \mathcal{R}).
  \]
- Geometric mean: $(\mathbb{R}_B \#_{\lambda} \mathbb{L}_A)X = A^\lambda XB^{1-\lambda}$ for $\lambda \in [0,1]$.
  Especially, $(\mathbb{R}_{|T|} \#_{\lambda} \mathbb{L}_{|T|})U = |T|^\lambda U |T|^{1-\lambda}$ is the $\lambda$ – Aluthge transform.

Introduction

◆ $\mathcal{P}$: The set of all positive definite operators on a Hilbert space.

**Definition 3 (Operator mean).** Let $\sigma: \mathcal{P}^2 \to \mathcal{P}$. If $\sigma$ satisfies the following conditions, then $\sigma$ is called an operator mean.

1. $\sigma(A, B) \leq \sigma(C, D)$ if $A \leq C$ and $B \leq D$,
2. $X^*\sigma(A, B)X \leq \sigma(X^*AX, X^*BX)$ for all bounded linear operator $X$,
3. $\sigma$ is upper semi-continuous on $\mathcal{P}^2$,
4. $\sigma(I, I) = I$.

Introduction

◆ $\mathcal{M}$: The set of all operator monotone functions on $(0, \infty)$.

**Theorem A (representing function).**
Let $\sigma$ be an operator mean. Then $\exists f \in \mathcal{M}$ such that $f(1) = 1$ and

$$
\sigma(A, B) = A^2 f \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\frac{1}{2}}
$$

for all $A, B \in \mathcal{P}$.

**Examples.** Let $\lambda \in [0,1]$.

● Arithmetic mean: $f(x) = 1 - \lambda + \lambda x$,

● Geometric mean: $f(x) = x^\lambda$,

● Harmonic mean: $f(x) = [1 - \lambda + \lambda x^{-1}]^{-1}$,

● Power mean: $f_r(x) = [1 - \lambda + \lambda x^r]\frac{1}{r}$ ($-1 \leq r \leq 1$).

Definition 4 (Extension of the Aluthge transform).
Let $T = U|T|$ be the polar decomposition. For an operator mean $\sigma$, the extension of the Aluthge transform $\Delta_{\sigma}(T)$ of $T$ is defined as follows.

$$\Delta_{\sigma}(T) = \sigma(\mathbb{R}|T|, \mathbb{L}|T|)U$$

Examples.

- Arithmetic mean case. $\Delta_A(T) = (1 - \lambda)U|T| + \lambda|T|U$
  (mean transform, S.H. Lee-W.Y. Lee-Yoon, 2014.)

- Geometric mean case. $\Delta_G(T) = |T|^{\lambda}U|T|^{1-\lambda}$
  ($\lambda$ – Aluthge transform, Furuta, 1996.)

Definition

Definition 4 (Extension of the Aluthge transform). Let $T = U|T|$ be the polar decomposition. For an operator mean $\sigma$, the extension of the Aluthge transform $\Delta_\sigma(T)$ of $T$ is defined as follows.

$$\Delta_\sigma(T) = \sigma(\mathbb{R}|T|, \mathbb{L}|T|)U$$

Matrix case

Let $T = U|T|$ be the polar decomposition and

$$U = (u_{ij}), \quad |T| = V^* \text{diag}(s_1, \ldots, s_n)V, \quad (V: \text{unitary}).$$

Then

$$\Delta_\sigma(T) = V^*\{VUV^* \circ [\sigma(s_i, s_j)]\}V.$$
### Today’s talk

| Formula | \(\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}\) | Theorem 2 |
|---------|---------------------------------|-----------|
| Fixed Point | \(\tilde{T} = T \iff |T|U = U|T|\) (quasinormal) | Theorem 3 |
| Iteration | Converge (finite dim.) Not converge (infinite dim.) | Theorem 4 (arithmetic mean case) |
| Numerical range | \(\overline{W(\tilde{T})} \subseteq \overline{W(T)}\) | Theorems 5 and 6 |
### Concrete formula (Harmonic mean case)

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<th>$\tilde{T}$</th>
<th>$\Delta_\sigma(T)$</th>
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<td>Formula</td>
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#### Proposition 1.
Let $T \in B(H)$ be invertible and $T = U|T|$ be the polar decomposition, and let $H_\lambda$ be the $\lambda$-weighted harmonic mean for $\lambda \in [0,1]$. Then

$$
\Delta_{H_\lambda}(T) = \int_0^\infty e^{-\lambda t|T|^{-1}}Ue^{-(1-\lambda)t|T|^{-1}} dt.
$$

The harmonic mean case for a **unilateral shift** is firstly considered by S.H. Lee, 2016.

Proof

Theorem A (Bhatia, Matrix Analysis, Theorem VII.2.3). Let $A$ and $B$ be operators whose spectra are contained in the open right half-plane and the open left-plane, respectively. Then the solution of the equation $AX - XB = Y$ can be expressed as

$$X = \int_0^\infty e^{-tA}Ye^{tB} \, dt.$$ 

Proof of Proposition 1. Let $X = \Delta_{H_\lambda}(T)$. Then we have

$$\left[(1 - \lambda)\mathbb{R}|T|^{-1} + \lambda\mathbb{L}|T|^{-1}\right]^{-1}U = X$$

It is equivalent to

$$U = \left[\lambda\mathbb{L}|T|^{-1} + (1 - \lambda)\mathbb{R}|T|^{-1}\right]X$$

$$= \lambda|T|^{-1}X + (1 - \lambda)X|T|^{-1}$$

$$= (\lambda|T|^{-1})X - X(-(1 - \lambda)|T|^{-1})$$

Hence

$$\Delta_{H_\lambda}(T) = \int_0^\infty e^{-\lambda t|T|^{-1}}Ue^{-(1-\lambda)t|T|^{-1}} \, dt.$$
**Concrete formula (general case)**

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**Theorem 2.**
Let $T \in B(H)$ be invertible and $T = U|T|$ be the polar decomposition and let $\sigma$ be an operator mean. Then there exists a probability measure $d\mu(\lambda)$ on $[0, 1]$, s.t.,

$$\Delta_{\sigma}(T) = \int_{0}^{1} \int_{0}^{\infty} e^{-\lambda t|T|^{-1}} U e^{-(1-\lambda) t|T|^{-1}} dt d\mu(\lambda).$$

**Corollary 3.** $\text{tr}(\Delta_{\sigma}(T)) = \text{tr}(T)$.

- Spectral of $T$ and $\Delta_{\sigma}(T)$ do not coincide, generally.
  
Other means cases

**Theorem 2.**
Let $T \in B(H)$ be invertible and $T = U|T|$ be the polar decomposition and let $\sigma$ be an operator mean. Then there exists a probability measure $d\mu(\lambda)$ on $[0, 1]$, s.t.,

$$\Delta_{\sigma}(T) = \int_0^1 \int_0^\infty e^{-\lambda t |T|^{-1}} U e^{-(1-\lambda)t |T|^{-1}} \, dt \, d\mu(\lambda).$$

**Proof.** Every representing function of operator mean can be given by

$$f(x) = \int_0^1 [1 - \lambda + \lambda x^{-1}]^{-1} d\mu(\lambda)$$

for a probability measure $d\mu(\lambda)$. Hence we have

$$\Delta_{\sigma}(T) = \int_0^1 \left[(1 - \lambda)R_{|T|^{-1}} + \lambda L_{|T|^{-1}}\right]^{-1} d\mu(\lambda)U$$

$$= \int_0^1 \Delta_{H_\lambda}(T') \, d\mu(\lambda) = \int_0^1 \int_0^\infty e^{-\lambda t |T|^{-1}} U e^{-(1-\lambda)t |T|^{-1}} \, dt \, d\mu(\lambda).$$
Theorem 3.
Let $T ∈ B(ℋ)$ and let $σ$ be an operator mean. Then

$$Δ_σ(T) = T ⇔ |T|U = U|T| \text{ (i.e., } T \text{ is quasinormal)}.$$

Fixed Point | $\tilde{T}$ | $Δ_σ(T)$ |
---|---|---|
Fixed Point | $\tilde{T} = T ⇔ |T|U = U|T|$ (quasinormal) | Theorem 3 |
### Iteration (finite dimensional case)

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<td>Not converge (infinite dim.)</td>
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#### Theorem 4.
Let $T$ be an invertible matrix and $A$ be an non-weighted arithmetic mean. Then the sequence $\{\Delta^n_A(T)\}$ of $n$-th iterated mean transform converges to a normal matrix.
Iteration (infinite dimensional case)

Remarks.
● There exists a weighted shift operator s.t., \( \{ \Delta^n_G(T) \} \) does not converge, where \( G \) means a non-weighted geometric mean.

● There exists a weighted shift operator s.t., \( \{ \Delta^n_A(T) \} \) does not converge, where \( A \) means a non-weighted geometric mean by modification of the geometric mean case.


Problem.
Let \( T \) be a matrix, and \( G \) be an non-weighted geometric mean. Then can you give another proof that \( \{ \Delta^n_G(T) \} \) converge to a normal matrix by modification of the arithmetic mean case?

Iteration (infinite dimensional case)

- $T$: semi-hyponormal $\iff |T| \geq |T^*|$
- $T$: hyponormal $\iff |T|^2 \geq |T^*|^2$

**Proposition 5.**
Let $T$ be an invertible semi-hyponormal operator and $A$ be an non-weighted arithmetic mean. Then the sequence $\{\Delta^n_A(T)\}$ of $n$-th iterated mean transform converges to a quasinormal operator, strongly.

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<th>If $T$ is semi-hyponormal</th>
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<td>$\tilde{T}$ is hyponormal</td>
<td>$\Delta_A(T)$ is semihypo</td>
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<td>Iteration</td>
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<td>Proposition 5 (arithmetic mean case)</td>
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Numerical range

**Definition 5 (Numerical range).**
Let $T \in B(H)$. Then the **numerical range** $W(T)$ of $T$ is defined by

$$W(T) = \{ \langle Tx, x \rangle \in \mathbb{C} : |x| = 1 \}.$$ 

**Theorem B (Y. 2002, Patel-Y., 2005).**
Let $T \in B(H)$. Then $\overline{W(\Delta_{G_\lambda}(T))} \subseteq W(T)$ for $\lambda \in [0, 1]$, where $G_\lambda$ is a $\lambda$–weighted geometric mean.

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<td>Numerical range</td>
<td>$\overline{W(\tilde{T})} \subseteq \overline{W(T)}$</td>
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Numerical range

Theorem 5.
Let $T \in B(H)$. Then $\mathcal{W}(\Delta_{\sigma_1}(T)) \subseteq \mathcal{W}(\Delta_{\sigma_2}(T))$ if $\sigma_1$ and $\sigma_2$ are symmetric operator means s.t., $\sigma_1 \preceq \sigma_2$, where $\sigma_1 \preceq \sigma_2$ means a matrix $\begin{bmatrix} \sigma_1(s_is_j) \\ \sigma_2(s_is_j) \end{bmatrix}$ is positive semi-definite for any $s_1, \ldots, s_n > 0$.

- $\sigma$ is symmetric $\iff \sigma(A, B) = \sigma(B, A)$

Examples of $\sigma_1 \preceq \sigma_2$.
A: non-weighted arithmetic mean
L: logarithmic mean
G: non-weighted geometric mean
H: non-weighted harmonic mean

Then $H \preceq G \preceq L \preceq A$
Key theorems

- \( w(T) = \sup\{ |\lambda| \mid \lambda \in W(T) \} \): numerical radius

**Theorem C.**

\[
W(T) = \bigcap_{\mu \in \mathbb{C}} \{ \lambda : |\lambda - \mu| \leq w(T - \mu I) \}.
\]

**Theorem D.** Let \( T \in B(\mathcal{H}) \). Then \( w(T) \leq 1 \) is equivalent to

\[
\| T - zI \| \leq 1 + \left( 1 + |z|^2 \right)^{\frac{1}{2}} \text{ for all } z \in \mathbb{C}.
\]

**Theorem E.** Let \( H, K \) be positive operators and \( \sigma_1, \sigma_2 \) are symmetric. Then **TFAE.**

1. \( \sigma_1 \preceq \sigma_2 \)
2. \( \| \sigma_1(H,K)X \| \leq \| \sigma_2(H,K)X \| \)

for all unitarily invariant norms and \( X \in B(\mathcal{H}) \).

Numerical range

**Theorem 6.**
Let $T \in B(H)$. Then $\overline{W(\Delta_\sigma(T))} \subseteq \overline{W(T)}$ if $\sigma$ is the symmetric mean s.t., $\sigma \preceq A$ ($A$ is a non-weighted arithmetic mean).

**Examples.**
- The non-weighted harmonic mean case. We have
  \[ \overline{W(\Delta_H(T))} \subseteq \overline{W(T)} \]
  However, the weighted harmonic mean case is not shown.

- If a representing function $f$ of an operator mean $\sigma$ can be represented as
  \[ f(x) = \int_0^1 x^\lambda d\mu(\lambda) \]
  for a probability vector, then
  \[ \overline{W(\Delta_\sigma(T))} \subseteq \int_0^1 \overline{W(|T|^{1-\lambda}U|T|^{\lambda})}d\mu(\lambda) \subseteq \overline{W(T)} \]
Numerical range

**Theorem 6.**
Let $T \in B(\mathcal{H})$. Then $\overline{W(\Delta_{\sigma}(T))} \subseteq \overline{W(T)}$ if $\sigma$ is the symmetric mean s.t., $\sigma \preceq A$ ($A$ is a non-weighted arithmetic mean).

**Problem.**
Let $T \in B(\mathcal{H})$. Then does $\overline{W(\Delta_{\sigma}(T))} \subseteq \overline{W(T)}$ hold for all operator mean $\sigma$?
Thanks!

Thank you for your attention!