The induced Aluthge transformations

Takeaki Yamazaki (Toyo Univ., Japan)

The International Workshop on
Matrix Analysis and Its Applications

June 4, 2021 @ Quy Nhon (Vietnam)

This talk is based on ``T. Yamazaki, LAA 628 (2021) 1-28''.

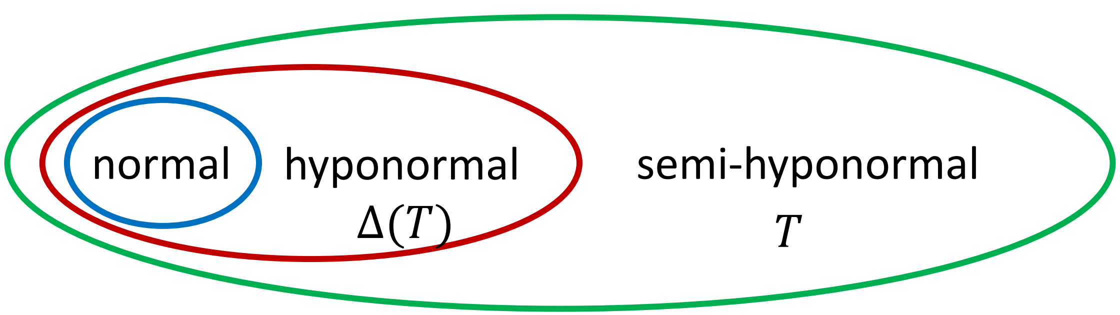
Introduction (Aluthge transformation)

- $B(\mathcal{H})$: C^*-algebra of all bounded linear operators on a Hilbert space

Definition (Aluthge transformation).
Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition. Then the Aluthge transformation $\Delta(T)$ of T is defined as follows.

$$\Delta(T) := |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$$

- $\sigma(T) = \sigma(\Delta(T))$
- If T is semi-hyponormal (i.e., $|T^*| \leq |T|$), then $\Delta(T)$ is hyponormal (i.e., $|\Delta(T)^*|^2 \leq |\Delta(T)|^2$).



- Aluthge, Integral Equations Operator Theory, **13** (1990), 307-315.

Introduction (Aluthge transformation)

Basic properties

- $\Delta(T)$ has an invariant subspace iff T does so.
- If T is a $n \times n$ matrix, then iteration of the Aluthge transformation converges to a normal matrix N such that $\sigma(N) = \sigma(T)$.
- $\lim_{n\to\infty} \|\Delta^n(T)\| = r(T)$,
  where $\Delta^n(T)$ means n-th iterated of the Aluthge transformation.
- $co\sigma(T) = \bigcap_{n\in\mathbb{N}} \overline{W(\Delta^n(T))}$.

- Jung, Ko, Pearcy, IEOT, **37** (2000), 437-448.
- Antezana, Pujals, Stojanoff, Adv. Math., **226** (2011), 1591-1620.
- Ando, Y., Linear Algebra Appl., **375** (2003), 299-309.
- Y., Proc. Amer. Math. Soc., **130** (2002), 1131-1137.
- Ando, Linear and Multilinear Algebra, **52** (2004), 281-292.

Introduction (Operator mean)

- $\mathcal{P}$: The set of all positive definite operators on a Hilbert space.

Operator mean.

Let $\mathfrak{M}: \mathcal{P}^2 \to \mathcal{P}$. If σ satisfies the following conditions, then $\mathfrak{M}$ is called an operator mean.

1. $\mathfrak{M}(A, B) \leq \mathfrak{M}(C, D)$ **if** $A \leq C$ **and** $B \leq D$,
2. $X^*\mathfrak{M}(A, B)X \leq \mathfrak{M}(X^*AX, X^*BX)$ **for all bounded linear operator** X,
3. $\mathfrak{M}$ **is upper semi-continuous on** $\mathcal{P}^2$,
4. $\mathfrak{M}(I, I) = I$.

- Kubo and Ando, Math. Ann., **246** (1980), 205-224.

Introduction

◆ $\mathcal{M}$: The set of all operator monotone functions on $(0,\infty)$.

Representing function.

Let $\mathfrak{M}$ be an operator mean. Then $\exists f \in \mathcal{M}$ such that $f(1) = 1$ and

$$\mathfrak{M}(A,B) = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}$$

for all $A, B \in \mathcal{P}$.

Examples. Let $\lambda \in [0,1]$.

- Arithmetic mean: $f(x) = 1 - \lambda + \lambda x$,
- Geometric mean: $f(x) = x^{\lambda}$,
- Harmonic mean: $f(x) = [1 - \lambda + \lambda x^{-1}]^{-1}$.

◆ $\mathfrak{M}_f$: Operator mean with a representing function f.

◆ Kubo and Ando, Math. Ann., **246** (1980), 205-224.

Motivation

- $\mathcal{M}_n$: $n \times n$ matrices
- $\mathcal{M}_n$ is a Hilbert space with inner product $\langle A, B \rangle := tr\,(AB^*)$

Left and right multiplications.

Let $T \in \mathcal{M}_n$. Define linear mappings $\mathcal{M}_n \to \mathcal{M}_n$ by

$$\mathbb{L}_T(X) := TX, \qquad \mathbb{R}_T(X) := XT \quad (X \in \mathcal{M}_n).$$

Remark

- $\mathbb{L}_T$ and $\mathbb{R}_S$ are commuting, i.e.,

$$\mathbb{L}_S\mathbb{R}_T(X) = \mathbb{L}_S(XT) = SXT = \mathbb{R}_T(SX) = \mathbb{R}_T\mathbb{L}_S(X).$$

- If T is positive semi-definite (or positive definite), then $\mathbb{L}_T$ and $\mathbb{R}_T$ are positive semi-definite (or positive definite). Especially

$$(\mathbb{L}_T)^\alpha = \mathbb{L}_{T^\alpha}, (\mathbb{R}_T)^\alpha = \mathbb{R}_{T^\alpha} \;\; (\alpha > 0 \; or \; \alpha \in \mathcal{R}).$$

- Geometric mean $(\mathbb{L}_S \,\#\, \mathbb{R}_T)(X) = S^{\frac{1}{2}}XT^{\frac{1}{2}}$.

  Especially, $(\mathbb{R}_{|T|} \,\#\, \mathbb{L}_{|T|})(U) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ **is the Aluthge transformation.**

Matrix case

Definition 1 (Induced Aluthge transformation).
Let $\mathfrak{M}_f$ be an operator mean, and $T = U|T| \in \mathcal{M}_n$ be the polar decomposition of an invertible T. Then the **induced Aluthge transformation** $\Delta_{\mathfrak{M}_f}(\mathrm{T})$ respect to an operator mean $\mathfrak{M}_f$ is defined by

$$\Delta_{\mathfrak{M}_f}(T) := \mathfrak{M}_f\left(\mathbb{L}_{|T|}, \mathbb{R}_{|T|}\right)(U) = \mathbb{L}_{|T|} f\left(\mathbb{L}_{|T|}^{-1} \mathbb{R}_{|T|}\right)(\mathbf{U}).$$

Examples.

- Arithmetic mean case. $\Delta_{\mathfrak{M}}(\mathrm{T}) = (1-\lambda)|T|U + \lambda U|T|$
  (mean transform, S.H. Lee-W.Y. Lee-Yoon, 2014.)
- Geometric mean case. $\Delta_{\mathfrak{M}}(\mathrm{T}) = |T|^{1-\lambda} U |T|^{\lambda}$
  (λ –Aluthge transform, Huruya, 1997.)

- S.H. Lee, W.Y. Lee Yoon, J. Math. Anal. Appl., **410** (2014), 70-81.
- Furuta, Proc. Amer. Math. Soc., **125** (1997), 3617-3624.

Matrix case

Definition 1 (Induced Aluthge transformation).
Let $\mathfrak{M}_f$ be an operator mean, and $T = U|T| \in \mathcal{M}_n$ be the polar decomposition of an invertible T. Then the **induced Aluthge transformation** $\Delta_{\mathfrak{M}_f}(\mathrm{T})$ respect to an operator mean $\mathfrak{M}_f$ is defined by

$$\Delta_{\mathfrak{M}_f}(T) := \mathfrak{M}_f\left(\mathbb{L}_{|T|}, \mathbb{R}_{|T|}\right)(U) = \mathbb{L}_{|T|} f\left(\mathbb{L}_{|T|}^{-1} \mathbb{R}_{|T|}\right)(\mathbf{U}).$$

Let $T = U|T|$ be the polar decomposition and

$$U = \left(u_{ij}\right), \ \ |T| = V^* diag(s_1, \dots, s_n)V, \text{ (V: unitary)}.$$

Then

$$\Delta_{\mathfrak{M}_f}(T) = V^*\left\{VUV^* \circ \left[\mathcal{P}_f\left(s_i, s_j\right)\right]\right\}V,$$

where $\mathcal{P}_f(s,t) \coloneqq sf\left(\frac{t}{s}\right)$ (perspective or solidarity).

Harmonic mean (matrix case)

Theorem 1.

Let $f(t) = \left[1-\lambda+\lambda t^{-1}\right]^{-1} (\lambda \in [0,1])$, and let $T = U|T| \in \mathcal{M}_n$ be the polar decomposition of an invertible matrix T. Then

$$\Delta_{\mathfrak{M}_f}(T) = \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}}\, dt.$$

The harmonic mean case for a **unilateral shift** is firstly considered by S.H. Lee, 2016.

◆ S.H. Lee, J. Chungcheong Math. Soc., **29** (2016), 123-135.

Proof

Theorem (Heinz).
Let A and B be operators whose spectra are contained in the open right half-plane and the open left-plane, respectively. Then the solution of the equation $AX - XB = Y$ can be expressed as

$$X = \int_0^\infty e^{-tA} Y e^{tB}\, dt.$$

Proof of Theorem 1. Let $X = \Delta_{\mathfrak{M}_\lambda}(T)$. Then we have

$$\left[(1-\lambda)\mathbb{L}_{|T|^{-1}} + \lambda\mathbb{R}_{|T|^{-1}}\right]^{-1}(U) = X.$$

It is equivalent to

$$U = \left[(1-\lambda)\mathbb{L}_{|T|^{-1}} + \lambda\mathbb{R}_{|T|^{-1}}\right](X)$$
$$= ((1-\lambda)|T|^{-1})X - X(-\lambda|T|^{-1})$$

Hence $$\Delta_{\mathfrak{M}_\lambda}(T) = \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}}\, dt.$$

- Heinz, Math. Ann., **123** (1951), 415-438.

Operator mean (matrix case)

Theorem 2.
Let $\mathfrak{M}$ be an operator mean, and $T = U|T|$ be the polar decomposition of an invertible matrix T. Then there exists a probability measure $d\mu(\lambda)$ on $[0,1]$, s.t.,

$$\Delta_{\mathfrak{M}}(T) = \int_0^1 \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}} \, dt \, d\mu(\lambda).$$

Corollary 1. $\mathrm{tr}\left(\Delta_{\mathfrak{M}}(T)\right) = \mathrm{tr}(T)$.

- Spectral of T and $\Delta_{\mathfrak{M}}(T)$ are not coincide, generally.
  Cf. S.H. Lee, W.Y. Lee Yoon, J. Math. Anal. Appl., **410** (2014), 70-81.

Operator mean (matrix case)

Theorem 2.

Let $\mathfrak{M}$ be an operator mean, and $T = U|T|$ be the polar decomposition of an invertible matrix T. Then there exists a probability measure $d\mu(\lambda)$ on $[0, 1]$, s.t.,

$$\Delta_{\mathfrak{M}}(T) = \int_0^1 \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}} \, dt d\mu(\lambda).$$

Proof. Every representing function of operator mean can be given by

$$f(x) = \int_0^1 [1 - \lambda + \lambda x^{-1}]^{-1} d\mu(\lambda)$$

for a probability measure $d\mu(\lambda)$ on $[0,1]$. Hence, we have

$$\Delta_{\mathfrak{M}_f}(T) = \int_0^1 \left[(1-\lambda)\mathbb{L}_{|T|^{-1}} + \lambda \mathbb{R}_{|T|^{-1}}\right]^{-1} d\mu(\lambda)(U)$$

Harmonic mean

$$= \int_0^1 \Delta_{\mathfrak{H}_\lambda}(T) \, d\mu(\lambda) = \int_0^1 \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}} dt \, d\mu(\lambda).$$

Another formula

Let $|T| = \sum_{i=1}^{n} s_i P_i$ be the spectral decomposition. Since

$$\mathcal{P}_f(s,t) := sf\left(\frac{t}{s}\right) = \int_0^1 [(1-\lambda)s^{-1} + \lambda t^{-1}]^{-1}\, d\mu(\lambda),$$

we have

$$\begin{aligned}
\Delta_{\mathfrak{M}_f}(T) &= \int_0^1 \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}}\, dt d\mu(\lambda) \\
&= \int_0^1 \int_0^\infty \left(\sum_i e^{-(1-\lambda)ts_i^{-1}} P_i\right) U \left(\sum_j e^{-\lambda ts_j^{-1}} P_j\right) dt d\mu(\lambda) \\
&= \sum_{i,j} \int_0^1 \int_0^\infty e^{-\{(1-\lambda)s_i^{-1} + \lambda s_j^{-1}\}t}\, dt d\mu(\lambda) P_i U P_j \\
&= \sum_{i,j} \int_0^1 \left[(1-\lambda)s_i^{-1} + \lambda\, s_j^{-1}\right]^{-1} d\mu(\lambda) P_i U P_j \\
&= \sum_{i,j} \mathcal{P}_f(s_i, s_j) P_i U P_j \longrightarrow \iint_{\sigma(|T|)^2} \mathcal{P}_f(s,t) dE_s U dE_t\ ?
\end{aligned}$$

Operator case?

Examples

Let $|T| = \sum_{i=1}^{n} s_i P_i$ be the spectral decomposition. Then

$$\Delta_{\mathfrak{M}_f}(T) = \sum_{i,j} \mathcal{P}_f\left(s_i, s_j\right) P_i U P_j \,.$$

Example.

1. Arithmetic mean: $\mathcal{P}_f(s,t) = (1-\lambda)s + \lambda t$.

$$\sum_i P_i = I$$

$$\Delta_{\mathfrak{M}_f}(T) = \sum_{i,j} [(1-\lambda)s_i + \lambda s_j] P_i U P_j$$

$$= \sum_{i,j} (1-\lambda) s_i\, P_i U P_j + \sum_{i,j} \lambda s_j\, P_i U P_j = (1-\lambda)|T|U + \lambda U|T|.$$

2. Geometric mean: $\mathcal{P}_f(s,t) = s^{1-\lambda} t^{\lambda}$.

$$\Delta_{\mathfrak{M}_f}(T) = \sum_{i,j} s_i^{1-\lambda} s_j^{\lambda} P_i U P_j = \sum_{i,j} (s_i^{1-\lambda} P_i) U (s_j^{\lambda} P_j) = |T|^{1-\lambda} U |T|^{\lambda}.$$

Examples

3. Power mean: $\mathcal{P}_f(s,t) = \left[(1-\lambda)s^{\frac{1}{2}} + \lambda t^{\frac{1}{2}}\right]^2$.

$$\begin{aligned}\Delta_{\mathfrak{M}_f}(T) &= \sum_{i,j}\left[(1-\lambda)s_i^{\frac{1}{2}} + \lambda s_j^{\frac{1}{2}}\right]^2 P_i U P_j \\ &= \sum_{i,j}\left[(1-\lambda)^2 s_i + 2\lambda(1-\lambda)s_i^{\frac{1}{2}}s_j^{\frac{1}{2}} + \lambda^2 s_j\right] P_i U P_j \\ &= (1-\lambda)^2|T|U + \lambda^2 U|T| + 2\lambda(1-\lambda)|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}.\end{aligned}$$

Especially, if $\lambda = \frac{1}{2}$, then

$$\Delta_{\mathfrak{M}_f}(T) = \frac{1}{2}\left[\frac{|T|U + U|T|}{2} + |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}\right].$$

Another formula

Let $|T| = \sum_{i=1}^{n} s_i P_i$ be the spectral decomposition. Since

$$\mathcal{P}_f(s,t) \coloneqq sf\left(\frac{t}{s}\right) = \int_0^1 [(1-\lambda)s^{-1} + \lambda t^{-1}]^{-1}\, d\mu(\lambda),$$

we have

$$\begin{aligned}
\Delta_{\mathfrak{M}_f}(T) &= \int_0^1 \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}}\, dt d\mu(\lambda) \\
&= \int_0^1 \int_0^\infty \left(\sum_i e^{-(1-\lambda)ts_i^{-1}} P_i\right) U \left(\sum_j e^{-\lambda t s_j^{-1}} P_j\right) dt d\mu(\lambda) \\
&= \sum_{i,j} \int_0^1 \int_0^\infty e^{-\left\{(1-\lambda)s_i^{-1} + \lambda s_j^{-1}\right\}t}\, dt d\mu(\lambda) P_i U P_j \\
&= \sum_{i,j} \int_0^1 \left[(1-\lambda)s_i^{-1} + \lambda\, s_j^{-1}\right]^{-1} d\mu(\lambda) P_i U P_j \\
&= \sum_{i,j} \mathcal{P}_f(s_i, s_j) P_i U P_j \longrightarrow \int\int_{\sigma(|T|)^2} \mathcal{P}_f(s,t) dE_s U dE_t\ ?
\end{aligned}$$

Operator case?

Double operator integrals

Definition.
Let $H = \int_{\sigma(H)} s dE_s$, $K = \int_{\sigma(K)} t dF_t$ be the spectral decompositions of positive definite operators. For $\varphi \in L^\infty(\sigma(H) \times \sigma(K))$ and $X \in B(\mathcal{H})$, define the double operator integrals $\Phi(X)$ as follows:

$$\Phi(X) := \int_{\sigma(H)} \int_{\sigma(K)} \varphi(s,t) dE_s X dE_t .$$

Question. For each $X \in B(\mathcal{H})$, does $\Phi(X) \in B(\mathcal{H})$ always hold?
Answer. **No!** If φ is a **Schur multiplier**, then $\Phi(X) \in B(\mathcal{H})$ holds.

Definition.
For a function $\varphi \in L^\infty(\sigma(H) \times \sigma(K))$, if $\Phi(X)\ (:= \Phi(X)|_{C_1(\mathcal{H})}) \in C_1(\mathcal{H})$, then φ is called the Schur multiplier.

- Hiai and Kosaki, Lecture Notes in Math. 1820 (2003).

Schur multiplier

Definition.
For a function $\varphi \in L^\infty(\sigma(H) \times \sigma(K))$, if $\Phi(X)\ (:= \Phi(X)|_{C_1(\mathcal{H})}) \in C_1(\mathcal{H})$, then φ is called a **Schur multiplier**.

Theorem.
Let $\varphi \in L^\infty(\sigma(H) \times \sigma(K))$. Then **TFAE**.

(1) φ is a Schur multiplier,

(2) there exists finite measure space (Ω, σ'), and there exists $\alpha \in L^\infty(\sigma(H) \times \Omega)$ and $\beta \in L^\infty(\sigma(K) \times \Omega)$ such that

$$\varphi(s,t) = \int_\Omega \alpha(s,x)\beta(t,x)\, d\sigma'(x).$$

Proposition 1.
Let $\mathfrak{M}_f$ be an operator mean with a representing function f.
Then $\mathcal{P}_f(s,t) := sf\left(\frac{t}{s}\right)$ is a Schur multiplier.

- Hiai and Kosaki, Lecture Notes in Math. 1820 (2003).
- Peller, Funct. Anal. Appl. 19 (1985) 111–123.

Proof

Proposition 1.
Let $\mathfrak{M}_f$ be an operator mean with a representing function f.
Then $\mathcal{P}_f(s,t) \coloneqq sf\left(\frac{t}{s}\right)$ is a Schur multiplier.

Proof. There exists a probably measure $d\mu$ on [0,1], s.t.,

$$f(x) = \int_0^1 [1-\lambda+\lambda x^{-1}]^{-1}\, d\mu(\lambda).$$

Then we have

$$\mathcal{P}_f(s,t) \coloneqq sf\left(\frac{t}{s}\right) = \int_0^1 [(1-\lambda)s^{-1}+\lambda t^{-1}]^{-1}\, d\mu(\lambda)$$

$$= \int_0^1 \int_0^\infty e^{-[(1-\lambda)s^{-1}+\lambda t^{-1}]x}\, dx\, d\mu(\lambda)$$

$$= \int_0^1 \int_0^\infty e^{-(1-\lambda)s^{-1}x} e^{-\lambda t^{-1}x}\, dx\, d\mu(\lambda) = \int_0^1 \int_0^\infty \alpha(s,\lambda,x)\beta(t,\lambda,x)\, dx\, d\mu(\lambda).$$

Operator case

Definition 1 (Induced Aluthge transformation).
Let $T \in B(\mathcal{H})$ be invertible with the polar decomposition $T = U|T|$. Let $|T| = \int_{\sigma(|T|)} s dE_s$ be the spectral decomposition. For each operator mean $\mathfrak{M}_f$, s.t., $f'(1) \in (0,1)$, **Induced Aluthge transformation** $\Delta_{\mathfrak{M}_f}(\mathrm{T})$ respect to an operator mean $\mathfrak{M}_f$ is defined by

$$\Delta_{\mathfrak{M}_f}(T) := \int_{\sigma(|T|)} \int_{\sigma(|T|)} \mathcal{P}_f(s,t)\, dE_s U dE_t.$$

Matrix case

$$\Delta_{\mathfrak{M}_f}(T) = \sum_{i,j} \mathcal{P}_f(s_i, s_j) P_i U P_j\,,$$

where $|T| = \sum_i s_i P_i$ is the spectral decomposition.

- If $\ker T \subseteq \ker T^*$, then $\Delta_{\mathfrak{M}_f}(T)$ can be defined. However, it is not known to define $\Delta_{\mathfrak{M}_f}(T)$, in generally.

Iteration (finite dimensional case)

$\Delta_{\mathfrak{M}}^{n}(T) := \Delta_{\mathfrak{M}}^{n-1}\big(\Delta(T)\big), \Delta_{\mathfrak{M}}^{0}(T) = T.$

Theorem 3.

Let $\mathfrak{M}$ be a non-weighted arithmetic mean, and $T = U|T|$ be the polar decomposition of an invertible matrix T. Then the sequence $\{\Delta_{\mathfrak{M}}^{n}(T)\}$ converges to a normal matrix.

Proof

Theorem 3.
Let $\mathfrak{M}$ be a non-weighted arithmetic mean, and $T = U|T|$ be the polar decomposition of an invertible matrix T. Then the sequence $\{\Delta_{\mathfrak{M}}^n(T)\}$ converges to a normal matrix.

Proof. Since U is unitary, polar decompositions are

- $\Delta_{\mathfrak{M}}(T) = \dfrac{|T|U + U|T|}{2} = U\dfrac{|T| + U^*|T|U}{2} = U|\Delta_{\mathfrak{M}}(T)|,$
- $\Delta_{\mathfrak{M}}^2(T) = \dfrac{|\Delta_{\mathfrak{M}}(T)|U + U|\Delta_{\mathfrak{M}}(T)|}{2} = U\dfrac{|\Delta_{\mathfrak{M}}(T)| + U^*|\Delta_{\mathfrak{M}}(T)|U}{2}$

$$= U\frac{|T| + 2U^*|T|U + (U^*)^2|T|U^2}{2^2} = U\left|\Delta_{\mathfrak{M}}^2(T)\right|,$$

- $\Delta_{\mathfrak{M}}^n(T) = U\dfrac{1}{2^n}\displaystyle\sum_{k=0}^{n}\binom{n}{k}(U^*)^k|T|U^k = U\left|\Delta_{\mathfrak{M}}^n(T)\right|.$

Prove convergence!

◆ Chabbabi, Curto and Mbekhta, Proc. AMS 147 (2019) 1119-1133.

Proof

- $\Delta_{\mathfrak{M}}^{n}(T) = U\,\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(U^*)^k|T|U^k = U\left|\Delta_{\mathfrak{M}}^{n}(T)\right|.$ **Prove convergence!**

Let

$$U = V^*DV, \text{where } V \text{ is unitary and } D = diag\left(e^{\theta_1\sqrt{-1}}, \ldots, e^{\theta_n\sqrt{-1}}\right).$$

For any unitary V,

$$\Delta_{\mathfrak{M}}(V^*TV) = V^*\Delta_{\mathfrak{M}}(T)V,$$

$$V\left|\Delta_{\mathfrak{M}}^{n}(T)\right|V^* = \left|\Delta_{\mathfrak{M}}^{n}(VTV^*)\right| = \left|\Delta_{\mathfrak{M}}^{n}(DV|T|V^*)\right|.$$

Proof

- $\Delta^n_{\mathfrak{M}}(T) = U\,\frac{1}{2^{\mathrm{n}}}\sum_{k=0}^{n}\binom{n}{k}(U^*)^k|T|U^k = U\left|\Delta^n_{\mathfrak{M}}(T)\right|$. **Prove convergence!**

Let

$$U = V^*DV, \text{where } V \text{ is unitary and } D = diag\left(e^{\theta_1\sqrt{-1}}, \dots, e^{\theta_n\sqrt{-1}}\right).$$

Put $P := V|T|V^*$. Then $(D^*)^k V|T|V D^k = \left[e^{k(\theta_j-\theta_i)\sqrt{-1}}\right] \circ P$ and

$$V\left|\Delta^n_{\mathfrak{M}}(T)\right|V^* = \left|\Delta^n_{\mathfrak{M}}(DV|T|V^*)\right| = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(D)^k V|T|V^*D^k$$

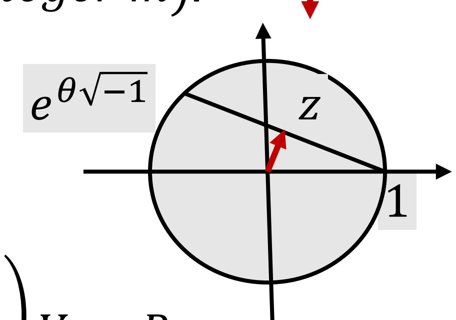
$$= \frac{1}{2^n}\left[\sum_{k=0}^{n}\binom{n}{k}e^{k(\theta_j-\theta_i)\sqrt{-1}}\right] \circ P = \left[\left(\frac{1+e^{(\theta_j-\theta_i)\sqrt{-1}}}{2}\right)^n\right] \circ P$$

Proof

$$V\left|\Delta_{\mathfrak{M}}^{n}(T)\right|V^{*} = \left|\Delta_{\mathfrak{M}}^{n}(DV|T|V^{*})\right| = \left[\left(\frac{1+e^{(\theta_j-\theta_i)\sqrt{-1}}}{2}\right)^{n}\right] \circ P$$

Notice that

$$\begin{cases} \left|\dfrac{1+e^{(\theta_j-\theta_i)\sqrt{-1}}}{2}\right| < 1 \ \ (\theta_j \neq \theta_i + 2m\pi \ for\ all\ integer\ m), \\ \dfrac{1+e^{(\theta_j-\theta_i)\sqrt{-1}}}{2} = 1 \ \ (\theta_j = \theta_i + 2m\pi \ for\ some\ integer\ m). \end{cases}$$



Hence

$$\exists \lim_{n\to\infty}\left|\Delta_{\mathfrak{M}}^{n}(T)\right| = V^{*}\left(\exists \lim_{n\to\infty}\left|\Delta_{\mathfrak{M}}^{n}(DV|T|V^{*})\right|\right)V$$
$$= V^{*}\left(\exists \lim_{n\to\infty}\left[\left(\frac{1+e^{(\theta_j-\theta_i)\sqrt{-1}}}{2}\right)^{n}\right] \circ P\right)V := P_0.$$

$$z = \frac{1+e^{\theta\sqrt{-1}}}{2}$$

Moreover

$$\lim_{n\to\infty}\Delta_{\mathfrak{M}}^{n}(T) = UP_0 = \Delta_{\mathfrak{M}}(UP_0) = \frac{P_0U+UP_0}{2} \Longrightarrow UP_0 = P_0U \ (normal).$$

Iteration (infinite dimensional case)

Theorem 4.
Let $\mathfrak{M}_f$ be an operator mean, s.t., $f'(1) \in (0,1)$. Then there exists an operator in $B(\mathcal{H})$, s.t., the sequence $\{\Delta^n_{\mathfrak{M}_f}(T)\}$ does not converge in a week operator topology.

Let $\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots) \in \ell^\infty$ and W_α be a weighted unilateral shift with a weight sequence α, i.e.,

$$W_\alpha \boldsymbol{e_n} = \alpha_n \boldsymbol{e_{n+1}},$$

where $(\boldsymbol{e_0}, \boldsymbol{e_1}, \boldsymbol{e_2}, \dots)$ is a canonical base of ℓ^2.

Sketch of a proof.

Step 1. Let

$$W_{\alpha^{(1)}} := \Delta_{\mathfrak{M}_f}(W_\alpha).$$

Then $\alpha^{(1)} := (\alpha_0^{(1)}, \alpha_1^{(1)}, \dots, \alpha_n^{(1)}, \dots)$ and $\alpha_n^{(1)} = \mathcal{P}_f(\alpha_{n+1}, \alpha_n)$.

Step 2.

- Let

$$\mathfrak{A} \cdots \textit{arithmetic mean,} \qquad \mathfrak{H} \cdots \textit{harmonic mean}$$

  with weight $f'(1)$.

- Let W_α be a weighted unilateral shift whose weights are either a or b $(a \neq b, a, b > 0)$.

- If there are only finitely many weights of W_α are equal to a.

- Then the sequence of the first weights of $\Delta_{\mathfrak{A}}^n(W_\alpha)$ and $\Delta_{\mathfrak{H}}^n(W_\alpha)$ are converge to b.

Sketch of a proof.

Step 3. Let the sequence of the first weights of $\Delta^n_{\mathfrak{A}}(W_\alpha)$ and $\Delta^n_{\mathfrak{H}}(W_\alpha)$ are $\{\mathfrak{a}_n\}$ and $\{\mathfrak{h}_n\}$.

- Let $\alpha = (a, b, b, b, \dots)$. Then there exists n_1, s.t.,

$$\left|\mathfrak{a}_{n_1} - b\right| < \frac{1}{2} \text{ and } \left|\mathfrak{h}_{n_1} - b\right| < \frac{1}{2}.$$

- Let $\alpha = (\overbrace{a, b, \dots, b}^{n_1}, a, a, \dots)$. Then there exists n_2, s.t.,

$$\left|\mathfrak{a}_{n_2} - a\right| < \frac{1}{2^2} \text{ and } \left|\mathfrak{h}_{n_2} - a\right| < \frac{1}{2^2}.$$

- Let $\alpha = (\underbrace{\overbrace{a, b, \dots, b}^{n_1}, a, \dots, a}_{n_2}, b, \dots)$. Then there exists n_3, s.t.,

$$\left|\mathfrak{a}_{n_3} - b\right| < \frac{1}{2^3} \text{ and } \left|\mathfrak{h}_{n_3} - b\right| < \frac{1}{2^3}.$$

Repeating the procedure, we obtain that there exists a weighted unilateral shift W_α such that $\{\mathfrak{a}_n\}$ and $\{\mathfrak{h}_n\}$ do not converge.

Sketch of a proof.

Step 4.

- Let $\mathfrak{M}_f$ be an arbitrary operator mean, s.t.,

$$[1 - f'(1) + f'(1)x^{-1}]^{-1} \leq f(x) \leq 1 - f'(1) + f'(1)x.$$

- Let the sequence of the first weights of $\Delta^n_{\mathfrak{M}_f}(W_\alpha)$ be $\{\mathfrak{m}_n\}$. Then by

$$\mathfrak{h}_n \leq \mathfrak{m}_n \leq \mathfrak{a}_n,$$

$\{\mathfrak{m}_n\}$ does not converge.

Problem

Problem.

Let $T \in B(\mathcal{H})$ be invertible and let $\mathfrak{M}_f$ be an operator mean , s.t., $f'(1) \in (0,1)$. Then TFAE?

(i) T **is quasinormal (i.e.,** $|T|U = U|T|$**),**

(ii) $\Delta_{\mathfrak{M}_f}(T) = T$.

- The above problem is true in the matrix case.
- **Any application or any extension is welcome.**

Thanks!

Thank you for your attention!