Stability of $\mathcal{AN}$-property for the induced Aluthge transformations

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$\mathcal{AN}(H)$-operator

- $B(H)$: C^*-algebra of all bounded linear operators on a Hilbert space

Definition (Norm attained operator).

Let $T \in B(H,K)$.

(i) T is called a norm attained operator ($T \in \mathcal{N}(H,K)$), if there exists a unit vector $x \in H$ such that $\|Tx\| = \|T\|$.

(ii) T is called an absolutely norm attained operator, if for any non-zero closed subspace $M \subseteq H$, $T|_M \in \mathcal{N}(M,K)$.

If $T \in B(H)$ is absolutely norm attained operator, we write $T \in \mathcal{AN}(H)$, T is $\mathcal{AN}(H)$-operator or T has $\mathcal{AN}$-property.

Compact operators and isometry are $\mathcal{AN}(H)$-operator.

- Carvajal and Neves, IEOT, **72** (2012), 179-195.
- Pandy and Paulsen, J Aust Math Soc, **102** (2017) 369-391.
- Ramesh, J Aust Math Soc, **96** (2014) 386-395.

Induced Aluthge transformation

- For $f(t)$, define

$$\mathcal{P}_f(s,t) := sf\left(\frac{t}{s}\right).$$

Definition (Induced Aluthge transformation).
Let $T \in B(H)$ be invertible with the polar decomposition $T = U|T|$. Let $|T| = \int_{\sigma(|T|)} s dE_s$ be the spectral decomposition. For each operator mean $\mathfrak{M}_f$ with a representing function f, the Induced Aluthge transformation $\Delta_{\mathfrak{M}_f}(T)$ respect to an operator mean $\mathfrak{M}_f$ is defined by

$$\Delta_{\mathfrak{M}_f}(T) := \int_{\sigma(|T|)} \int_{\sigma(|T|)} \mathcal{P}_f(s,t)\, dE_s U dE_t.$$

If $\ker T \subseteq \ker T^*$, then we can define $\Delta_{\mathfrak{M}_f}(T)$. However, it is not known to define $\Delta_{\mathfrak{M}_f}(T)$, in generally.

◆ Y., LAA, **628** (2021) 1-28.

Induced Aluthge transformation

Induced Aluthge transformation.

$$\Delta_{\mathfrak{M}_f}(T) := \int_{\sigma(|T|)} \int_{\sigma(|T|)} \mathcal{P}_f(s,t)\, dE_s U dE_t.$$

Example.

- $f(t) = 1 - \lambda + \lambda t$ のとき (Mean transformation).

$$\begin{aligned}\Delta_{\mathfrak{M}_f}(T) &= \int\int [(1-\lambda)s + \lambda t]\, dE_s U dE_t \\ &= (1-\lambda)\int\int s\, dE_s U dE_t + \lambda \int\int t\, dE_s U dE_t \\ &= (1-\lambda)|T|U + \lambda U|T|.\end{aligned}$$

- $f(t) = t^\lambda$ のとき (Generalized Aluthge transformation).

$$\begin{aligned}\Delta_{\mathfrak{M}_f}(T) &= \int\int s^{1-\lambda} t^\lambda\, dE_s U dE_t \\ &= \left(\int s^{1-\lambda} dE_s\right) U \left(\int t^\lambda dE_t\right) = |T|^{1-\lambda} U |T|^\lambda.\end{aligned}$$

Induced Aluthge transformation

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$$\Delta_{\mathfrak{M}_f}(T) := \int_{\sigma(|T|)} \int_{\sigma(|T|)} \mathcal{P}_f(s,t)\, dE_s U dE_t.$$

Example.

- $f(t) = \left(\frac{1+t^{1/2}}{2}\right)^2$ のとき (Power mean).

$$\begin{aligned}
\Delta_{\mathfrak{M}_f}(T) &= \int\int \left(\frac{s^{1/2}+t^{1/2}}{2}\right)^2 dE_s U dE_t \\
&= \frac{1}{4}\int\int (s + t + 2s^{1/2}t^{1/2})\, dE_s U dE_t \\
&= \frac{1}{4}\left(|T|U + U|T| + 2|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}\right).
\end{aligned}$$

Results

Theorem 1.

Let f be a non-negative positive operator monotone function on $[0,\infty)$ with $f(1)=1$. Suppose that $T \in \mathcal{AN}(H)$ is invertible.

Then $\Delta_{\mathfrak{M}_f}(T) \in \overline{\mathcal{AN}(H)}$.

Notice: $\mathcal{AN}(H) \subsetneq \overline{\mathcal{AN}(H)}$

Example.

$$T_n := \begin{pmatrix} 1/2 & & & & & & \\ & 1-1/3 & & & & & \\ & & 1-1/4 & & & & \\ & & & \ddots & & & \\ & & & & 1-1/(n+1) & & \\ & & & & & 1 & \\ & & & & & & \ddots \end{pmatrix}.$$

Then $T_n \in \mathcal{AN}(H)$ but $T_n \to T \notin \mathcal{AN}(H)$.

Results

Theorem 2.

For $\lambda \in [0,1]$, let $f_\lambda(t) = 1-\lambda+\lambda t$ and $g_\lambda(t) = t^\lambda$. If $T \in \mathcal{AN}(H)$, then $\Delta_{\mathfrak{M}_{f_\lambda}}(T), \Delta_{\mathfrak{M}_{g_\lambda}}(T) \in \mathcal{AN}(H)$. Especially, $\Delta(T) \in \mathcal{AN}(H)$.

- $f_\lambda(t) = 1-\lambda+\lambda t$ のとき、$\Delta_{\mathfrak{M}_{f_\lambda}}(T) = (1-\lambda)|T|U + \lambda U|T|$.
- $g_\lambda(t) = t^\lambda$ のとき、$\Delta_{\mathfrak{M}_{g_\lambda}}(T) = |T|^{1-\lambda}U|T|^\lambda$.

  とくに、$\Delta(T) \coloneqq \Delta_{\mathfrak{M}_{g_{\frac{1}{2}}}}(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ (Aluthge transformation).
- $S, T \in \mathcal{AN}(H)$ であっても、$S+T \in \mathcal{AN}(H)$ とは限らないため、

  $f(t) = \left(\frac{1+t^{1/2}}{2}\right)^2$, $T \in \mathcal{AN}(H)$ のときに、

$$\Delta_{\mathfrak{M}_f}(T) = \frac{1}{4}(|T|U + U|T| + 2|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}) \in \mathcal{AN}(H)$$

であるかは**不明**。

Results

T is semi-hyponormal $\Leftrightarrow |T| \geq |T^*| = u|T|u^*$. Then

$$|T| \leq u^*|T|u \leq u^{*2}|T|u^2 \leq \cdots \leq u^{*n}|T|u^n \leq \cdots \leq \|T\|I$$

Hence $L := s - \lim_{n\to\infty} u^{*n}|T|u^n$ exists.

Theorem 3.

Let $f_{1/2}(t) = \frac{1+t}{2}$ and $T \in B\,(H)$ be a semi-hyponormal operator with the polar decomposition $T = u|T|$. If $ker\,(T^*) = ker(T)$, then

$$s - \lim_{n\to\infty} \Delta^n_{m_{f_{1/2}}}(T) = uL$$

in the strong operator topology. Moreover, uL is a normal operator and $\sigma(T) = \sigma(uL)$.

$T = u|T|$ と極分解したとき、$\Delta^n_{f_{1/2}}(T)$ の極分解は次のようになる。

$$\Delta^n_{f_{1/2}}(T) = u\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} u^{*k}|T|u^k\right]$$

◆ Chabbabi, Curto and Mbekhta, Proc AMS **147** (2019) 1119-1133.

Results

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$$|T| \leq u^*|T|u \leq u^{*2}|T|u^2 \leq \cdots \leq u^{*n}|T|u^n \leq \cdots \leq \|T\|I$$

Hence $L := s - \lim_{n\to\infty} u^{*n}|T|u^n$ exists.

Theorem 4.

If $T \in \mathcal{AN}(H)$ is a semi-hyponormal operator such that

$ker\,(T^*) = ker(T)$, then $s - \lim_{n\to\infty} \Delta^n_{m_{f_{1/2}}}(T) \in \mathcal{AN}(H)$.