Recent research for the Aluthge transformations

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Introduction

$B(H)$: C^*-algebra of bounded linear operators on a Hilbert space H.
Let $T \in B(H)$.

- $\sigma(T)$: Spectrum
- $W(T) = \{\langle Tx, x\rangle;\ \|x\| = 1, x \in H\}$: Numerical range
- $r(T) = \sup\{|\lambda|; \lambda \in \sigma(T)\}$: Spectral radius
- $w(T) = \sup\{|\lambda|; \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|;\ \|x\| = 1\}$ Numerical radius
- $\|T\| = \sup\{\|Tx\|;\ \|x\| = 1\}$: Operator norm

Basic Properties.

- $co\ \sigma(T) \subseteq \overline{W(T)}$, where $co\ \sigma(T)$ is a convex hull of $\sigma(T)$.
- $r(T) \leq w(T) \leq \|T\|$ and $\frac{1}{2}\|T\| \leq w(T) \leq \|T\|$
- $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = r(T)$

Introduction (Aluthge transformation)

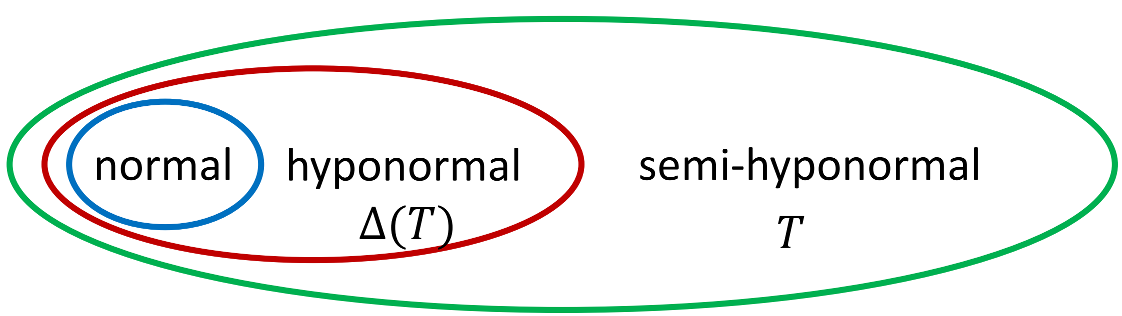
◆ $B(\mathcal{H})$: C^*-algebra of all bounded linear operators on a Hilbert space

Definition 0.1 (Aluthge transformation).

Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition. Then the Aluthge transformation $\Delta(T)$ of T is defined as follows.

$$\Delta(T) := |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$$

- $\sigma(T) = \sigma(\Delta(T))$
- If T is semi-hyponormal (i.e., $|T^*| \leq |T|$), then $\Delta(T)$ is hyponormal (i.e., $|\Delta(T)^*|^2 \leq |\Delta(T)|^2$).



◆ Aluthge, Integral Equations Operator Theory, **13** (1990), 307-315.

Introduction (Aluthge transformation)

Basic properties

- $\Delta(T)$ has an invariant subspace iff T does so.
- If T is a $n \times n$ matrix, then iteration of the Aluthge transformation converges to a normal matrix N such that $\sigma(N) = \sigma(T)$.
- $\lim_{n\to\infty} \|\Delta^n(T)\| = r(T)$,
  where $\Delta^n(T)$ means n-th iterated of the Aluthge transformation.
- $co\sigma(T) = \cap_{n\in\mathbb{N}} \overline{W(\Delta^n(T))}$.
- $\|\tilde{T}\| \le \|T\|, w(\tilde{T}) \le w(T)$ and $r(\tilde{T}) = r(T)$.

- Jung, Ko, Pearcy, IEOT, **37** (2000), 437-448.
- Antezana, Pujals, Stojanoff, Adv. Math., **226** (2011), 1591-1620.
- Ando, Y., Linear Algebra Appl., **375** (2003), 299-309.
- Y., Proc. Amer. Math. Soc., **130** (2002), 1131-1137.
- Ando, Linear and Multilinear Algebra, **52** (2004), 281-292.
- Wu, LAA, 357 (2002), 295-298.

2-Types of extensions

1) Extension to $\boldsymbol{n}$-tuple of operators

$\Longrightarrow$ **Spherical Aluthge transform**

◆ Curto, Yoon, C. R. Acad. Sci. Paris 354 (2016), 1200-1204.

2) Extension in the viewpoint of means.

- $\tilde{T}_\lambda = |T|^{1-\lambda}U|T|^\lambda$ $\lambda \in [0,1]$, $\tilde{T}_{s,t} = |T|^s U|T|^t$ $s,t \in \mathbb{R}$ (geometric mean)
- $\hat{T} = \frac{|\boldsymbol{T}|\boldsymbol{U}+\boldsymbol{U}|\boldsymbol{T}|}{\boldsymbol{2}}$ (arithmetic mean)

$\Longrightarrow$ **Induced Aluthge transform**

◆ Huruya, Proc. Amer. Math. Soc. **125** (1997), 3617–3624.
◆ Furuta, Proc. Amer. Math. Soc., **125** (1997), 3617-3624.
◆ S.H. Lee, W.Y. Lee Yoon, J. Math. Anal. Appl., **410** (2014), 70-81.

Spherical Aluthge transformations

◆ Y., Feki, Math. Inequal. Appl. 24 (2021), 405–420.

Polar decomposition

Consider

$$S = \begin{pmatrix} T_1 \\ \vdots \\ T_d \end{pmatrix}, x \in H \mapsto Sx = \begin{pmatrix} T_1 x \\ \vdots \\ T_d x \end{pmatrix} \in H \oplus \cdots \oplus H.$$

Then we have

$$P^2 := S^* S = (T_1^* \dots T_d^*) \begin{pmatrix} T_1 \\ \vdots \\ T_d \end{pmatrix} = T_1^* T_1 + \cdots + T_d^* T_d.$$

The **polar decomposition** of S is

$$\begin{pmatrix} T_1 \\ \vdots \\ T_d \end{pmatrix} = \begin{pmatrix} V_1 \\ \vdots \\ V_d \end{pmatrix} P = \begin{pmatrix} V_1 P \\ \vdots \\ V_d P \end{pmatrix}.$$

Remark.

$V = \begin{pmatrix} V_1 \\ \vdots \\ V_d \end{pmatrix}$ is a partial isometry with $\ker(V) = \ker(P)$ and

$$P = V^* V P = (V_1^* V_1 + \cdots + V_d^* V_d) P.$$

Spherical Aluthge transform

Definition 1.1 (Spherical Aluthge transform)

Let $\mathbb{T} = (T_1, \ldots, T_d) \in B(H)^d$. Then the spherical Aluthge transform $\widehat{\mathbb{T}} \in B(H)^d$ is defined by

$$\widehat{\mathbb{T}} := \left(P^{\frac{1}{2}} V_1 P^{\frac{1}{2}}, \ldots, P^{\frac{1}{2}} V_d P^{\frac{1}{2}}\right).$$

- Curto, Yoon, C. R. Acad. Sci. Paris 354 (2016), 1200-1204.

Theorem 1.A

Let $\mathbb{T} = (T_1, \ldots, T_d) \in B(H)^d$ be a commuting tuple.
Then

$$\sigma_T(\widehat{\mathbb{T}}) = \sigma_T(\mathbb{T}),$$

where σ_T is the Taylor spectrum.

- Benhida, Curto, Lee, Yoon, C. R. Math. Acad. Sci. Paris 357 (2019), 799-802.

The aim of this section.

We want to extend the following results.

- T has a non-trivial invariant subspace iff $\tilde{T}$ does so.
- $\|\tilde{T}\| \leq \|T\|$, $w(\tilde{T}) \leq w(T)$ and $r(\tilde{T}) = r(T)$
- $w(T) \leq \frac{1}{2}\|T\| + \frac{1}{2}w(\tilde{T})$
- $\lim_{n\to\infty}\|\widetilde{T_n}\| = r(T)$, where $\widetilde{T_n} := (\widetilde{\widetilde{T_{n-1}}}), \widetilde{T_0} = T$.
- There exists a normal matrix N such that $\widetilde{T_n} \to N$.

They will be Extended!

Joint Numerical Range

Let $\mathbb{T} = (T_1, \dots, T_d) \in B(H)^d$.

- **Joint Numerical Range**

  $JtW(\mathbb{T}) = \{(\langle T_1 x, x\rangle, \dots, \langle T_d x, x\rangle) \in \mathbb{C}^d;\ x \in H, \|x\| = 1\}.$

- **Joint Numerical Radius**

$$w(\mathbb{T}) = \sup\{\|\lambda\|_2\,; \lambda = (\lambda_1, \dots, \lambda_d) \in JtW(\mathbb{T})\}$$

$$= \sup\left\{\left(\sum |\langle T_i x, x\rangle|^2\right)^{\frac{1}{2}}\ ; x \in H, \|x\| = 1\right\}$$

◆ Dash, Glasnik Mat. 7 (1972), 75-81.
◆ Cho, Takaguchi, Pacific J. Math. 95 (1981), 27—35.

Joint Numerical Radius

$\mathbb{B}_d = \{\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{C}^d;\ \sum|\lambda_k|^2 \leq 1\}$ とする。

Theorem 1.B

$$\boldsymbol{w}(\mathbb{T}) = \sup_{(\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_d)\in\mathbb{B}_d} \boldsymbol{w}(\boldsymbol{\lambda}_1 \boldsymbol{T}_1 + \cdots + \boldsymbol{\lambda}_d \boldsymbol{T}_d)$$

***d*-tuple of operators**

single operator

◆ Baklouti, Feki, LAA, 557 (2018), 455-463.

Joint Spectrum

Let $\mathbb{T} = (T_1, \dots, T_d) \in B(H)^d$.

- **Taylor Spectrum** $\sigma_T(\mathbb{T}) \subset \mathbb{C}^d$.
  Definition is complicated.
  It is defined for commuting tuples.
  We can consider functional calculus on the neighborhood of $\sigma_T(\mathbb{T})$.

- **Harte Spectrum** $\sigma_H(\mathbb{T}) \subset \mathbb{C}^d$.
  It can be defined for non-commuting tuples.
  Definition is easier than the Taylor spectrum.

◆ Muller, ``Spectral theory of linear operators and spectral systems in Banach algebras'' Birkhäuser Verlag, Basel, 2007.

Joint Spectral Radius

Let $\mathbb{T} = (T_1, \dots, T_d) \in B(H)^d$ be commuting.

- **Joint (Taylor) spectral radius** is defined by

$$r(\mathbb{T}) = \sup\{\|\lambda\|_2 \ |\lambda = (\lambda_1, \dots, \lambda_d) \in \sigma_T(\mathbb{T})\}$$

Theorem 1.C

$r(\mathbb{T})$ is independent of the choice of the joint spectrum of $\mathbb{T}$.

◆ Cho-Zelasko, Hokkaido Math. J., 21 (1992), 251-258.

Joint Operator Norm

Let $\mathbb{T} = (T_1, \dots, T_d) \in B(H)^d$.

- **Joint Operator Norm**

$$\|\mathbb{T}\| = \sup\left\{\left(\sum \|T_k x\|^2\right)^{\frac{1}{2}} ;\ x \in H, \|x\| = 1\right\}$$

Theorem 1.D

$$\frac{\mathbf{1}}{\mathbf{2}\sqrt{\boldsymbol{d}}}\|\mathbb{T}\| \leq \boldsymbol{w}(\mathbb{T}) \leq \|\mathbb{T}\|$$

- Baklouti, Feki, Ahmed, LAA, 555 (2018), 266-284.
- Popescu, Memories of AMS, 200 (2009)

Spectral radius formula

Let $\mathbb{T} = (T_1, \dots, T_d) \in B(H)^{\boldsymbol{d}}, \mathbb{S} = (S_1, \dots, S_e), \in B(H)^{\boldsymbol{e}}$.

$$\mathbb{T}\mathbb{S} := (T_1S_1, \dots, T_1S_e, T_2S_1, \dots, T_2S_e, \dots, T_dS_1, \dots, T_dS_e) \in B(H)^{\boldsymbol{de}}$$

Theorem 1.E

Let $\mathbb{T}$ be a commuting tuple. Then

$$\lim_{n\to\infty} \|\mathbb{T}^n\|^{\frac{1}{n}} = r(\mathbb{T})$$

- Bunce, J. Funct. Anal., 57 (1984), 21-30.
- Muller, Soltysiak, Studia Math., 103 (1992), 329-333.

Results

Theorem 1.1

Let $\mathbb{T} = (T_1, \dots, T_d) \in B(H)^d$**. Then**

$$\|\widehat{\mathbb{T}}\| \leq \|\mathbb{T}\|.$$

Theorem 1.2

Let $\mathbb{T} = (T_1, \dots, T_d) \in B(H)^d$**. Then**

$$w(\widehat{\mathbb{T}}) \leq \frac{1}{2} w(\mathbb{T}) + \frac{1}{2} w(\mathbb{T}_1),$$

where $\mathbb{T}_1 \coloneqq (PV_1, \dots, PV_d)$.

Results

Theorem 1.2

Let $\mathbb{T} = (T_1, \dots, T_d) \in B(H)^d$. Then

$$w(\widehat{\mathbb{T}}) \le \frac{1}{2} w(\mathbb{T}) + \frac{1}{2} w(\mathbb{T}_1),$$

where $\mathbb{T}_1 := (PV_1, \dots, PV_d)$.

Proof. Let $U_\lambda = \lambda_1 V_1 + \cdots + \lambda_d V_d$, $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{B}^d$. Since $\lambda_k T_k = \lambda_k V_k P$,

$$w(\mathbb{T}) = \sup w(\lambda_1 T_1 + \cdots + \lambda_d T_d) = \sup w(U_\lambda P),$$

By Th. 1.B

$$w(\widehat{\mathbb{T}}) = \sup w(\lambda_1 P^{\frac{1}{2}} V_1 P^{\frac{1}{2}} + \cdots + \lambda_d P^{\frac{1}{2}} V_d P^{\frac{1}{2}}) = \sup w(P^{\frac{1}{2}} U_\lambda P^{\frac{1}{2}})$$

$$w(\mathbb{T}_1) = \sup w(\lambda_1 PV_1 + \cdots + \lambda_d PV_d) = \sup w(PU_\lambda)$$

We shall prove

$$w\left(P^{\frac{1}{2}} U_\lambda P^{\frac{1}{2}}\right) \le \frac{1}{2} w(U_\lambda P) + \frac{1}{2} w(PU_\lambda).$$

Single operator!

Results

Theorem 1.3

Let $\mathbb{T} = (T_1, \dots, T_d) \in B(H)^d$. If $\mathbb{T}$ is commuting tuple, then

$$w(\widehat{\mathbb{T}}) \leq w(\mathbb{T}).$$

Proof. We note that if $\mathbb{T}$ is commuting. Then

$$T_i T_j = T_j T_i \Leftrightarrow V_i P V_j P = V_j P V_i P \Leftrightarrow V_i P V_j = V_j P V_i$$

on $\mathrm{Ker}(P) \oplus \mathrm{Ker}(P)^\perp$

We shall prove

$$w(\mathbb{T}_1) \leq w(\mathbb{T}), i.e., w(PU_\lambda) \leq w(U_\lambda P)$$

$$\langle PU_\lambda x, x\rangle = \langle \left(\sum V_k^* V_k\right) PU_\lambda x, x\rangle = \sum \langle V_k PU_\lambda x, V_k x\rangle.$$

Here

$$V_k PU_\lambda = V_k P(\lambda_1 V_1 + \cdots + \lambda_d V_d)$$

$$= (\lambda_1 V_1 + \cdots + \lambda_d V_d) PV_k = U_\lambda PV_k$$

$$\boldsymbol{T_i T_j = T_j T_i}$$
$$\boldsymbol{\Leftrightarrow V_i P V_j = V_j P V_i}$$

Results

Hence

$$\langle PU_\lambda x, x\rangle = \sum \langle V_k PU_\lambda x, V_k x\rangle = \sum \langle U_\lambda P V_k x, V_k x\rangle$$

Put $y_k = \frac{V_k x}{\|V_k x\|}$. Then

$$|\langle PU_\lambda x, x\rangle| = \left| \sum \|V_k x\|^2 \langle U_\lambda P y_k, y_k\rangle \right|$$

$$\leq \sum \|V_k x\|^2 |\langle U_\lambda P y_k, y_k\rangle|$$

$$\leq \langle \sum V_k^* V_k\, x, x\rangle\, w(U_\lambda P) \leq w(U_\lambda P)$$

projection

Results

Theorem 1.4 **Let $\mathbb{T} = (T_1, \dots, T_d) \in B(H)^d$. Then**

$$\boldsymbol{w(\mathbb{T}) \leq \frac{1}{2}\|\mathbb{T}\| + \frac{1}{2} w\left(\widehat{\mathbb{T}}\right).}$$

Proof. We shall prove

$$w(U_\lambda P) \leq \frac{1}{2}\|P\| + \frac{1}{2} w(P^{\frac{1}{2}} U_\lambda P^{\frac{1}{2}})$$

Results

Theorem 1.5 Let $\mathbb{T} = (T_1, \dots, T_d) \in B(H)^d$ be a commuting tuple. Then

$$\lim_{n\to\infty} \left\|\widehat{\mathbb{T}_n}\right\| = r(\mathbb{T}).$$

Theorem 1.E

Let $\mathbb{T}$ be a commuting tuple. Then

$$\lim_{n\to\infty} \|\mathbb{T}^n\|^{\frac{1}{n}} = r(\mathbb{T})$$

- $\widehat{\mathbb{T}_n} := (\widehat{\widehat{\mathbb{T}_{n-1}}}), \widehat{\mathbb{T}_0} := \mathbb{T}$
  (n-th iteration of spherical Aluthge transform)
- $\widehat{\mathbb{T}_n}$ is a d-tuple of operators.
- $\mathbb{T}^n$ is a d^n-tuple of operators.

Induced Aluthge transformations

- Y., Linear Algebra Appl. 628 (2021), 1–28.

Introduction (Operator mean)

- $\mathcal{P}$: The set of all positive definite operators on a Hilbert space.

Definition 2.1 (Operator mean)

Let $\mathfrak{M}: \mathcal{P}^2 \to \mathcal{P}$. If σ satisfies the following conditions, then $\mathfrak{M}$ is called an operator mean.

1. $\mathfrak{M}(A, B) \leq \mathfrak{M}(C, D)$ if $A \leq C$ and $B \leq D$,

2. $X^*\mathfrak{M}(A, B)X \leq \mathfrak{M}(X^*AX, X^*BX)$ for all bounded linear operator X,

3. $\mathfrak{M}$ is upper semi-continuous on $\mathcal{P}^2$,

4. $\mathfrak{M}(I, I) = I$.

- Kubo and Ando, Math. Ann., **246** (1980), 205-224.

Introduction

◆ $\mathcal{M}$: The set of all operator monotone functions on $(0, \infty)$.

Theorem 2.A (Representing function)

Let $\mathfrak{M}$ be an operator mean. Then $\exists f \in \mathcal{M}$ such that $f(1) = 1$ and

$$\mathfrak{M}(A, B) = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}$$

for all $A, B \in \mathcal{P}$.

Examples. Let $\lambda \in [0,1]$.

- Arithmetic mean: $f(x) = 1 - \lambda + \lambda x$,
- Geometric mean: $f(x) = x^{\lambda}$,
- Harmonic mean: $f(x) = [1 - \lambda + \lambda x^{-1}]^{-1}$.

■ $\mathfrak{M}_f$: Operator mean with a representing function f.

◆ Kubo and Ando, Math. Ann., **246** (1980), 205-224.

Motivation

- $\mathcal{M}_n$: $n \times n$ matrices
- $\mathcal{M}_n$ is a Hilbert space with inner product $\langle A, B\rangle := tr\,(AB^*)$

Definition 2.2 (Left and right multiplications)

Let $T \in \mathcal{M}_n$. Define linear mappings $\mathcal{M}_n \to \mathcal{M}_n$ by

$$\mathbb{L}_T(X) := TX, \qquad \mathbb{R}_T(X) := XT \quad (X \in \mathcal{M}_n).$$

Remark

- $\mathbb{L}_T$ and $\mathbb{R}_S$ are commuting, i.e.,

$$\mathbb{L}_S\mathbb{R}_T(X) = \mathbb{L}_S(XT) = SXT = \mathbb{R}_T(SX) = \mathbb{R}_T\mathbb{L}_S(X).$$

- If T is positive semi-definite (or positive definite), then $\mathbb{L}_T$ and $\mathbb{R}_T$ are positive semi-definite (or positive definite). Especially

$$(\mathbb{L}_T)^\alpha = \mathbb{L}_{T^\alpha}, (\mathbb{R}_T)^\alpha = \mathbb{R}_{T^\alpha} \ \ (\alpha > 0 \ or \ \alpha \in \mathcal{R}).$$

- Geometric mean $(\mathbb{L}_S \# \mathbb{R}_T)(X) = S^{\frac{1}{2}}XT^{\frac{1}{2}}$.

  Especially, $\left(\mathbb{R}_{|T|} \# \mathbb{L}_{|T|}\right)(U) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ **is the Aluthge transformation.**

Matrix case

Definition 2.3 (Induced Aluthge transformation).
Let $\mathfrak{M}_f$ be an operator mean, and $T = U|T| \in \mathcal{M}_n$ be the polar decomposition of an invertible T. Then the induced Aluthge transformation $\Delta_{\mathfrak{M}_f}(\mathrm{T})$ respect to an operator mean $\mathfrak{M}_f$ is defined by

$$\Delta_{\mathfrak{M}_f}(T) := \mathfrak{M}_f\left(\mathbb{L}_{|T|}, \mathbb{R}_{|T|}\right)(U) = \mathbb{L}_{|T|} f\left(\mathbb{L}_{|T|}^{-1} \mathbb{R}_{|T|}\right)(\mathbf{U}).$$

Examples.

- Arithmetic mean case. $\Delta_{\mathfrak{M}}(\mathrm{T}) = (1-\lambda)|T|U + \lambda U|T|$
  (mean transform, S.H. Lee-W.Y. Lee-Yoon, 2014.)
- Geometric mean case. $\Delta_{\mathfrak{M}}(\mathrm{T}) = |T|^{1-\lambda} U |T|^{\lambda}$
  (λ –Aluthge transform, Huruya, 1997.)

- S.H. Lee, W.Y. Lee Yoon, J. Math. Anal. Appl., **410** (2014), 70-81.
- Furuta, Proc. Amer. Math. Soc., **125** (1997), 3617-3624.

Operator mean (matrix case)

Theorem 2.1

Let $\mathfrak{M}$ be an operator mean, and $T = U|T|$ be the polar decomposition of an invertible matrix T. Then there exists a probability measure $d\mu(\lambda)$ on $[0,1]$, s.t.,

$$\Delta_{\mathfrak{M}}(T) = \int_0^1 \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}} \, dt d\mu(\lambda).$$

Corollary 2.1 $\mathrm{tr}\big(\Delta_{\mathfrak{M}}(T)\big) = \mathrm{tr}(T)$.

- Spectral of T and $\Delta_{\mathfrak{M}}(T)$ are not coincide, generally.

  Cf. S.H. Lee, W.Y. Lee Yoon, J. Math. Anal. Appl., **410** (2014), 70-81.

Another formula

Let $|T| = \sum_{i=1}^{n} s_i P_i$ be the spectral decomposition. Since

$$\mathcal{P}_f(s,t) := sf\left(\frac{t}{s}\right) = \int_0^1 [(1-\lambda)s^{-1} + \lambda t^{-1}]^{-1}\, d\mu(\lambda),$$

we have

$$\begin{aligned}
\Delta_{\mathfrak{M}_f}(T) &= \int_0^1 \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}}\, dt d\mu(\lambda) \\
&= \int_0^1 \int_0^\infty \left(\sum_i e^{-(1-\lambda)ts_i^{-1}} P_i\right) U \left(\sum_j e^{-\lambda t s_j^{-1}} P_j\right) dt d\mu(\lambda) \\
&= \sum_{i,j} \int_0^1 \int_0^\infty e^{-\left\{(1-\lambda)s_i^{-1} + \lambda s_j^{-1}\right\}t}\, dt d\mu(\lambda) P_i U P_j \\
&= \sum_{i,j} \int_0^1 \left[(1-\lambda)s_i^{-1} + \lambda\, s_j^{-1}\right]^{-1} d\mu(\lambda) P_i U P_j \\
&= \sum_{i,j} \mathcal{P}_f(s_i, s_j) P_i U P_j \longrightarrow \int\int_{\sigma(|T|)^2} \mathcal{P}_f(s,t) dE_s U dE_t\,?
\end{aligned}$$

Operator case?

Double operator integrals

Definition 2.4

Let $H=\int_{\sigma(H)} s dE_s$, $K=\int_{\sigma(K)} t dF_t$ be the spectral decompositions of positive definite operators. For $\varphi \in L^\infty(\sigma(H)\times\sigma(K))$ and $X \in B(\mathcal{H})$, define the double operator integrals $\Phi(X)$ as follows:

$$\Phi(X) := \int_{\sigma(H)}\int_{\sigma(K)} \varphi(s,t)\, dE_s X dE_t .$$

Question. For each $X \in B(\mathcal{H})$, does $\Phi(X) \in B(\mathcal{H})$ always hold?
Answer. No! If φ is a **Schur multiplier**, then $\Phi(X) \in B(\mathcal{H})$ holds.

Definition 2.5

For a function $\varphi \in L^\infty(\sigma(H)\times\sigma(K))$, if $\Phi(X)\ (:= \Phi(X)|_{C_1(\mathcal{H})}) \in C_1(\mathcal{H})$, then φ is called the Schur multiplier.

◆ Hiai and Kosaki, Lecture Notes in Math. 1820 (2003).

Schur multiplier

Definition 2.5

For a function $\varphi \in L^\infty\big(\sigma(H) \times \sigma(K)\big)$, if $\Phi(X)\ (:= \Phi(X)|_{C_1(\mathcal{H})}) \in C_1(\mathcal{H})$, then φ is called a **Schur multiplier**.

Theorem 2.C

Let $\varphi \in L^\infty\big(\sigma(H) \times \sigma(K)\big)$. Then **TFAE**.

(1) φ is a Schur multiplier,

(2) there exists finite measure space (Ω, σ'), and there exists $\alpha \in L^\infty(\sigma(H) \times \Omega)$ and $\beta \in L^\infty(\sigma(K) \times \Omega)$ such that

$$\varphi(s,t) = \int_\Omega \alpha(s,x)\beta(t,x)\, d\sigma'(x).$$

Proposition 2.1

Let $\mathfrak{M}_f$ be an operator mean with a representing function f.

Then $\mathcal{P}_f(s,t) := sf\left(\frac{t}{s}\right)$ is a Schur multiplier.

- Hiai and Kosaki, Lecture Notes in Math. 1820 (2003).
- Peller, Funct. Anal. Appl. 19 (1985) 111–123.

Operator case

Definition 2.3 (Induced Aluthge transformation).
Let $T \in B(\mathcal{H})$ be invertible with the polar decomposition $T = U|T|$. Let $|T| = \int_{\sigma(|T|)} s dE_s$ be the spectral decomposition. For each operator mean $\mathfrak{M}_f$, s.t., $f'(1) \in (0,1)$, **Induced Aluthge transformation** $\Delta_{\mathfrak{M}_f}(\mathrm{T})$ respect to an operator mean $\mathfrak{M}_f$ is defined by

$$\Delta_{\mathfrak{M}_f}(T) := \int_{\sigma(|T|)} \int_{\sigma(|T|)} \mathcal{P}_f(s,t)\, dE_s U dE_t.$$

Matrix case

$$\Delta_{\mathfrak{M}_f}(T) = \sum_{i,j} \mathcal{P}_f(s_i, s_j) P_i U P_j\,,$$

where $|T| = \sum_i s_i P_i$ is the spectral decomposition.

- If $\ker T \subseteq \ker T^*$, then $\Delta_{\mathfrak{M}_f}(T)$ can be defined. However, it is not known to define $\Delta_{\mathfrak{M}_f}(T)$, in generally.

Examples

Let $|T| = \sum_{i=1}^{n} s_i P_i$ be the spectral decomposition. Then

$$\Delta_{\mathfrak{M}_f}(T) = \sum_{i,j} \mathcal{P}_f(s_i, s_j) P_i U P_j \,.$$

Example.

1. Arithmetic mean: $\mathcal{P}_f(s,t) = (1-\lambda)s + \lambda t$.

$$\boxed{\sum_i P_i = I}$$

$$\begin{aligned}\Delta_{\mathfrak{M}_f}(T) &= \sum_{i,j} [(1-\lambda)s_i + \lambda s_j] P_i U P_j \\ &= \sum_{i,j} (1-\lambda) s_i \, P_i U P_j + \sum_{i,j} \lambda s_j \, P_i U P_j = (1-\lambda)|T|U + \lambda U |T|.\end{aligned}$$

2. Geometric mean: $\mathcal{P}_f(s,t) = s^{1-\lambda} t^{\lambda}$.

$$\Delta_{\mathfrak{M}_f}(T) = \sum_{i,j} s_i^{1-\lambda} s_j^{\lambda} P_i U P_j = \sum_{i,j} (s_i^{1-\lambda} P_i) U (s_j^{\lambda} P_j) = |T|^{1-\lambda} U |T|^{\lambda}.$$

Examples

3. Power mean: $\mathcal{P}_f(s,t) = \left[(1-\lambda)s^{\frac{1}{2}} + \lambda t^{\frac{1}{2}}\right]^2$.

$$
\begin{aligned}
\Delta_{\mathfrak{M}_f}(T) &= \sum_{i,j} \left[(1-\lambda)s_i^{\frac{1}{2}} + \lambda s_j^{\frac{1}{2}}\right]^2 P_i U P_j \\
&= \sum_{i,j} \left[(1-\lambda)^2 s_i + 2\lambda(1-\lambda)s_i^{\frac{1}{2}} s_j^{\frac{1}{2}} + \lambda^2 s_j\right] P_i U P_j \\
&= (1-\lambda)^2 |T|U + \lambda^2 U|T| + 2\lambda(1-\lambda)|T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}.
\end{aligned}
$$

Especially, if $\lambda = \frac{1}{2}$, then

$$
\Delta_{\mathfrak{M}_f}(T) = \frac{1}{2}\left[\frac{|T|U + U|T|}{2} + |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}\right].
$$

Iteration

$\Delta_{\mathfrak{M}}^{n}(T) \coloneqq \Delta_{\mathfrak{M}}^{n-1}\big(\Delta(T)\big), \Delta_{\mathfrak{M}}^{0}(T) = T.$

Theorem 2.2 (Finite dimensional)

Let $\mathfrak{M}$ be a non-weighted arithmetic mean, and $T = U|T|$ be the polar decomposition of an invertible matrix T. Then the sequence $\{\Delta_{\mathfrak{M}}^{n}(T)\}$ converges to a normal matrix.

Theorem 2.3 (Infinite dimensional)

Let $\mathfrak{M}_f$ be an operator mean, s.t., $f'(1) \in (0, 1)$. Then there exists an operator in $B(\mathcal{H})$, s.t., the sequence $\{\Delta_{\mathfrak{M}_f}^{n}(\mathrm{T})\}$ does not converge in a week operator topology.

$\mathcal{AN}(H)$-operator

Definition 2.4 (Norm attained operator).

Let $T \in B(H,K)$.

(i) T is called a norm attained operator ($T \in \mathcal{N}(H,K)$), if there exists a unit vector $x \in H$ such that $\|Tx\| = \|T\|$.

(ii) T is called an absolutely norm attained operator, if for any non-zero closed subspace $M \subseteq H$, $T|_M \in \mathcal{N}(M,K)$.

If $T \in B(H)$ is absolutely norm attained operator, we write $T \in \mathcal{AN}(H)$, T is $\mathcal{AN}(H)$-operator or T has $\mathcal{AN}$-property.

Compact operators and isometry are $\mathcal{AN}(H)$-operator.

- Carvajal and Neves, IEOT, **72** (2012), 179-195.
- Pandy and Paulsen, J Aust Math Soc, **102** (2017) 369-391.
- Ramesh, J Aust Math Soc, **96** (2014) 386-395.

Results

Theorem 2.4

Let f be a non-negative positive operator monotone function on $[0,\infty)$ with $f(1)=1$. Suppose that $T \in \mathcal{AN}(H)$ is invertible.

Then $\Delta_{\mathfrak{M}_f}(T) \in \overline{\mathcal{AN}(H)}$.

Notice: $\mathcal{AN}(H) \subsetneq \overline{\mathcal{AN}(H)}$

Example.

$$T_n := \begin{pmatrix} 1/2 & & & & & & \\ & 1-1/3 & & & & & \\ & & 1-1/4 & & & & \\ & & & \ddots & & & \\ & & & & 1-1/(n+1) & & \\ & & & & & 1 & \\ & & & & & & \ddots \end{pmatrix}.$$

Then $T_n \in \mathcal{AN}(H)$ but $T_n \to T \notin \mathcal{AN}(H)$.

- Osaka Ramesh Udagawa Yamazaki, preprint.

Results

Theorem 2.5

For $\lambda \in [0,1]$, let $f_\lambda(t) = 1 - \lambda + \lambda t$ and $g_\lambda(t) = t^\lambda$. If $T \in \mathcal{AN}(H)$, then $\Delta_{\mathfrak{M}_{f_\lambda}}(T), \Delta_{\mathfrak{M}_{g_\lambda}}(T) \in \mathcal{AN}(H)$. Especially, $\Delta(T) \in \mathcal{AN}(H)$.

- $f_\lambda(t) = 1 - \lambda + \lambda t$ のとき、$\Delta_{\mathfrak{M}_{f_\lambda}}(T) = (1-\lambda)|T|U + \lambda U|T|$.
- $g_\lambda(t) = t^\lambda$ のとき、$\Delta_{\mathfrak{M}_{g_\lambda}}(T) = |T|^{1-\lambda}U|T|^\lambda$.
  とくに、$\Delta(T) := \Delta_{\mathfrak{M}_{g_{\frac{1}{2}}}}(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ (Aluthge transformation).
- $S, T \in \mathcal{AN}(H)$ であっても、$S + T \in \mathcal{AN}(H)$ とは限らないため、
  $f(t) = \left(\frac{1+t^{1/2}}{2}\right)^2$,$T \in \mathcal{AN}(H)$ のときに、

$$\Delta_{\mathfrak{M}_f}(T) = \frac{1}{4}(|T|U + U|T| + 2|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}) \in \mathcal{AN}(H)$$

であるかは**不明**。

Results

T is semi-hyponormal $\Leftrightarrow |T| \geq |T^*| = u|T|u^*$. Then

$$|T| \leq u^*|T|u \leq u^{*2}|T|u^2 \leq \cdots \leq u^{*n}|T|u^n \leq \cdots \leq \|T\|I$$

Hence $L := s - \lim_{n\to\infty} u^{*n}|T|u^n$ exists.

Theorem 2.6

Let $f_{1/2}(t) = \frac{1+t}{2}$ and $T \in B(H)$ be a semi-hyponormal operator with the polar decomposition $T = u|T|$. If $ker(T^*) = ker(T)$, then

$$s - \lim_{n\to\infty} \Delta^n_{m_{f_{1/2}}}(T) = uL$$

in the strong operator topology. Moreover, uL is a normal operator and $\sigma(T) = \sigma(uL)$.

$T = u|T|$ と極分解したとき、$\Delta^n_{f_{1/2}}(T)$ の極分解は次のようになる。

$$\Delta^n_{f_{1/2}}(T) = u\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} u^{*k}|T|u^k\right]$$

◆ Chabbabi, Curto and Mbekhta, Proc AMS **147** (2019) 1119-1133.

Results

T is semi-hyponormal $\Longleftrightarrow |T| \geq |T^*| = u|T|u^*$. Then

$$|T| \leq u^*|T|u \leq u^{*2}|T|u^2 \leq \cdots \leq u^{*n}|T|u^n \leq \cdots \leq \|T\|I$$

Hence $L \coloneqq s - \lim_{n\to\infty} u^{*n}|T|u^n$ exists.

Theorem 2.7

If $\boldsymbol{T} \in \mathcal{AN}(\boldsymbol{H})$ is a semi-hyponormal operator such that $\boldsymbol{ker}\ (\boldsymbol{T}^*) = \boldsymbol{ker}(\boldsymbol{T})$, then $\boldsymbol{s} - \lim_{\boldsymbol{n}\to\infty} \boldsymbol{\Delta}^{\boldsymbol{n}}_{\boldsymbol{m}_{\boldsymbol{f}_{1/2}}}(\boldsymbol{T}) \in \mathcal{AN}(\boldsymbol{H})$.

Notice: $\mathcal{AN}(H) \subsetneq \overline{\mathcal{AN}(H)}$

Thanks!

ご清聴ありがとうございました！