Extensions of the Aluthge transformations

山崎　丈明　（東洋大学）

2022年1２月1１日
作用素論・作用素環論研究集会@大阪教育大学

Introduction

$B(H)$: C^*-algebra of bounded linear operators on a Hilbert space H.
Let $T \in B(H)$.

- $\sigma(T)$: Spectrum
- $W(T) = \{\langle Tx, x\rangle;\ \|x\| = 1, x \in H\}$: Numerical range
- $r(T) = \sup\{|\lambda|; \lambda \in \sigma(T)\}$: Spectral radius
- $w(T) = \sup\{|\lambda|; \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|;\ \|x\| = 1\}$ Numerical radius
- $\|T\| = \sup\{\|Tx\|;\ \|x\| = 1\}$: Operator norm

Basic Properties.

- $co\ \sigma(T) \subseteq \overline{W(T)}$, where $co\ \sigma(T)$ is a convex hull of $\sigma(T)$.
- $r(T) \leq w(T) \leq \|T\|\ and\ \frac{1}{2}\|T\| \leq w(T) \leq \|T\|$
- $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = r(T)$

Introduction (Aluthge transformation)

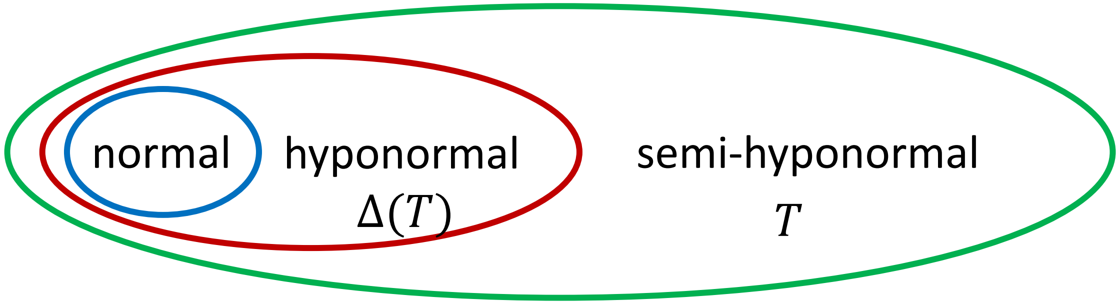
◆ $B(\mathcal{H})$: C^*-algebra of all bounded linear operators on a Hilbert space

Definition 0.1 (Aluthge transformation).

Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition. Then the Aluthge transformation $\Delta(T)$ of T is defined as follows.

$$\Delta(T) := |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$$

- $\sigma(T) = \sigma(\Delta(T))$
- If T is semi-hyponormal (i.e., $|T^*| \le |T|$), then $\Delta(T)$ is hyponormal (i.e., $|\Delta(T)^*|^2 \le |\Delta(T)|^2$).



◆ Aluthge, Integral Equations Operator Theory, **13** (1990), 307-315.

Introduction (Aluthge transformation)

Basic properties

- $\Delta(T)$ has an invariant subspace iff T does so.
- If T is a $n \times n$ matrix, then iteration of the Aluthge transformation converges to a normal matrix N such that $\sigma(N) = \sigma(T)$.
- $\lim_{n\to\infty} \|\Delta^n(T)\| = r(T)$,
  where $\Delta^n(T)$ means n-th iterated of the Aluthge transformation.
- $co\sigma(T) = \bigcap_{n\in\mathbb{N}} \overline{W(\Delta^n(T))}$.
- $\|\Delta(T)\| \leq \|T\|, w(\Delta(T)) \leq w(T)$ and $r(\Delta(T)) = r(T)$.

- Jung, Ko, Pearcy, IEOT, 37 (2000), 437-448.
- Antezana, Pujals, Stojanoff, Adv. Math., 226 (2011), 1591-1620.
- Ando, Y., Linear Algebra Appl., 375 (2003), 299-309.
- Y., Proc. Amer. Math. Soc., 130 (2002), 1131-1137.
- Ando, Linear and Multilinear Algebra, 52 (2004), 281-292.
- Wu, LAA, 357 (2002), 295-298.

2-Types of extensions

1）Extension to n-tuple of operators

⟹ **Spherical Aluthge transform**

◆ Curto, Yoon, C. R. Acad. Sci. Paris **354** (2016), 1200-1204.

2）Extension in the viewpoint of means.

- $\Delta_\lambda(T) = |T|^{1-\lambda}U|T|^{\lambda}$ $\lambda \in [0,1]$, $\Delta_{s,t}(T) = |T|^{s}U|T|^{t}$ $s,t \in \mathbb{R}$ (geometric mean)
- $\hat{T} = \frac{|T|U+U|T|}{2}$ (arithmetic mean)

⟹ **Induced Aluthge transform**

◆ Huruya, Proc. Amer. Math. Soc. 125 (1997), 3617–3624.
◆ Furuta, Proc. Amer. Math. Soc., 125 (1997), 3617-3624.
◆ S.H. Lee, W.Y. Lee Yoon, J. Math. Anal. Appl., 410 (2014), 70-81.

Spherical Aluthge transformations

◆ Y., Feki, Math. Inequal. Appl. 24 (2021), 405–420.

Polar decomposition

Consider

$$S = \begin{pmatrix} T_1 \\ \vdots \\ T_d \end{pmatrix}, x \in H \mapsto Sx = \begin{pmatrix} T_1 x \\ \vdots \\ T_d x \end{pmatrix} \in H \oplus \cdots \oplus H.$$

Then we have

$$P^2 := S^* S = (T_1^* \dots T_d^*) \begin{pmatrix} T_1 \\ \vdots \\ T_d \end{pmatrix} = T_1^* T_1 + \cdots + T_d^* T_d.$$

The **polar decomposition** of S is

$$\begin{pmatrix} T_1 \\ \vdots \\ T_d \end{pmatrix} = \begin{pmatrix} V_1 \\ \vdots \\ V_d \end{pmatrix} P = \begin{pmatrix} V_1 P \\ \vdots \\ V_d P \end{pmatrix}.$$

Remark.

$V = \begin{pmatrix} V_1 \\ \vdots \\ V_d \end{pmatrix}$ is a partial isometry with $\ker(V) = \ker(P)$ $(\ker(V) \subseteq \ker(V_k))$ and

$$P = V^* V P = (V_1^* V_1 + \cdots + V_d^* V_d) P.$$

Spherical Aluthge transform

Definition 1.1 (Spherical Aluthge transform)

Let $\mathbb{T} = (T_1, \dots, T_d) \in B(H)^d$. Then the spherical Aluthge transform $\widehat{\mathbb{T}} \in B(H)^d$ is defined by

$$\widehat{\mathbb{T}} := \left(P^{\frac{1}{2}} V_1 P^{\frac{1}{2}}, \dots, P^{\frac{1}{2}} V_d P^{\frac{1}{2}}\right).$$

- Curto, Yoon, C. R. Acad. Sci. Paris 354 (2016), 1200-1204.

Theorem 1.A

Let $\mathbb{T} = (T_1, \dots, T_d) \in B(H)^d$ be a commuting tuple.
Then

$$\sigma_T\left(\widehat{\mathbb{T}}\right) = \sigma_T(\mathbb{T}),$$

where σ_T is the Taylor spectrum.

- Benhida, Curto, Lee, Yoon, C. R. Math. Acad. Sci. Paris 357 (2019), 799-802.

The aim of this section.

We want to extend the following results.

- T has a non-trivial invariant subspace iff $\Delta(T)$ does so.
- $\|\Delta(T)\| \leq \|T\|$, $w\big(\Delta(T)\big) \leq w(T)$ and $r(\Delta(T)) = r(T)$
- $w(T) \leq \frac{1}{2}\|T\| + \frac{1}{2}w(\Delta(T))$
- $\lim_{n\to\infty}\|\Delta^n(T)\| = r(T)$, where $\Delta^n(T) := \Delta\big(\Delta^{n-1}(T)\big), \Delta^0(T) = T$.
- There exists a normal matrix N such that $\Delta^n(T) \to N$.

Joint Numerical Range

Let $\mathbb{T} = (T_1, \dots, T_d) \in B(H)^d$.

- **Joint Numerical Range**

  $JtW(\mathbb{T}) = \{(\langle T_1 x, x\rangle, \dots, \langle T_d x, x\rangle) \in \mathbb{C}^d;\ x \in H, \|x\| = 1\}.$

- **Joint Numerical Radius**

$$w(\mathbb{T}) = \sup\{\|\lambda\|_2\,; \lambda = (\lambda_1, \dots, \lambda_d) \in JtW(\mathbb{T})\}$$

$$= \sup\left\{\left(\sum |\langle T_i x, x\rangle|^2\right)^{\frac{1}{2}}\,; x \in H, \|x\| = 1\right\}$$

- Dash, Glasnik Mat. 7 (1972), 75-81.
- Cho, Takaguchi, Pacific J. Math. 95 (1981), 27—35.

Joint Numerical Radius

$\mathbb{B}_d = \{\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d;\ \sum |\lambda_k|^2 \leq 1\}$ とする。

Theorem 1.B

$$\boldsymbol{w}(\mathbb{T}) = \sup_{(\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_d) \in \mathbb{B}_d} \boldsymbol{w}(\boldsymbol{\lambda}_1 \boldsymbol{T}_1 + \cdots + \boldsymbol{\lambda}_d \boldsymbol{T}_d)$$

***d*-tuple of operators**

single operator

◆ Baklouti, Feki, LAA, 557 (2018), 455-463.

Joint Spectrum

Let $\mathbb{T} = (T_1, \dots, T_d) \in B(H)^d$.

- **Taylor Spectrum** $\sigma_T(\mathbb{T}) \subset \mathbb{C}^d$.
  Definition is complicated.
  It is defined for commuting tuples.
  We can consider functional calculus on the neighborhood of $\sigma_T(\mathbb{T})$.

- **Harte Spectrum** $\sigma_H(\mathbb{T}) \subset \mathbb{C}^d$.
  It can be defined for non-commuting tuples.
  Definition is easier than the Taylor spectrum.

◆ Muller, ``Spectral theory of linear operators and spectral systems in Banach algebras'' Birkhäuser Verlag, Basel, 2007.

Joint Spectral Radius

Let $\mathbb{T} = (T_1, \dots, T_d) \in B(H)^d$ be commuting.

- **Joint (Taylor) spectral radius** is defined by

$$r(\mathbb{T}) = \sup\{\|\lambda\|_2 \ |\lambda = (\lambda_1, \dots, \lambda_d) \in \sigma_T(\mathbb{T})\}$$

Theorem 1.C

$r(\mathbb{T})$ is independent of the choice of the joint spectrum of $\mathbb{T}$.

◆ Cho-Zelasko, Hokkaido Math. J., 21 (1992), 251-258.

Joint Operator Norm

Let $\mathbb{T} = (T_1, \dots, T_d) \in B(H)^d$.

- **Joint Operator Norm**

$$\|\mathbb{T}\| = \sup\left\{\left(\sum \|T_k x\|^2\right)^{\frac{1}{2}}; \; x \in H, \|x\| = 1\right\}$$

Theorem 1.D

$$\frac{\mathbf{1}}{\mathbf{2}\sqrt{\boldsymbol{d}}}\|\mathbb{T}\| \leq \boldsymbol{w}(\mathbb{T}) \leq \|\mathbb{T}\|$$

- Baklouti, Feki, Ahmed, LAA, 555 (2018), 266-284.
- Popescu, Memories of AMS, 200 (2009)

Spectral radius formula

Let $\mathbb{T} = (T_1, \dots, T_d) \in B(H)^{d}, \mathbb{S} = (S_1, \dots, S_e), \in B(H)^{e}$.

$$\mathbb{T}\mathbb{S} := (T_1S_1, \dots, T_1S_e, T_2S_1, \dots, T_2S_e, \dots, T_dS_1, \dots, T_dS_e) \in B(H)^{de}$$

Theorem 1.E

Let $\mathbb{T}$ be a commuting tuple. Then

$$\lim_{n\to\infty} \|\mathbb{T}^n\|^{\frac{1}{n}} = r(\mathbb{T})$$

- Bunce, J. Funct. Anal., 57 (1984), 21-30.
- Muller, Soltysiak, Studia Math., 103 (1992), 329-333.

Results

Theorem 1.1

Let $\mathbb{T} = (T_1, \ldots, T_d) \in B(H)^d$. Then

$$\|\widehat{\mathbb{T}}\| \leq \|\mathbb{T}\|.$$

Lemma 1.1

Let $\mathbb{T} = (T_1, \ldots, T_d) \in B(H)^d$. Then

$$\|\mathbb{T}\| = \left\|\sum T_k^* T_k\right\|^{\frac{1}{2}} = \|P\|.$$

Lemma 1.2

Let $A, X_k \in B(H)$ $(k = 1, \ldots, d)$. Then

$$\left\|\sum X_k^* A X_k\right\| \leq \left\|\sum X_k^* X_k\right\| \|A\|.$$

Results

Theorem 1.1

Let $\mathbb{T} = (T_1, \dots, T_d) \in B(H)^d$**. Then**

$$\|\widehat{\mathbb{T}}\| \le \|\mathbb{T}\|.$$

Proof.

$$\|\widehat{\mathbb{T}}\|^2 = \left\|\sum \widehat{T_k}^* \widehat{T_k}\right\| = \left\|\sum P^{\frac{1}{2}} V_k^* P V_k P^{\frac{1}{2}}\right\|$$

By Lem. 1.1

By Lem. 1.2

$$\le \|P\| \left\|\sum P^{\frac{1}{2}} V_k^* V_k P^{\frac{1}{2}}\right\| = \|P\|\|P\| = \|\mathbb{T}\|^2$$

$\sum V_k^* V_k$ **is a projection onto** $R(P)$.

Results

Theorem 1.2

Let $\mathbb{T} = (T_1, \dots, T_d) \in B(H)^d$. Then

$$w(\widehat{\mathbb{T}}) \le \frac{1}{2} w(\mathbb{T}) + \frac{1}{2} w(\mathbb{T}_1),$$

Where $\mathbb{T}_1 := (PV_1, \dots, PV_d)$.

Proof. Let $U_\lambda = \lambda_1 V_1 + \cdots + \lambda_d V_d$, $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{B}^d$. Since $\lambda_k T_k = \lambda_k V_k P$,

$$w(\mathbb{T}) = \sup w(\lambda_1 T_1 + \cdots + \lambda_d T_d)$$

$$= \sup w(\lambda_1 V_1 P + \cdots + \lambda_d V_d P) = \sup w(U_\lambda P),$$

By Th. 1.B

$$w(\widehat{\mathbb{T}}) = \sup w(\lambda_1 P^{\frac{1}{2}} V_1 P^{\frac{1}{2}} + \cdots + \lambda_d P^{\frac{1}{2}} V_d P^{\frac{1}{2}}) = \sup w(P^{\frac{1}{2}} U_\lambda P^{\frac{1}{2}})$$

$$w(\mathbb{T}_1) = \sup w(\lambda_1 PV_1 + \cdots + \lambda_d PV_d) = \sup w(PU_\lambda)$$

We shall prove $w\left(P^{\frac{1}{2}} U_\lambda P^{\frac{1}{2}}\right) \le \frac{1}{2} w(U_\lambda P) + \frac{1}{2} w(PU_\lambda)$.

Single operator!

Results

Theorem 1.3

Let $\mathbb{T} = (T_1, \dots, T_d) \in B(H)^d$. If $\mathbb{T}$ is commuting tuple, then

$$w(\widehat{\mathbb{T}}) \leq w(\mathbb{T}).$$

Proof. We note that if $\mathbb{T}$ is commuting. Then

$$T_iT_j = T_jT_i \Leftrightarrow V_iPV_jP = V_jPV_iP \Leftrightarrow V_iPV_j = V_jPV_i$$

on $\mathrm{Ker}(P) \oplus \mathrm{Ker}(P)^\perp$

We shall prove

$$w(\mathbb{T}_1) \leq w(\mathbb{T}), i.e., w(PU_\lambda) \leq w(U_\lambda P)$$

$$\langle PU_\lambda x, x\rangle = \langle\left(\sum V_k^* V_k\right) PU_\lambda x, x\rangle = \sum \langle V_k PU_\lambda x, V_k x\rangle.$$

Here

$$V_kPU_\lambda = V_kP(\lambda_1V_1 + \cdots + \lambda_dV_d)$$
$$= (\lambda_1V_1 + \cdots + \lambda_dV_d)PV_k = U_\lambda PV_k$$

$T_iT_j = T_jT_i$
$\Leftrightarrow V_iPV_j = V_jPV_i$

Results

Hence

$$\langle PU_\lambda x, x\rangle = \sum \langle V_k PU_\lambda x, V_k x\rangle = \sum \langle U_\lambda PV_k x, V_k x\rangle$$

Put $y_k = \frac{V_k x}{\|V_k x\|}$. Then

$$|\langle PU_\lambda x, x\rangle| = \left|\sum \|V_k x\|^2 \langle U_\lambda P y_k, y_k\rangle\right|$$

$$\leq \sum \|V_k x\|^2 |\langle U_\lambda P y_k, y_k\rangle|$$

$$\leq \langle \sum V_k^* V_k\, x, x\rangle w(U_\lambda P) \leq w(U_\lambda P)$$

projection

Results

Theorem 1.4 **Let $\mathbb{T} = (T_1, \ldots, T_d) \in B(H)^d$. Then**

$$w(\mathbb{T}) \le \frac{1}{2}\|\mathbb{T}\| + \frac{1}{2} w\big(\widehat{\mathbb{T}}\big).$$

Proof. We shall prove

$$w(U_\lambda P) \le \frac{1}{2}\|P\| + \frac{1}{2} w(P^{\frac{1}{2}} U_\lambda P^{\frac{1}{2}})$$

Results

Theorem 1.5 Let $\mathbb{T} = (T_1, \dots, T_d) \in B(H)^d$ be a commuting tuple. Then

$$\lim_{n\to\infty} \left\| \widehat{\mathbb{T}_n} \right\| = r(\mathbb{T}).$$

Let $\mathbb{T}$ be a commuting tuple. Then

$$\lim_{n\to\infty} \|\mathbb{T}^n\|^{\frac{1}{n}} = r(\mathbb{T})$$

- $\widehat{\mathbb{T}_n} := (\widehat{\widehat{\mathbb{T}_{n-1}}}), \widehat{\mathbb{T}_0} := \mathbb{T}$
  (n-th iteration of spherical Aluthge transform)
- $\widehat{\mathbb{T}_n}$ is a d-tuple of operators.
- $\mathbb{T}^n$ is a d^n-tuple of operators.

The sketch of proof

The proof is a modification of
Wang, Math Inequal. Appl 6 (2003), 121-124.

Key Idea.

- $\widehat{\mathbb{T}} = (\hat{T}_1, \ldots, \hat{T}_d)$ is a commuting tuple.
- $\left\| \widehat{\mathbb{T}_{n+1}}^{k} \right\| \le \left\| \widehat{\mathbb{T}_n}^{k} \right\|$
- $\left\| \widehat{\mathbb{T}_{n+1}}^{k} \right\| \le \left\| \widehat{\mathbb{T}_n}^{k+1} \right\|^{\frac{1}{2}} \left\| \widehat{\mathbb{T}_n}^{k-1} \right\|^{\frac{1}{2}}$
- $\exists s\ such\ that \left\| \widehat{\mathbb{T}_n}^{k} \right\| \to s^k$ for all $k = 1,2, \ldots$
- $\|\mathbb{T}^n\|^{\frac{1}{n}} \to r(\mathbb{T})$ as $n \to \infty$

The sketch of proof

- $\left\|\widehat{\mathbb{T}_{n+1}}^{k}\right\| \leq \left\|\widehat{\mathbb{T}_{n}}^{k}\right\| \Rightarrow \left\{\left\|\widehat{\mathbb{T}_{n}}^{k}\right\|^{\frac{1}{k}}\right\}_{n=0}^{\infty}$ is decreasing

- $\exists s\ such\ that \left\|\widehat{\mathbb{T}_{n}}^{k}\right\| \to s^{k}\ as\ n \to \infty$ for all $k = 1,2,\ldots$

$$\Rightarrow s \leq \left\|\widehat{\mathbb{T}_{n}}^{k}\right\|^{\frac{1}{k}} \to s\ as\ n \to \infty$$

- Assume $r(\mathbb{T}) < s$. By $\|\mathbb{T}^{n}\|^{\frac{1}{n}} \to r(\mathbb{T})$ for any $\mathbb{T}$,

$$\left\|\widehat{\mathbb{T}_{n}}^{k}\right\|^{\frac{1}{k}} \leq \left\|\mathbb{T}^{k}\right\|^{\frac{1}{k}} < s$$

holds for sufficiently large k. It is a contradiction.

- Hence $r(\mathbb{T}) = s$.

Induced Aluthge transformations

◆ Y., Linear Algebra Appl. 628 (2021), 1–28.

Introduction (Operator mean)

- $\mathcal{P}$: The set of all positive definite operators on a Hilbert space.

Definition 2.1 (Operator mean)

Let $\mathfrak{M}: \mathcal{P}^2 \to \mathcal{P}$. If σ satisfies the following conditions, then $\mathfrak{M}$ is called an operator mean.

1. $\mathfrak{M}(A, B) \leq \mathfrak{M}(C, D)$ if $A \leq C$ and $B \leq D$,

2. $X^*\mathfrak{M}(A, B)X \leq \mathfrak{M}(X^*AX, X^*BX)$ for all bounded linear operator X,

3. $\mathfrak{M}$ is upper semi-continuous on $\mathcal{P}^2$,

4. $\mathfrak{M}(I, I) = I$.

- Kubo and Ando, Math. Ann., 246 (1980), 205-224.

Introduction

- $\mathcal{M}$: The set of all operator monotone functions on $(0,\infty)$.

Theorem 2.A (Representing function)

Let $\mathfrak{M}$ be an operator mean. Then $\exists f \in \mathcal{M}$ such that $f(1) = 1$ and

$$\mathfrak{M}(A,B) = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}$$

for all $A, B \in \mathcal{P}$.

Examples. Let $\lambda \in [0,1]$.

- Arithmetic mean: $A \nabla_\lambda B = (1-\lambda)A + \lambda B$ $\;(f(x) = 1 - \lambda + \lambda x)$,
- Geometric mean: $A \,\sharp_\lambda\, B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\lambda} A^{\frac{1}{2}}$ $\;(f(x) = x^\lambda)$,
- Harmonic mean: $A !_\lambda B = [(1-\lambda)A^{-1} + \lambda B^{-1}]^{-1}$ $\;(f(x) = [1 - \lambda + \lambda x^{-1}]^{-1})$.
- $\mathfrak{M}_f$: Operator mean with a representing function f.

- Kubo and Ando, Math. Ann., 246 (1980), 205-224.

Motivation

- $\mathcal{M}_n$: $n \times n$ matrices
- $\mathcal{M}_n$ is a Hilbert space with inner product $\langle A, B \rangle := tr\,(AB^*)$

Definition 2.2 (Left and right multiplications)

Let $T \in \mathcal{M}_n$. Define linear mappings $\mathcal{M}_n \to \mathcal{M}_n$ by

$$\mathbb{L}_T(X) := TX, \qquad \mathbb{R}_T(X) := XT \quad (X \in \mathcal{M}_n).$$

Remark

- $\mathbb{L}_T$ and $\mathbb{R}_S$ are commuting, i.e.,

$$\mathbb{L}_S \mathbb{R}_T(X) = \mathbb{L}_S(XT) = SXT = \mathbb{R}_T(SX) = \mathbb{R}_T \mathbb{L}_S(X).$$

- If T is positive semi-definite (or positive definite), then $\mathbb{L}_T$ and $\mathbb{R}_T$ are positive semi-definite (or positive definite). Especially

$$(\mathbb{L}_T)^\alpha = \mathbb{L}_{T^\alpha}, (\mathbb{R}_T)^\alpha = \mathbb{R}_{T^\alpha} \ \ (\alpha > 0 \ or \ \alpha \in \mathcal{R}).$$

- Geometric mean $\left(\mathbb{L}_S \#_{1/2} \mathbb{R}_T\right)(X) = S^{\frac{1}{2}} X T^{\frac{1}{2}}$.

  Especially $\left(\mathbb{L}_{|T|} \#_{1/2} \mathbb{R}_{|T|}\right)(U) = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}} = \Delta(T)$

Matrix case

Definition 2.1 (Induced Aluthge transformation).
Let $\mathfrak{M}_f$ be an operator mean, and $T = U|T| \in \mathcal{M}_n$ be the polar decomposition of an invertible T. Then the induced Aluthge transformation $\Delta_{\mathfrak{M}_f}(\mathrm{T})$ respect to an operator mean $\mathfrak{M}_f$ is defined by

$$\Delta_{\mathfrak{M}_f}(T) := \mathfrak{M}_f\left(\mathbb{L}_{|T|}, \mathbb{R}_{|T|}\right)(U) = \mathbb{L}_{|T|} f\left(\mathbb{L}_{|T|}^{-1}\mathbb{R}_{|T|}\right)(\mathbf{U}).$$

Examples.

- Arithmetic mean case. $\Delta_{\mathfrak{M}}(\mathrm{T}) = (1-\lambda)|T|U + \lambda U|T|$
  (mean transform, S.H. Lee-W.Y. Lee-Yoon, 2014.)
- Geometric mean case. $\Delta_{\mathfrak{M}}(\mathrm{T}) = |T|^{1-\lambda}U|T|^{\lambda}$
  (λ –Aluthge transform, Huruya, 1997.)

- S.H. Lee, W.Y. Lee Yoon, J. Math. Anal. Appl., 410 (2014), 70-81.
- Furuta, Proc. Amer. Math. Soc., 125 (1997), 3617-3624.

Matrix case

Definition 2.3 (Induced Aluthge transformation).
Let $\mathfrak{M}_f$ be an operator mean, and $T = U|T| \in \mathcal{M}_n$ be the polar decomposition of an invertible T. Then the induced Aluthge transformation $\Delta_{\mathfrak{M}_f}(\mathrm{T})$ respect to an operator mean $\mathfrak{M}_f$ is defined by

$$\Delta_{\mathfrak{M}_f}(T) := \mathfrak{M}_f\left(\mathbb{L}_{|T|}, \mathbb{R}_{|T|}\right)(U) = \mathbb{L}_{|T|} f\left(\mathbb{L}_{|T|}^{-1} \mathbb{R}_{|T|}\right)(\mathrm{U}).$$

Let $T = U|T|$ be the polar decomposition and

$$U = \left(u_{ij}\right), \ \ |T| = V^* diag(s_1, \dots, s_n)V, \text{ (V: unitary).}$$

Then

$$\Delta_{\mathfrak{M}_f}(T) = V^*\left\{VUV^* \circ \left[\mathcal{P}_f\left(s_i, s_j\right)\right]\right\}V,$$

where $\mathcal{P}_f(s,t) \coloneqq sf\left(\frac{t}{s}\right)$ (perspective or solidarity).

Harmonic mean (matrix case)

Theorem 2.1

Let $f(t) = \left[1 - \lambda + \lambda t^{-1}\right]^{-1} (\lambda \in [0,1])$, and let $T = U|T| \in \mathcal{M}_n$ be the polar decomposition of an invertible matrix T. Then

$$\Delta_{\mathfrak{M}_f}(T) = \int_0^{\infty} e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}} \, dt.$$

The harmonic mean case for a **unilateral shift** is firstly considered by S.H. Lee, 2016.

- S.H. Lee, J. Chungcheong Math. Soc., 29 (2016), 123-135.

Proof

Theorem 2.B
Let A and B be operators whose spectra are contained in the open right half-plane and the open left-plane, respectively. Then the solution of the equation $AX - XB = Y$ can be expressed as

$$\boldsymbol{X = \int_0^\infty e^{-tA} Y e^{tB}\, dt.}$$

Proof of Theorem 1. Let $X = \Delta_{\mathfrak{M}_\lambda}(T)$. Then we have

$$\left[(1-\lambda)\mathbb{L}_{|T|^{-1}} + \lambda \mathbb{R}_{|T|^{-1}}\right]^{-1}(U) = X.$$

It is equivalent to

$$U = \left[(1-\lambda)\mathbb{L}_{|T|^{-1}} + \lambda \mathbb{R}_{|T|^{-1}}\right](X)$$
$$= ((1-\lambda)|T|^{-1})X - X(-\lambda|T|^{-1})$$

Hence
$$\Delta_{\mathfrak{M}_\lambda}(T) = \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}}\, dt.$$

- Heinz, Math. Ann., 123 (1951), 415-438.

Operator mean (matrix case)

Theorem 2.2

Let $\mathfrak{M}$ be an operator mean, and $T = U|T|$ be the polar decomposition of an invertible matrix T. Then there exists a probability measure $d\mu(\lambda)$ on $[0,1]$, s.t.,

$$\Delta_{\mathfrak{M}}(T) = \int_0^1 \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}} \, dt d\mu(\lambda) \, .$$

Corollary 2.1 $\mathrm{tr}\big(\Delta_{\mathfrak{M}}(T)\big) = \mathrm{tr}(T)$.

- Spectral of T and $\Delta_{\mathfrak{M}}(T)$ are not coincide, generally.

  Cf. S.H. Lee, W.Y. Lee Yoon, J. Math. Anal. Appl., 410 (2014), 70-81.

Operator mean (matrix case)

Theorem 2.2

Let $\mathfrak{M}$ be an operator mean, and $T = U|T|$ be the polar decomposition of an invertible matrix T. Then there exists a probability measure $d\mu(\lambda)$ on $[0,1]$, s.t.,

$$\boldsymbol{\Delta_{\mathfrak{M}}(T) = \int_0^1 \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}}\, dt d\mu(\lambda)}.$$

Proof. Every representing function of operator mean can be given by

$$f(x) = \int_0^1 [1 - \lambda + \lambda x^{-1}]^{-1} d\mu(\lambda)$$

for a probability measure $d\mu(\lambda)$ on $[0,1]$. Hence, we have

$$\Delta_{\mathfrak{M}_f}(T) = \int_0^1 \left[(1-\lambda)\mathbb{L}_{|T|^{-1}} + \lambda \mathbb{R}_{|T|^{-1}}\right]^{-1} d\mu(\lambda)(U)$$

Harmonic mean

$$= \int_0^1 \Delta_{\mathfrak{H}_\lambda}(T)\, d\mu(\lambda) = \int_0^1 \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}} dt\, d\mu(\lambda).$$

Another formula

Let $|T| = \sum_{i=1}^{n} s_i P_i$ be the spectral decomposition. Since

$$\mathcal{P}_f(s,t) := sf\left(\frac{t}{s}\right) = \int_0^1 [(1-\lambda)s^{-1} + \lambda t^{-1}]^{-1}\, d\mu(\lambda),$$

we have

$$\begin{aligned}
\Delta_{\mathfrak{M}_f}(T) &= \int_0^1 \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}}\, dt d\mu(\lambda) \\
&= \int_0^1 \int_0^\infty \left(\sum_i e^{-(1-\lambda)ts_i^{-1}} P_i\right) U \left(\sum_j e^{-\lambda t s_j^{-1}} P_j\right) dt d\mu(\lambda) \\
&= \sum_{i,j} \int_0^1 \int_0^\infty e^{-\left\{(1-\lambda)s_i^{-1} + \lambda s_j^{-1}\right\}t}\, dt d\mu(\lambda) P_i U P_j \\
&= \sum_{i,j} \int_0^1 \left[(1-\lambda)s_i^{-1} + \lambda\, s_j^{-1}\right]^{-1} d\mu(\lambda) P_i U P_j \\
&= \sum_{i,j} \mathcal{P}_f(s_i, s_j) P_i U P_j \longrightarrow \int\int_{\sigma(|T|)^2} \mathcal{P}_f(s,t)\, dE_s U dE_t\ ?
\end{aligned}$$

Operator case?

Examples

Let $|T| = \sum_{i=1}^{n} s_i P_i$ be the spectral decomposition. Then

$$\Delta_{\mathfrak{M}_f}(T) = \sum_{i,j} \mathcal{P}_f\left(s_i, s_j\right) P_i U P_j \,.$$

Example.

1. Arithmetic mean: $\mathcal{P}_f(s,t) = (1-\lambda)s + \lambda t$.

$$\Delta_{\mathfrak{M}_f}(T) = \sum_{i,j} [(1-\lambda)s_i + \lambda s_j] P_i U P_j$$

$$= \sum_{i,j} (1-\lambda) s_i \, P_i U P_j + \sum_{i,j} \lambda s_j \, P_i U P_j = (1-\lambda)|T|U + \lambda U|T|.$$

$$\sum_i P_i = I$$

2. Geometric mean: $\mathcal{P}_f(s,t) = s^{1-\lambda} t^{\lambda}$.

$$\Delta_{\mathfrak{M}_f}(T) = \sum_{i,j} s_i^{1-\lambda} s_j^{\lambda} P_i U P_j = \sum_{i,j} (s_i^{1-\lambda} P_i) U (s_j^{\lambda} P_j) = |T|^{1-\lambda} U |T|^{\lambda}.$$

Examples

3. Power mean: $\mathcal{P}_f(s,t) = \left[(1-\lambda)s^{\frac{1}{2}} + \lambda t^{\frac{1}{2}}\right]^2$.

$$\Delta_{\mathfrak{M}_f}(T) = \sum_{i,j}\left[(1-\lambda)s_i^{\frac{1}{2}} + \lambda s_j^{\frac{1}{2}}\right]^2 P_i U P_j$$

$$= \sum_{i,j}\left[(1-\lambda)^2 s_i + 2\lambda(1-\lambda)s_i^{\frac{1}{2}}s_j^{\frac{1}{2}} + \lambda^2 s_j\right] P_i U P_j$$

$$= (1-\lambda)^2|T|U + \lambda^2 U|T| + 2\lambda(1-\lambda)|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}.$$

Especially, if $\lambda = \frac{1}{2}$, then

$$\Delta_{\mathfrak{M}_f}(T) = \frac{1}{2}\left[\frac{|T|U + U|T|}{2} + |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}\right].$$

Another formula

Let $|T| = \sum_{i=1}^{n} s_i P_i$ be the spectral decomposition. Since

$$\mathcal{P}_f(s,t) := sf\left(\frac{t}{s}\right) = \int_0^1 [(1-\lambda)s^{-1} + \lambda t^{-1}]^{-1}\, d\mu(\lambda),$$

we have

$$\begin{aligned}
\Delta_{\mathfrak{M}_f}(T) &= \int_0^1 \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}}\, dt d\mu(\lambda) \\
&= \int_0^1 \int_0^\infty \left(\sum_i e^{-(1-\lambda)ts_i^{-1}} P_i\right) U \left(\sum_j e^{-\lambda t s_j^{-1}} P_j\right) dt d\mu(\lambda) \\
&= \sum_{i,j} \int_0^1 \int_0^\infty e^{-\left\{(1-\lambda)s_i^{-1} + \lambda s_j^{-1}\right\}t}\, dt d\mu(\lambda) P_i U P_j \\
&= \sum_{i,j} \int_0^1 \left[(1-\lambda)s_i^{-1} + \lambda\, s_j^{-1}\right]^{-1} d\mu(\lambda) P_i U P_j \\
&= \sum_{i,j} \mathcal{P}_f(s_i, s_j) P_i U P_j \longrightarrow \int\int_{\sigma(|T|)^2} \mathcal{P}_f(s,t)\, dE_s U dE_t\ ?
\end{aligned}$$

Operator case?

Double operator integrals

Definition 2.4
Let $H = \int_{\sigma(H)} s dE_s$, $K = \int_{\sigma(K)} t dF_t$ be the spectral decompositions of positive definite operators. For $\varphi \in L^\infty(\sigma(H) \times \sigma(K))$ and $X \in B(\mathcal{H})$, define the double operator integrals $\Phi(X)$ as follows:

$$\Phi(X) := \int_{\sigma(H)} \int_{\sigma(K)} \varphi(s,t) dE_s X dE_t .$$

Question. For each $X \in B(\mathcal{H})$, does $\Phi(X) \in B(\mathcal{H})$ always hold?
Answer. No! If φ is a **Schur multiplier**, then $\Phi(X) \in B(\mathcal{H})$ holds.

Definition 2.5
For a function $\varphi \in L^\infty(\sigma(H) \times \sigma(K))$, if $\Phi(X)$ $(:= \Phi(X)|_{C_1(\mathcal{H})}) \in C_1(\mathcal{H})$, then φ is called the Schur multiplier.

◆ Hiai and Kosaki, Lecture Notes in Math. 1820 (2003).

Schur multiplier

Definition 2.5

For a function $\varphi \in L^\infty\big(\sigma(H) \times \sigma(K)\big)$, if $\Phi(X)\ (:= \Phi(X)|_{C_1(\mathcal{H})}) \in C_1(\mathcal{H})$, then φ is called a **Schur multiplier**.

Theorem 2.C

Let $\varphi \in L^\infty\big(\sigma(H) \times \sigma(K)\big)$. Then **TFAE**.

(1) φ is a Schur multiplier,

(2) there exists finite measure space (Ω, σ'), and there exists $\alpha \in L^\infty(\sigma(H) \times \Omega)$ and $\beta \in L^\infty(\sigma(K) \times \Omega)$ such that

$$\varphi(s,t) = \int_\Omega \alpha(s,x)\beta(t,x)\, d\sigma'(x).$$

Proposition 2.1

Let $\mathfrak{M}_f$ be an operator mean with a representing function f.

Then $\mathcal{P}_f(s,t) := sf\left(\frac{t}{s}\right)$ is a Schur multiplier.

- Hiai and Kosaki, Lecture Notes in Math. 1820 (2003).
- Peller, Funct. Anal. Appl. 19 (1985) 111–123.

Proof

Proposition 2.1
Let $\mathfrak{M}_f$ be an operator mean with a representing function f.
Then $\mathcal{P}_f(s,t) \coloneqq sf\left(\frac{t}{s}\right)$ is a Schur multiplier.

Proof. There exists a probably measure $d\mu$ on [0,1], s.t.,

$$f(x) = \int_0^1 [1-\lambda+\lambda x^{-1}]^{-1}\, d\mu(\lambda).$$

Then we have

$$\mathcal{P}_f(s,t) \coloneqq sf\left(\frac{t}{s}\right) = \int_0^1 [(1-\lambda)s^{-1}+\lambda t^{-1}]^{-1}\, d\mu(\lambda)$$

$$= \int_0^1 \int_0^\infty e^{-[(1-\lambda)s^{-1}+\lambda t^{-1}]x}\, dx\, d\mu(\lambda)$$

$$= \int_0^1 \int_0^\infty e^{-(1-\lambda)s^{-1}x} e^{-\lambda t^{-1}x}\, dx\, d\mu(\lambda) = \int_0^1 \int_0^\infty \alpha(s,\lambda,x)\beta(t,\lambda,x)\, dx\, d\mu(\lambda).$$

Operator case

Definition 2.3 (Induced Aluthge transformation).
Let $T \in B(\mathcal{H})$ be invertible with the polar decomposition $T = U|T|$. Let $|T| = \int_{\sigma(|T|)} s dE_s$ be the spectral decomposition. For each operator mean $\mathfrak{M}_f$, s.t., $f'(1) \in (0,1)$, induced Aluthge transformation $\Delta_{\mathfrak{M}_f}(\mathrm{T})$ respect to an operator mean $\mathfrak{M}_f$ is defined by

$$\Delta_{\mathfrak{M}_f}(T) := \int_{\sigma(|T|)} \int_{\sigma(|T|)} \mathcal{P}_f(s,t)\, dE_s U dE_t.$$

Matrix case

$$\Delta_{\mathfrak{M}_f}(T) = \sum_{i,j} \mathcal{P}_f(s_i, s_j) P_i U P_j,$$

where $|T| = \sum_i s_i P_i$ is the spectral decomposition.

- If $\ker T \subseteq \ker T^*$, then $\Delta_{\mathfrak{M}_f}(T)$ can be defined. However, it is not known to define $\Delta_{\mathfrak{M}_f}(T)$, in generally.

Iteration (finite dimensional case)

$\Delta_{\mathfrak{M}}^{n}(T) := \Delta_{\mathfrak{M}}^{n-1}\big(\Delta(T)\big), \Delta_{\mathfrak{M}}^{0}(T) = T.$

Theorem 2.3

Let $\mathfrak{M}$ be a non-weighted arithmetic mean, and $T = U|T|$ be the polar decomposition of an invertible matrix T. Then the sequence $\{\Delta_{\mathfrak{M}}^{n}(T)\}$ converges to a normal matrix.

Non-weighted arithmetic mean

$$\mathfrak{M}(A, B) = \frac{A + B}{2}, \qquad \Delta_{\mathfrak{M}}(T) = \frac{|T|U + U|T|}{2}$$

Proof

Theorem 2.3

Let $\mathfrak{M}$ be a non-weighted arithmetic mean, and $T = U|T|$ be the polar decomposition of an invertible matrix T. Then the sequence $\{\Delta_{\mathfrak{M}}^n(T)\}$ converges to a normal matrix.

Proof. Since U is unitary, polar decompositions are

- $\Delta_{\mathfrak{M}}(T) = \dfrac{|T|U + U|T|}{2} = U\dfrac{|T| + U^*|T|U}{2} = U|\Delta_{\mathfrak{M}}(T)|,$
- $\Delta_{\mathfrak{M}}^2(T) = \dfrac{|\Delta_{\mathfrak{M}}(T)|U + U|\Delta_{\mathfrak{M}}(T)|}{2} = U\dfrac{|\Delta_{\mathfrak{M}}(T)| + U^*|\Delta_{\mathfrak{M}}(T)|U}{2}$

  $= U\dfrac{|T| + 2U^*|T|U + (U^*)^2|T|U^2}{2^2} = U\left|\Delta_{\mathfrak{M}}^2(T)\right|,$

- $\Delta_{\mathfrak{M}}^n(T) = U\dfrac{1}{2^{\mathrm{n}}}\sum_{k=0}^{n}\binom{n}{k}(U^*)^k|T|U^k = U\left|\Delta_{\mathfrak{M}}^n(T)\right|.$

Prove convergence!

◆ Chabbabi, Curto and Mbekhta, Proc. AMS 147 (2019) 1119-1133.

Proof

- $\Delta_{\mathfrak{M}}^{n}(T) = U \frac{1}{2^{\mathrm{n}}} \sum_{k=0}^{n} \binom{n}{k} (U^*)^k |T| U^k = U \left| \Delta_{\mathfrak{M}}^{n}(T) \right|.$ **Prove convergence!**

Let

$$U = V^* D V, \text{where } V \text{ is unitary and } D = diag\left(e^{\theta_1 \sqrt{-1}}, \dots, e^{\theta_n \sqrt{-1}}\right).$$

For any unitary V,

$$\Delta_{\mathfrak{M}}(V^* T V) = V^* \Delta_{\mathfrak{M}}(T) V,$$

$$V \left| \Delta_{\mathfrak{M}}^{n}(T) \right| V^* = \left| \Delta_{\mathfrak{M}}^{n}(V T V^*) \right| = \left| \Delta_{\mathfrak{M}}^{n}(D V |T| V^*) \right|.$$

Proof

- $\Delta_{\mathfrak{M}}^{n}(T) = U\frac{1}{2^{\mathrm{n}}}\sum_{k=0}^{n}\binom{n}{k}(U^*)^k|T|U^k = U\left|\Delta_{\mathfrak{M}}^{n}(T)\right|.$ **Prove convergence!**

Let

$$U = V^*DV, \text{where } V \text{ is unitary and } D = diag\left(e^{\theta_1\sqrt{-1}}, \dots, e^{\theta_n\sqrt{-1}}\right).$$

Put $P := V|T|V^*$. Then $(D^*)^k V|T|V D^k = \left[e^{k(\theta_j-\theta_i)\sqrt{-1}}\right] \circ P$ and

$$V\left|\Delta_{\mathfrak{M}}^{n}(T)\right|V^* = \left|\Delta_{\mathfrak{M}}^{n}(DV|T|V^*)\right| = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(D)^k V|T|V^* D^k$$

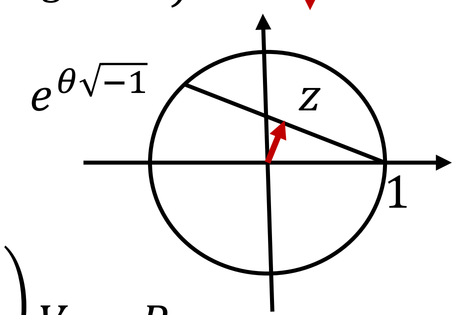
$$= \frac{1}{2^n}\left[\sum_{k=0}^{n}\binom{n}{k}e^{k(\theta_j-\theta_i)\sqrt{-1}}\right] \circ P = \left[\left(\frac{1+e^{(\theta_j-\theta_i)\sqrt{-1}}}{2}\right)^n\right] \circ P$$

Proof

$$V\left|\Delta_{\mathfrak{M}}^{n}(T)\right|V^{*} = \left|\Delta_{\mathfrak{M}}^{n}(DV|T|V^{*})\right| = \left[\left(\frac{1+e^{(\theta_j-\theta_i)\sqrt{-1}}}{2}\right)^{n}\right] \circ P$$

Notice that

$$\begin{cases} \left|\dfrac{1+e^{(\theta_j-\theta_i)\sqrt{-1}}}{2}\right| < 1 \ \ (\theta_j \neq \theta_i + 2m\pi \ for\ all\ integer\ m), \\ \dfrac{1+e^{(\theta_j-\theta_i)\sqrt{-1}}}{2} = 1 \ \ (\theta_j = \theta_i + 2m\pi \ for\ some\ integer\ m). \end{cases}$$



Hence

$$\exists \lim_{n\to\infty}\left|\Delta_{\mathfrak{M}}^{n}(T)\right| = V^{*}\left(\exists \lim_{n\to\infty}\left|\Delta_{\mathfrak{M}}^{n}(DV|T|V^{*})\right|\right)V$$

$$= V^{*}\left(\exists \lim_{n\to\infty}\left[\left(\frac{1+e^{(\theta_j-\theta_i)\sqrt{-1}}}{2}\right)^{n}\right] \circ P\right)V := P_0.$$

$$z = \frac{1+e^{\theta\sqrt{-1}}}{2}$$

Moreover

$$\lim_{n\to\infty}\Delta_{\mathfrak{M}}^{n}(T) = UP_0 = \Delta_{\mathfrak{M}}(UP_0) = \frac{P_0U + UP_0}{2} \Longrightarrow UP_0 = P_0U \ (normal).$$

Iteration (infinite dimensional case)

Theorem 2.4

Let $\mathfrak{M}_f$ be an operator mean, s.t., $f'(1) \in (0, 1)$. Then there exists an operator in $B(\mathcal{H})$, s.t., the sequence $\{\Delta^n_{\mathfrak{M}_f}(\mathrm{T})\}$ does not converge in a week operator topology.

Let $\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots) \in \ell^\infty$ and W_α be a weighted unilateral shift with a weight sequence α, i.e.,

$$W_\alpha \boldsymbol{e_n} = \alpha_n \boldsymbol{e_{n+1}},$$

where $(\boldsymbol{e_0}, \boldsymbol{e_1}, \boldsymbol{e_2}, \dots)$ is a canonical base of ℓ^2.

Sketch of a proof.

Step 1. Let

$$W_{\alpha^{(1)}} := \Delta_{\mathfrak{M}_f}(W_\alpha).$$

Then $\alpha^{(1)} := (\alpha_0^{(1)}, \alpha_1^{(1)}, \ldots, \alpha_n^{(1)}, \ldots)$ and $\alpha_n^{(1)} = \mathcal{P}_f(\alpha_{n+1}, \alpha_n)$.

Step 2.

- Let

  $$\mathfrak{A} \cdots \textit{arithmetic mean}, \qquad \mathfrak{H} \cdots \textit{harmonic mean}$$

  with weight $f'(1)$.

- Let W_α be a weighted unilateral shift whose weights are either a or b $(a \neq b, a, b > 0)$.

- If there are only finitely many weights of W_α are equal to a.

- Then the sequence of the first weights of $\Delta_{\mathfrak{A}}^n(W_\alpha)$ and $\Delta_{\mathfrak{H}}^n(W_\alpha)$ are converge to b.

Sketch of a proof.

Step 3. Let the sequence of the first weights of $\Delta_{\mathfrak{A}}^{n}(W_\alpha)$ and $\Delta_{\mathfrak{H}}^{n}(W_\alpha)$ are $\{\mathfrak{a}_n\}$ and $\{\mathfrak{h}_n\}$.

- Let $\alpha = (a, b, b, b, \ldots)$. Then there exists n_1, s.t.,

$$\left|\mathfrak{a}_{n_1} - b\right| < \frac{1}{2} \text{ and } \left|\mathfrak{h}_{n_1} - b\right| < \frac{1}{2}.$$

- Let $\alpha = (\overbrace{a, b, \ldots, b}^{n_1}, a, a, \ldots)$. Then there exists n_2, s.t.,

$$\left|\mathfrak{a}_{n_2} - a\right| < \frac{1}{2^2} \text{ and } \left|\mathfrak{h}_{n_2} - a\right| < \frac{1}{2^2}.$$

- Let $\alpha = (\underbrace{\overbrace{a, b, \ldots, b}^{n_1}, a, \ldots, a}_{n_2}, b, \ldots)$. Then there exists n_3, s.t.,

$$\left|\mathfrak{a}_{n_3} - b\right| < \frac{1}{2^3} \text{ and } \left|\mathfrak{h}_{n_3} - b\right| < \frac{1}{2^3}.$$

Repeating the procedure, we obtain that there exists a weighted unilateral shift W_α such that $\{\mathfrak{a}_n\}$ and $\{\mathfrak{h}_n\}$ do not converge.

Sketch of a proof.

Step 4.

- Let $\mathfrak{M}_f$ be an arbitrary operator mean, s.t.,

$$[1 - f'(1) + f'(1)x^{-1}]^{-1} \leq f(x) \leq 1 - f'(1) + f'(1)x.$$

- Let the sequence of the first weights of $\Delta^n_{\mathfrak{M}_f}(W_\alpha)$ be $\{\mathfrak{m}_n\}$. Then by

$$\mathfrak{h}_n \leq \mathfrak{m}_n \leq \mathfrak{a}_n,$$

$\{\mathfrak{m}_n\}$ does not converge.

Problem

Problem.

Let $T \in B(\mathcal{H})$ be invertible and let $\mathfrak{M}_f$ be an operator mean , s.t., $f'(1) \in (0, 1)$. Then TFAE?

(i) T is quasinormal (i.e., $|T|U = U|T|$),

(ii) $\Delta_{\mathfrak{M}_f}(T) = T$.

- The above problem is true in the matrix case.
- **Any application or any extension is welcome.**

$\mathcal{AN}(H)$-operator

Definition 2.4 (Norm attained operator).

Let $T \in B(H, K)$.

(i) T is called a norm attained operator ($T \in \mathcal{N}(H, K)$), if there exists a unit vector $x \in H$ such that $\|Tx\| = \|T\|$.

(ii) T is called an absolutely norm attained operator, if for any non-zero closed subspace $M \subseteq H$, $T|_M \in \mathcal{N}(M, K)$.

If $T \in B(H)$ is absolutely norm attained operator, we write $T \in \mathcal{AN}(H)$, T is $\mathcal{AN}(H)$-operator or T has $\mathcal{AN}$-property.

Compact operators and isometry are $\mathcal{AN}(H)$-operator.

- Carvajal and Neves, IEOT, **72** (2012), 179-195.
- Pandy and Paulsen, J Aust Math Soc, **102** (2017) 369-391.
- Ramesh, J Aust Math Soc, **96** (2014) 386-395.

Results

Theorem 2.5

Let f be a non-negative positive operator monotone function on $[0,\infty)$ with $f(1)=1$. Suppose that $T \in \mathcal{AN}(H)$ is invertible. Then $\Delta_{\mathfrak{M}_f}(T) \in \overline{\mathcal{AN}(H)}$.

Notice: $\mathcal{AN}(H) \subsetneq \overline{\mathcal{AN}(H)}$

Example.

$$T_n := \begin{pmatrix} 1/2 & & & & & & \\ & 1-1/3 & & & & & \\ & & 1-1/4 & & & & \\ & & & \ddots & & & \\ & & & & 1-1/(n+1) & & \\ & & & & & 1 & \\ & & & & & & \ddots \end{pmatrix}.$$

Then $T_n \in \mathcal{AN}(H)$ but $T_n \to T \notin \mathcal{AN}(H)$.

- Osaka Ramesh Udagawa Yamazaki, preprint.

Results

Theorem 2.6

For $\lambda \in [0,1]$, let $f_\lambda(t) = 1 - \lambda + \lambda t$ and $g_\lambda(t) = t^\lambda$. If $T \in \mathcal{AN}(H)$, then $\Delta_{\mathfrak{M}_{f_\lambda}}(T), \Delta_{\mathfrak{M}_{g_\lambda}}(T) \in \mathcal{AN}(H)$. Especially, $\Delta(T) \in \mathcal{AN}(H)$.

- $f_\lambda(t) = 1 - \lambda + \lambda t$ のとき、$\Delta_{\mathfrak{M}_{f_\lambda}}(T) = (1-\lambda)|T|U + \lambda U|T|$.
- $g_\lambda(t) = t^\lambda$ のとき、$\Delta_{\mathfrak{M}_{g_\lambda}}(T) = |T|^{1-\lambda}U|T|^\lambda$.
  とくに、$\Delta(T) := \Delta_{\mathfrak{M}_{g_{\frac{1}{2}}}}(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ (Aluthge transformation).
- $S, T \in \mathcal{AN}(H)$ であっても、$S + T \in \mathcal{AN}(H)$ とは限らないため、
  $f(t) = \left(\frac{1+t^{1/2}}{2}\right)^2$,$T \in \mathcal{AN}(H)$ のときに、

$$\Delta_{\mathfrak{M}_f}(T) = \frac{1}{4}(|T|U + U|T| + 2|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}) \in \mathcal{AN}(H)$$

であるかは**不明**。

Results

T is semi-hyponormal $\Leftrightarrow |T| \geq |T^*| = U|T|U^*$. Then

$$|T| \leq U^*|T|U \leq U^{*2}|T|U^2 \leq \cdots \leq U^{*n}|T|U^n \leq \cdots \leq \|T\|I$$

Hence $L := s - \lim_{n\to\infty} U^{*n}|T|U^n$ exists.

Theorem 2.7

Let $f_{1/2}(t) = \frac{1+t}{2}$ and $T \in B(H)$ be a semi-hyponormal operator with the polar decomposition $T = U|T|$. If $ker(T^*) = ker(T)$, then

$$s - \lim_{n\to\infty} \Delta^n_{m_{f_{1/2}}}(T) = UL$$

in the strong operator topology. Moreover, UL is a normal operator and $\sigma(T) = \sigma(UL)$.

$T = U|T|$ と極分解したとき、$\Delta^n_{f_{1/2}}(T)$ の極分解は次のようになる。

$$\Delta^n_{f_{1/2}}(T) = U\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} U^{*k}|T|U^k\right]$$

◆ Chabbabi, Curto and Mbekhta, Proc AMS 147 (2019) 1119-1133.

Results

T is semi-hyponormal $\Leftrightarrow |T| \geq |T^*| = U|T|U^*$. Then

$$|T| \leq U^*|T|U \leq U^{*2}|T|U^2 \leq \cdots \leq U^{*n}|T|U^n \leq \cdots \leq \|T\|I$$

Hence $L := s - \lim_{n\to\infty} U^{*n}|T|U^n$ exists.

Theorem 2.8

If $T \in \mathcal{AN}(H)$ is a semi-hyponormal operator such that $ker\,(T^*) = ker(T)$, then $s - \lim_{n\to\infty} \Delta^n_{m_{f_{1/2}}}(T) \in \mathcal{AN}(H)$.

Notice: $\mathcal{AN}(H) \subsetneq \overline{\mathcal{AN}(H)}$

Thanks!

ご清聴ありがとうございました！