# The induced Aluthge transformations

**Takeaki YAMAZAKI (Toyo Univ., Japan)**

**7, January 2023**
**ILAS Special Session on Matrices and Operators, II**
**in the 2023 Joint Mathematics Meetings**

- Y., Linear Algebra Appl. 628 (2021), 1–28.

# Introduction

$B(H)$: $C^*$-algebra of bounded linear operators on a Hilbert space $H$.
Let $T \in B(H)$.

- $\sigma(T)$: Spectrum
- $W(T) = \{\langle Tx, x\rangle;\ \|x\| = 1, x \in H\}$: Numerical range
- $r(T) = \sup\{|\lambda|; \lambda \in \sigma(T)\}$: Spectral radius
- $w(T) = \sup\{|\lambda|; \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|;\ \|x\| = 1\}$ Numerical radius
- $\|T\| = \sup\{\|Tx\|;\ \|x\| = 1\}$ : Operator norm

**Basic Properties.**

- $co\, \sigma(T) \subseteq \overline{W(T)}$ , where $co\, \sigma(T)$ is a convex hull of $\sigma(T)$.
- $r(T) \le w(T) \le \|T\|\ and\ \frac{1}{2}\|T\| \le w(T) \le \|T\|$
- $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = r(T)$

# Introduction (Aluthge transformation)

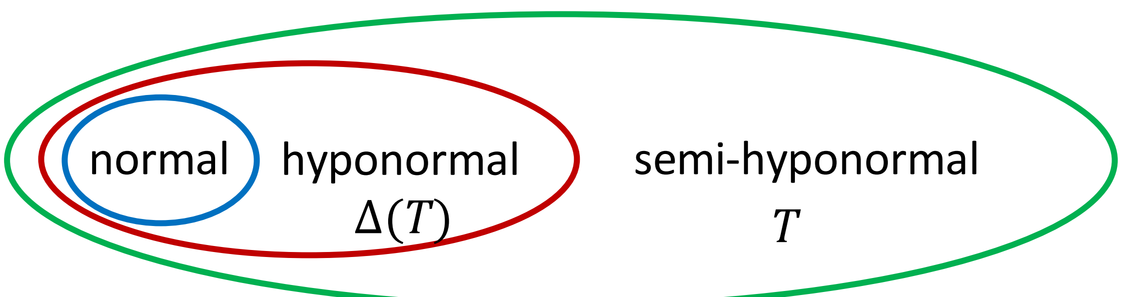
- $B(\mathcal{H})$: $C^*$-algebra of all bounded linear operators on a Hilbert space

**Definition 0.1 (Aluthge transformation).**

**Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition. Then the Aluthge transformation $\Delta(T)$ of $T$ is defined as follows.**

$$\Delta(T) := |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$$

- $\sigma(T) = \sigma(\Delta(T))$
- If $T$ is semi-hyponormal (i.e., $|T^*| \leq |T|$), then $\Delta(T)$ is hyponormal (i.e., $|\Delta(T)^*|^2 \leq |\Delta(T)|^2$).




**Loewner-Heinz ineq.**

$0 \leq A \leq B$

$\Longrightarrow A^\alpha \leq B^\alpha \quad \alpha \in [0,1]$

- Aluthge, Integral Equations Operator Theory, **13** (1990), 307-315.

# Introduction (Aluthge transformation)

## Basic properties

- $\Delta(T)$ has an invariant subspace iff $T$ does so.
- If $T$ is a $n \times n$ matrix, then iteration of the Aluthge transformation converges to a normal matrix $N$ such that $\sigma(N) = \sigma(T)$.
- $\lim_{n\to\infty} \|\Delta^n(T)\| = r(T)$,
  where $\Delta^n(T)$ means $n$-th iterated of the Aluthge transformation.
- $co\sigma(T) = \bigcap_{n\in\mathbb{N}} \overline{W(\Delta^n(T))}$.
- $\|\Delta(T)\| \leq \|T\|, w(\Delta(T)) \leq w(T)$ and $r(\Delta(T)) = r(T)$.


◆ Jung, Ko, Pearcy, IEOT, 37 (2000), 437-448.
◆ Antezana, Pujals, Stojanoff, Adv. Math., 226 (2011), 1591-1620.
◆ Ando, Y., Linear Algebra Appl., 375 (2003), 299-309.
◆ Y., Proc. Amer. Math. Soc., 130 (2002), 1131-1137.
◆ Ando, Linear and Multilinear Algebra, 52 (2004), 281-292.
◆ Wu, LAA, 357 (2002), 295-298.

# 2-Types of extensions

**1）Extension to $n$-tuple of operators**

$\Longrightarrow$ Spherical Aluthge transform

◆ Curto, Yoon, C. R. Acad. Sci. Paris **354** (2016), 1200-1204.

**2）Extension in the viewpoint of means.**

- $\Delta_\lambda(T) = |T|^{1-\lambda}U|T|^\lambda$ $\lambda \in [0,1]$, $\Delta_{s,t}(T) = |T|^s U|T|^t$ $s,t \in \mathbb{R}$ (geometric mean)
- $\hat{T} = \frac{|T|U+U|T|}{2}$ (arithmetic mean)

$\Longrightarrow$ Induced Aluthge transform ← **Today's talk**

◆ Huruya, Proc. Amer. Math. Soc. 125 (1997), 3617–3624.
◆ Furuta, Proc. Amer. Math. Soc., 125 (1997), 3617-3624.
◆ S.H. Lee, W.Y. Lee Yoon, J. Math. Anal. Appl., 410 (2014), 70-81.

# Introduction (Operator mean)

- $\mathcal{P}$: The set of all positive definite operators on a Hilbert space.

**Definition 1.1 (Operator mean)**

**Let $\mathfrak{M}$: $\mathcal{P}^2 \to \mathcal{P}$. If $\sigma$ satisfies the following conditions, then $\mathfrak{M}$ is called an operator mean.**

**1. $\mathfrak{M}(A,B) \le \mathfrak{M}(C,D)$ if $A \le C$ and $B \le D$,**

**2. $X^*\mathfrak{M}(A,B)X \le \mathfrak{M}(X^*AX, X^*BX)$ for all bounded linear operator $X$,**

**3. $\mathfrak{M}$ is upper semi-continuous on $\mathcal{P}^2$,**

**4. $\mathfrak{M}(I,I) = I$.**

- Kubo and Ando, Math. Ann., 246 (1980), 205-224.

# Introduction

◆ $\mathcal{M}$: The set of all operator monotone functions on $(0, \infty)$.

**Theorem 1.A (Representing function)**

**Let $\mathfrak{M}$ be an operator mean. Then $\exists f \in \mathcal{M}$ such that $f(1) = 1$ and**

$$\mathfrak{M}(A, B) = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}$$

**for all $A, B \in \mathcal{P}$.**

**Examples.** Let $\lambda \in [0,1]$.

- Arithmetic mean: $A \nabla_\lambda B = (1-\lambda)A + \lambda B$ $\;(f(x) = 1 - \lambda + \lambda x)$,
- Geometric mean: $A \sharp_\lambda B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\lambda} A^{\frac{1}{2}}$ $\;(f(x) = x^{\lambda})$,

$AB = BA$
$\Longrightarrow A \sharp_\lambda B = A^{1-\lambda} B^{\lambda}$

- Harmonic mean: $A !_\lambda B = [(1-\lambda)A^{-1} + \lambda B^{-1}]^{-1}$ $\;(f(x) = [1 - \lambda + \lambda x^{-1}]^{-1})$.

■ $\mathfrak{M}_f$: Operator mean with a representing function $f$.

◆ Kubo and Ando, Math. Ann., 246 (1980), 205-224.

# Motivation

- $\mathcal{M}_n$: $n \times n$ matrices
- $\mathcal{M}_n$ is a Hilbert space with inner product $\langle A, B \rangle \coloneqq tr\,(AB^*)$

**Definition 1.2 (Left and right multiplications)**

**Let $T \in \mathcal{M}_n$. Define linear mappings $\mathcal{M}_n \to \mathcal{M}_n$ by**

$$\mathbb{L}_T(X) \coloneqq TX, \qquad \mathbb{R}_T(X) \coloneqq XT \quad (X \in \mathcal{M}_n).$$

## Remark

- $\mathbb{L}_T$ and $\mathbb{R}_S$ are commuting, i.e.,

$$\mathbb{L}_T\mathbb{R}_S(X) = \mathbb{L}_T(XS) = TXS = \mathbb{R}_S(TX) = \mathbb{R}_S\mathbb{L}_T(X).$$

- If $T$ is positive semi-definite (or positive definite), then $\mathbb{L}_T$ and $\mathbb{R}_T$ are positive semi-definite (or positive definite). Especially

$$(\mathbb{L}_T)^\alpha = \mathbb{L}_{T^\alpha}, (\mathbb{R}_T)^\alpha = \mathbb{R}_{T^\alpha} \ \ (\alpha > 0 \ or \ \alpha \in \mathcal{R}).$$

- Geometric mean $\left(\mathbb{L}_T \#_{1/2} \mathbb{R}_S\right)(X) = T^{\frac{1}{2}}XS^{\frac{1}{2}}$.

Especially, if $T = U|T|$, then $\left(\mathbb{L}_{|T|} \#_{1/2} \mathbb{R}_{|T|}\right)(U) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = \Delta(T)$

# Matrices case

**Definition 1.3 (Induced Aluthge transformation).**
Let $\mathfrak{M}_f$ be an operator mean, and $T = U|T| \in \mathcal{M}_n$ be the polar decomposition of an invertible $T$. Then the **induced Aluthge transformation** $\Delta_{\mathfrak{M}_f}(\mathrm{T})$ respect to an operator mean $\mathfrak{M}_f$ is defined by

$$\Delta_{\mathfrak{M}_f}(T) := \mathfrak{M}_f\left(\mathbb{L}_{|T|}, \mathbb{R}_{|T|}\right)(U) = \mathbb{L}_{|T|} f\left(\mathbb{L}_{|T|}^{-1} \mathbb{R}_{|T|}\right)(\mathbf{U}).$$

**Examples.**

- Arithmetic mean case. $\Delta_{\mathfrak{M}}(\mathrm{T}) = (1-\lambda)|T|U + \lambda U|T|$
  (mean transform, S.H. Lee-W.Y. Lee-Yoon, 2014.)
- Geometric mean case. $\Delta_{\mathfrak{M}}(\mathrm{T}) = |T|^{1-\lambda}U|T|^{\lambda}$
  ($\lambda$ –Aluthge transform, Huruya, 1997.)


- S.H. Lee, W.Y. Lee Yoon, J. Math. Anal. Appl., 410 (2014), 70-81.
- Furuta, Proc. Amer. Math. Soc., 125 (1997), 3617-3624.

# Harmonic mean (matrices case)

**Theorem 1.1**

**Let $f(t) = \left[1 - \lambda + \lambda t^{-1}\right]^{-1} (\lambda \in [0,1])$, and let $T = U|T| \in \mathcal{M}_n$ be the polar decomposition of an invertible matrix $T$. Then**

$$\Delta_{\mathfrak{M}_f}(T) = \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}}\, dt.$$

The harmonic mean case for a **unilateral shift** is firstly considered by S.H. Lee, 2016.

- S.H. Lee, J. Chungcheong Math. Soc., 29 (2016), 123-135.

# Proof

**Theorem 1.B**
**Let $A$ and $B$ be operators whose spectra are contained in the open right half-plane and the open left-plane, respectively.**
**Then the solution of the equation $AX - XB = Y$ can be expressed as**

$$X = \int_0^\infty e^{-tA} Y e^{tB}\, dt.$$

**Proof of Theorem 1.1** Let $X = \Delta_{\mathfrak{M}_\lambda}(T)$. Then we have

$$\left[(1-\lambda)\mathbb{L}_{|T|^{-1}} + \lambda \mathbb{R}_{|T|^{-1}}\right]^{-1}(U) = X.$$

It is equivalent to

$$\begin{aligned} U &= \left[(1-\lambda)\mathbb{L}_{|T|^{-1}} + \lambda \mathbb{R}_{|T|^{-1}}\right](X) \\ &= ((1-\lambda)|T|^{-1})X - X(-\lambda|T|^{-1}) \end{aligned}$$

Hence $$\Delta_{\mathfrak{M}_\lambda}(T) = \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}}\, dt.$$

◆ Heinz, Math. Ann., 123 (1951), 415-438.

# Operator mean (matrices case)

**Theorem 1.2**
Let $\mathfrak{M}$ **be an operator mean**, and $T = U|T|$ be the polar decomposition of an invertible matrix $T$. Then **there exists a probability measure $d\mu(\lambda)$** on $[0,1]$, s.t.,

$$\Delta_{\mathfrak{M}}(T) = \int_0^1 \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}} \, dt \, d\mu(\lambda).$$

**Corollary 1.1** $\mathrm{tr}\big(\Delta_{\mathfrak{M}}(T)\big) = \mathrm{tr}(T)$.

- For **geometric mean** case, $\sigma(\Delta(T)) = \sigma(T)$ holds.
- For **arithmetic mean** case $\sigma(T)$ and $\sigma\big(\Delta_{\mathfrak{M}}(T)\big)$ are **not coincide**, generally.
  Cf. S.H. Lee, W.Y. Lee Yoon, J. Math. Anal. Appl., 410 (2014), 70-81.

# Operator mean (matrices case)

**Theorem 1.2**

**Let $\mathfrak{M}$ be an operator mean, and $T = U|T|$ be the polar decomposition of an invertible matrix $T$. Then there exists a probability measure $d\mu(\lambda)$ on $[0,1]$, s.t.,**

$$\boldsymbol{\Delta_{\mathfrak{M}}(T) = \int_0^1 \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}} dt d\mu(\lambda).}$$

**Proof.** Every representing function of operator mean can be given by

$$f(x) = \int_0^1 [1 - \lambda + \lambda x^{-1}]^{-1} d\mu(\lambda)$$

for a probability measure $d\mu(\lambda)$ on $[0,1]$. Hence, we have

$$\Delta_{\mathfrak{M}_f}(T) = \int_0^1 \left[(1-\lambda)\mathbb{L}_{|T|^{-1}} + \lambda \mathbb{R}_{|T|^{-1}}\right]^{-1} d\mu(\lambda)(U)$$

$$= \int_0^1 \Delta_{\mathfrak{H}_\lambda}(T)\, d\mu(\lambda) = \int_0^1 \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}} dt\, d\mu(\lambda).$$

Harmonic mean

# Another formula

Let $|T| = \sum_{i=1}^n s_i P_i$ be the spectral decomposition. Since

$$\mathcal{P}_f(s,t) := sf\left(\frac{t}{s}\right) = \int_0^1 [(1-\lambda)s^{-1} + \lambda t^{-1}]^{-1} d\mu(\lambda),$$

we have

$$\begin{aligned}\Delta_{\mathfrak{M}_f}(T) &= \int_0^1 \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}} \, dt d\mu(\lambda) \\ &= \int_0^1 \int_0^\infty \left(\sum_i e^{-(1-\lambda)ts_i^{-1}} P_i\right) U \left(\sum_j e^{-\lambda t s_j^{-1}} P_j\right) dt d\mu(\lambda) \\ &= \sum_{i,j} \int_0^1 \int_0^\infty e^{-\left\{(1-\lambda)s_i^{-1} + \lambda s_j^{-1}\right\}t} \, dt d\mu(\lambda) P_i U P_j \\ &= \sum_{i,j} \int_0^1 \left[(1-\lambda)s_i^{-1} + \lambda\, s_j^{-1}\right]^{-1} d\mu(\lambda) P_i U P_j \\ &= \sum_{i,j} \mathcal{P}_f(s_i, s_j) P_i U P_j \longrightarrow \int\int_{\sigma(|T|)^2} \mathcal{P}_f(s,t) dE_s U dE_t\,?\end{aligned}$$

**Operator case?**

# Examples

Let $|T| = \sum_{i=1}^{n} s_i P_i$ be the spectral decomposition. Then

$$\Delta_{\mathfrak{M}_f}(T) = \sum_{i,j} \mathcal{P}_f\left(s_i, s_j\right) P_i U P_j .$$

**Example.**

1. Arithmetic mean: $\mathcal{P}_f(s,t) = (1-\lambda)s + \lambda t$.

$$\Delta_{\mathfrak{M}_f}(T) = \sum_{i,j} [(1-\lambda)s_i + \lambda s_j] P_i U P_j$$

$$= \sum_{i,j} (1-\lambda) s_i\, P_i U P_j + \sum_{i,j} \lambda s_j\, P_i U P_j = (1-\lambda)|T|U + \lambda U |T|.$$

$$\sum_i P_i = I$$

2. Geometric mean: $\mathcal{P}_f(s,t) = s^{1-\lambda} t^{\lambda}$.

$$\Delta_{\mathfrak{M}_f}(T) = \sum_{i,j} s_i^{1-\lambda} s_j^{\lambda} P_i U P_j = \sum_{i,j} (s_i^{1-\lambda} P_i) U (s_j^{\lambda} P_j) = |T|^{1-\lambda} U |T|^{\lambda}.$$

# Examples

3. Power mean: $\mathcal{P}_f(s,t) = \left[(1-\lambda)s^{\frac{1}{2}} + \lambda t^{\frac{1}{2}}\right]^2$.

$$\Delta_{\mathfrak{M}_f}(T) = \sum_{i,j} \left[(1-\lambda)s_i^{\frac{1}{2}} + \lambda s_j^{\frac{1}{2}}\right]^2 P_i U P_j$$
$$= \sum_{i,j} \left[(1-\lambda)^2 s_i + 2\lambda(1-\lambda) s_i^{\frac{1}{2}} s_j^{\frac{1}{2}} + \lambda^2 s_j\right] P_i U P_j$$
$$= (1-\lambda)^2 |T| U + \lambda^2 U |T| + 2\lambda(1-\lambda) |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}.$$

Especially, if $\lambda = \frac{1}{2}$, then

$$\Delta_{\mathfrak{M}_f}(T) = \frac{1}{2}\left[\frac{|T|U + U|T|}{2} + |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}\right].$$

# Another formula

Let $|T| = \sum_{i=1}^{n} s_i P_i$ be the spectral decomposition. Since

$$\mathcal{P}_f(s,t) := sf\left(\frac{t}{s}\right) = \int_0^1 [(1-\lambda)s^{-1} + \lambda t^{-1}]^{-1} d\mu(\lambda),$$

we have

$$\begin{aligned}
\Delta_{\mathfrak{M}_f}(T) &= \int_0^1 \int_0^\infty e^{-(1-\lambda)t|T|^{-1}} U e^{-\lambda t|T|^{-1}} dt d\mu(\lambda) \\
&= \int_0^1 \int_0^\infty \left(\sum_i e^{-(1-\lambda)ts_i^{-1}} P_i\right) U \left(\sum_j e^{-\lambda t s_j^{-1}} P_j\right) dt d\mu(\lambda) \\
&= \sum_{i,j} \int_0^1 \int_0^\infty e^{-\left\{(1-\lambda)s_i^{-1} + \lambda s_j^{-1}\right\}t} dt d\mu(\lambda) P_i U P_j \\
&= \sum_{i,j} \int_0^1 \left[(1-\lambda)s_i^{-1} + \lambda\, s_j^{-1}\right]^{-1} d\mu(\lambda) P_i U P_j \\
&= \sum_{i,j} \mathcal{P}_f(s_i, s_j) P_i U P_j \longrightarrow \int\int_{\sigma(|T|)^2} \mathcal{P}_f(s,t) dE_s U dE_t\ ?
\end{aligned}$$

**Operator case?**

# Double operator integrals

**Definition 1.4**

**Let $H = \int_{\sigma(H)} s dE_s, K = \int_{\sigma(K)} t dF_t$ be the spectral decompositions of positive definite operators. For $\varphi \in L^\infty(\sigma(H) \times \sigma(K))$ and $X \in B(\mathcal{H})$, define the double operator integrals $\Phi(X)$ as follows:**

$$\Phi(X) := \int_{\sigma(H)} \int_{\sigma(K)} \varphi(s,t) dE_s X dE_t.$$

**Question.** For each $X \in B(\mathcal{H})$, does $\Phi(X) \in B(\mathcal{H})$ always hold?
**Answer.** **No!** If $\varphi$ is a **Schur multiplier**, then $\Phi(X) \in B(\mathcal{H})$ holds.

**Definition 1.5**

**For a function $\varphi \in L^\infty(\sigma(H) \times \sigma(K))$, if $\Phi(X)\ (:= \Phi(X)|_{C_1(\mathcal{H})}) \in C_1(\mathcal{H})$, then $\varphi$ is called the Schur multiplier.**

◆ Hiai and Kosaki, Lecture Notes in Math. 1820 (2003).

# Schur multiplier

**Definition 1.5**

For a function $\varphi \in L^{\infty}\big(\sigma(H) \times \sigma(K)\big)$, if $\Phi(X)\ (\coloneqq \Phi(X)|_{C_1(\mathcal{H})}) \in C_1(\mathcal{H})$, then $\varphi$ is called a **Schur multiplier**.

**Theorem 1.C**

Let $\varphi \in L^{\infty}\big(\sigma(H) \times \sigma(K)\big)$. Then **TFAE**.

(1) $\varphi$ is a Schur multiplier,

(2) there exists finite measure space $(\Omega, \sigma')$, and there exists $\alpha \in L^{\infty}(\sigma(H) \times \Omega)$ and $\beta \in L^{\infty}(\sigma(K) \times \Omega)$ such that

$$\varphi(s,t) = \int_{\Omega} \alpha(s,x)\beta(t,x)\, d\sigma'(x).$$

**Proposition 1.1**

Let $\mathfrak{M}_f$ be an operator mean with a representing function $f$.

Then $\mathcal{P}_f(s,t) \coloneqq sf\left(\frac{t}{s}\right)$ is a Schur multiplier.


- Hiai and Kosaki, Lecture Notes in Math. 1820 (2003).
- Peller, Funct. Anal. Appl. 19 (1985) 111–123.

# Proof

**Proposition 1.1**
**Let $\mathfrak{M}_f$ be an operator mean with a representing function $f$.**
**Then $\mathcal{P}_f(s,t) := sf\left(\frac{t}{s}\right)$ is a Schur multiplier.**

**Proof.** There exists a probably measure $d\mu$ on $[0,1]$, s.t.,

$$f(x) = \int_0^1 [1 - \lambda + \lambda x^{-1}]^{-1}\, d\mu(\lambda).$$

Then we have

$$\mathcal{P}_f(s,t) := sf\left(\frac{t}{s}\right) = \int_0^1 [(1-\lambda)s^{-1} + \lambda t^{-1}]^{-1}\, d\mu(\lambda)$$

$$= \int_0^1 \int_0^\infty e^{-[(1-\lambda)s^{-1} + \lambda t^{-1}]x}\, dx\, d\mu(\lambda)$$

$$= \int_0^1 \int_0^\infty e^{-(1-\lambda)s^{-1}x} e^{-\lambda t^{-1}x}\, dx\, d\mu(\lambda) = \int_0^1 \int_0^\infty \alpha(s,\lambda,x)\beta(t,\lambda,x)\, dx\, d\mu(\lambda).$$

$\alpha(s,\lambda,x)$ $\qquad$ $\beta(t,\lambda,x)$

# Operators case

**Definition 1.3 (Induced Aluthge transformation).**
Let $T \in B(\mathcal{H})$ be invertible with the polar decomposition $T = U|T|$. Let $|T| = \int_{\sigma(|T|)} s dE_s$ be the spectral decomposition. For each operator mean $\mathfrak{M}_f$, s.t., $f'(1) \in (0,1)$, **induced Aluthge transformation** $\Delta_{\mathfrak{M}_f}(\mathrm{T})$ respect to an operator mean $\mathfrak{M}_f$ is defined by

$$\Delta_{\mathfrak{M}_f}(T) := \int_{\sigma(|T|)} \int_{\sigma(|T|)} \mathcal{P}_f(s,t)\, dE_s U dE_t.$$

## Matrices case

$$\Delta_{\mathfrak{M}_f}(T) = \sum_{i,j} \mathcal{P}_f(s_i, s_j) P_i U P_j,$$

where $|T| = \sum_i s_i P_i$ is the spectral decomposition.

- If $\ker T \subseteq \ker T^*$ , then $\Delta_{\mathfrak{M}_f}(T)$ can be defined. However, it is not known to define $\Delta_{\mathfrak{M}_f}(T)$, in generally.

# Iteration (finite dimensional case)

$\Delta_{\mathfrak{M}}^{n}(T) \coloneqq \Delta_{\mathfrak{M}}^{n-1}\big(\Delta(T)\big), \Delta_{\mathfrak{M}}^{0}(T) = T.$

**Theorem 1.3**

**Let $\mathfrak{M}$ be a non-weighted arithmetic mean, and $T = U|T|$ be the polar decomposition of an invertible matrix $T$. Then the sequence $\{\Delta_{\mathfrak{M}}^{n}(T)\}$ converges to a normal matrix.**

Non-weighted arithmetic mean

$$\mathfrak{M}(A, B) = \frac{A + B}{2}, \qquad \Delta_{\mathfrak{M}}(T) = \frac{|T|U + U|T|}{2}$$

# Iteration (infinite dimensional case)

**Theorem 1.4**

Let $\mathfrak{M}_f$ **be an operator mean**, s.t., $f'(1) \in (0,1)$. Then there exists an operator in $B(\mathcal{H})$, s.t., the sequence $\{\Delta^n_{\mathfrak{M}_f}(T)\}$ **does not converge** in a week operator topology.

| | $\dim \mathcal{H} < +\infty$ | $\dim \mathcal{H} = +\infty$ |
|---|---|---|
| Arithmetic mean case | ○ | × |
| Geometric mean case (Aluthge transform) | ○ ◆ | × |
| General case | | × |

◆ Antezana, Pujals, Stojanoff, Adv. Math., 226 (2011), 1591-1620.

# Problem

Let $T = U|T| \in B(H)$. Then

Arithmetic mean case

$$T \text{ is quasinormal} \Leftrightarrow |T|U = U|T| \Leftrightarrow |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = T \Leftrightarrow \frac{|T|U+U|T|}{2} = T$$

Geometric mean case

**Problem.**

**Let $T \in B(\mathcal{H})$ be invertible and let $\mathfrak{M}_f$ be an operator mean , s.t., $f'(1) \in (0,1)$. Then TFAE?**

**(i) $T$ is quasinormal (i.e., $|T|U = U|T|$),**

**(ii) $\Delta_{\mathfrak{M}_f}(T) = T$.**

- The above problem is true in the matrices case.
- **Any application or any extension is welcome.**

# $\mathcal{AN}(H)$-operator

**Definition 1.6 (Norm attained operator).**

**Let $T \in B(H, K)$.**

**(i) $T$ is called a norm attained operator ($T \in \mathcal{N}(H, K)$), if there exists a unit vector $x \in H$ such that $\|Tx\| = \|T\|$.**

**(ii) $T$ is called an absolutely norm attained operator, if for any non-zero closed subspace $M \subseteq H$, $T|_M \in \mathcal{N}(M, K)$.**

If $T \in B(H)$ is absolutely norm attained operator, we write $T \in \mathcal{AN}(H)$, $T$ is $\mathcal{AN}(H)$-operator or $T$ has $\mathcal{AN}$-property.

Compact operators and isometry are $\mathcal{AN}(H)$-operator.


- Carvajal and Neves, IEOT, **72** (2012), 179-195.
- Pandy and Paulsen, J Aust Math Soc, **102** (2017) 369-391.
- Ramesh, J Aust Math Soc, **96** (2014) 386-395.

# Results

**Theorem 1.5**

**Let $f$ be a non-negative positive operator monotone function on $[0,\infty)$ with $f(1)=1$. Suppose that $T \in \mathcal{AN}(H)$ is invertible.**

**Then $\Delta_{\mathfrak{M}_f}(T) \in \overline{\mathcal{AN}(H)}$.**

**Notice:** $\mathcal{AN}(H) \subsetneq \overline{\mathcal{AN}(H)}$

**Example.**

$$T_n := \begin{pmatrix} 1/2 & & & & & & \\ & 1-1/3 & & & & & \\ & & 1-1/4 & & & & \\ & & & \ddots & & & \\ & & & & 1-1/(n+1) & & \\ & & & & & 1 & \\ & & & & & & \ddots \end{pmatrix}.$$

Then $T_n \in \mathcal{AN}(H)$ but $T_n \to T \notin \mathcal{AN}(H)$.

- Osaka Ramesh Udagawa Yamazaki, preprint.

# Results

**Theorem 1.6**

**For $\lambda \in [0,1]$, let $f_\lambda(t) = 1 - \lambda + \lambda t$ and $g_\lambda(t) = t^\lambda$. If $T \in \mathcal{AN}(H)$, then $\Delta_{\mathfrak{m}_{f_\lambda}}(T), \Delta_{\mathfrak{m}_{g_\lambda}}(T) \in \mathcal{AN}(H)$. Especially, $\Delta(T) \in \mathcal{AN}(H)$.**

- The case $f_\lambda(t) = 1 - \lambda + \lambda t$, $\Delta_{\mathfrak{m}_{f_\lambda}}(T) = (1-\lambda)|T|U + \lambda U|T|$.

- The case $g_\lambda(t) = t^\lambda$, $\Delta_{\mathfrak{m}_{g_\lambda}}(T) = |T|^{1-\lambda}U|T|^\lambda$.

  Especially, $\Delta(T) := \Delta_{\mathfrak{m}_{g_{\frac{1}{2}}}}(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ (Aluthge transformation).

- If $S, T \in \mathcal{AN}(H)$, we don't know whether $S + T \in \mathcal{AN}(H)$ or not.

  Hence if $f(t) = \left(\frac{1+t^{1/2}}{2}\right)^2$ $and\ T \in \mathcal{AN}(H)$, then we don't know

$$\Delta_{\mathfrak{m}_f}(T) = \frac{1}{2}\left(\frac{|T|U + U|T|}{2} + |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}\right) \in \mathcal{AN}(H)$$

  or not.

# Results

$T$ is semi-hyponormal $\Leftrightarrow |T| \geq |T^*| = U|T|U^*$. Then

$$|T| \leq U^*|T|U \leq U^{*2}|T|U^2 \leq \cdots \leq U^{*n}|T|U^n \leq \cdots \leq \|T\|I$$

Hence $L := s - \lim_{n\to\infty} U^{*n}|T|U^n$ exists.

**Theorem 1.7**

**Let $f_{1/2}(t) = \frac{1+t}{2}$ and $T \in B\ (H)$ be a semi-hyponormal operator with the polar decomposition $T = U|T|$. If $ker\ (T^*) = ker(T)$, then**

$$s - \lim_{n\to\infty} \Delta^n_{m_{f_{1/2}}}(T) = UL$$

**in the strong operator topology. Moreover, $UL$ is a normal operator and $\sigma(T) = \sigma(UL)$.**

◆ Osaka Ramesh Udagawa Yamazaki, preprint.

# Results

$T$ is semi-hyponormal $\Leftrightarrow |T| \geq |T^*| = U|T|U^*$. Then

$$|T| \leq U^*|T|U \leq U^{*2}|T|U^2 \leq \cdots \leq U^{*n}|T|U^n \leq \cdots \leq \|T\|I$$

Hence $L := s - \lim_{n\to\infty} U^{*n}|T|U^n$ exists.

**Theorem 1.8**

**If $T \in \mathcal{AN}(H)$ is a semi-hyponormal operator such that $ker\,(T^*) = ker(T)$, then $s - \lim_{n\to\infty} \Delta^n_{m_{f_{1/2}}}(T) \in \mathcal{AN}(H)$.**

**Notice:** $\mathcal{AN}(H) \subsetneq \overline{\mathcal{AN}(H)}$

◆ Osaka Ramesh Udagawa Yamazaki, preprint.

# Convergency of iteration

| | $\dim \mathcal{H} < +\infty$ | $\dim \mathcal{H} = +\infty$ | |
|---|---|---|---|
| | No condition | Semi-hyponormal | No condition |
| Arithmetic mean case | ○ | ○ | × |
| Geometric mean case (Aluthge transform) | ○ | | × |
| General case | | | × |

# Thanks!

Thank you for your attention!

# Matrices case

**Definition 1.3 (Induced Aluthge transformation).**
**Let $\mathfrak{M}_f$ be an operator mean, and $T = U|T| \in \mathcal{M}_n$ be the polar decomposition of an invertible $T$. Then the induced Aluthge transformation $\Delta_{\mathfrak{M}_f}(\mathrm{T})$ respect to an operator mean $\mathfrak{M}_f$ is defined by**

$$\Delta_{\mathfrak{M}_f}(T) := \mathfrak{M}_f\left(\mathbb{L}_{|T|}, \mathbb{R}_{|T|}\right)(U) = \mathbb{L}_{|T|} f\left(\mathbb{L}_{|T|}^{-1}\mathbb{R}_{|T|}\right)(\mathrm{U}).$$

Let $T = U|T|$ be the polar decomposition and

$$U = \left(u_{ij}\right), \ \ |T| = V^* diag(s_1, \ldots, s_n)V, \text{ (V: unitary).}$$

Then

$$\Delta_{\mathfrak{M}_f}(T) = V^*\{VUV^* \circ [\mathcal{P}_f(s_i, s_j)]\}V,$$

where $\mathcal{P}_f(s,t) \coloneqq sf\left(\frac{t}{s}\right)$ (perspective or solidarity).

# Proof

**Theorem 1.3**

**Let $\mathfrak{M}$ be a non-weighted arithmetic mean, and $T = U|T|$ be the polar decomposition of an invertible matrix $T$. Then the sequence $\{\Delta_{\mathfrak{M}}^{n}(T)\}$ converges to a normal matrix.**

**Proof.** Since $U$ is unitary, polar decompositions are

- $$\Delta_{\mathfrak{M}}(T) = \frac{|T|U + U|T|}{2} = U\frac{|T| + U^*|T|U}{2} = U|\Delta_{\mathfrak{M}}(T)|,$$
- $$\Delta_{\mathfrak{M}}^{2}(T) = \frac{|\Delta_{\mathfrak{M}}(T)|U + U|\Delta_{\mathfrak{M}}(T)|}{2} = U\frac{|\Delta_{\mathfrak{M}}(T)| + U^*|\Delta_{\mathfrak{M}}(T)|U}{2}$$
$$= U\frac{|T| + 2U^*|T|U + (U^*)^2|T|U^2}{2^2} = U\left|\Delta_{\mathfrak{M}}^{2}(T)\right|,$$
- $$\Delta_{\mathfrak{M}}^{n}(T) = U\,\frac{1}{2^{\mathrm{n}}}\sum_{k=0}^{n}\binom{n}{k}(U^*)^k|T|U^k = U\left|\Delta_{\mathfrak{M}}^{n}(T)\right|.$$

**Prove convergence!**

◆ Chabbabi, Curto and Mbekhta, Proc. AMS 147 (2019) 1119-1133.

# Proof

- $\Delta_{\mathfrak{M}}^{n}(T) = U \frac{1}{2^{\mathrm{n}}} \sum_{k=0}^{n} \binom{n}{k} (U^{*})^{k} |T| U^{k} = U \left| \Delta_{\mathfrak{M}}^{n}(T) \right|.$ **Prove convergence!**

Let

$$U = V^{*}DV, \text{where } V \text{ is unitary and } D = diag\left(e^{\theta_1 \sqrt{-1}}, \dots, e^{\theta_n \sqrt{-1}}\right).$$

For any unitary $V$,

$$\Delta_{\mathfrak{M}}(V^{*}TV) = V^{*}\Delta_{\mathfrak{M}}(T)V,$$

$$V\left|\Delta_{\mathfrak{M}}^{n}(T)\right|V^{*} = \left|\Delta_{\mathfrak{M}}^{n}(VTV^{*})\right| = \left|\Delta_{\mathfrak{M}}^{n}(DV|T|V^{*})\right|.$$

# Proof

- $\Delta_{\mathfrak{M}}^{n}(T) = U \frac{1}{2^{\mathrm{n}}} \sum_{k=0}^{n} \binom{n}{k} (U^*)^k |T| U^k = U \left| \Delta_{\mathfrak{M}}^{n}(T) \right|$. **Prove convergence!**

Let

$$U = V^* D V, \text{where } V \text{ is unitary and } D = diag\left(e^{\theta_1 \sqrt{-1}}, \dots, e^{\theta_n \sqrt{-1}}\right).$$

Put $P := V|T|V^*$. Then $(D^*)^k V|T|V D^k = \left[e^{k(\theta_j - \theta_i)\sqrt{-1}}\right] \circ P$ and

$$V\left|\Delta_{\mathfrak{M}}^{n}(T)\right|V^* = \left|\Delta_{\mathfrak{M}}^{n}(DV|T|V^*)\right| = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (D)^k V|T|V^* D^k$$

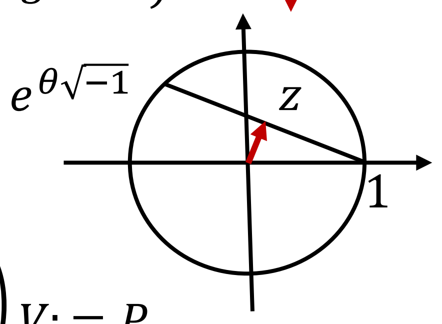
$$= \frac{1}{2^n} \left[ \sum_{k=0}^{n} \binom{n}{k} e^{k(\theta_j - \theta_i)\sqrt{-1}} \right] \circ P = \left[ \left( \frac{1 + e^{(\theta_j - \theta_i)\sqrt{-1}}}{2} \right)^n \right] \circ P$$

# Proof

$$V\left|\Delta_{\mathfrak{M}}^{n}(T)\right|V^{*}=\left|\Delta_{\mathfrak{M}}^{n}(DV|T|V^{*})\right|=\left[\left(\frac{1+e^{(\theta_j-\theta_i)\sqrt{-1}}}{2}\right)^{n}\right]\circ P$$

Notice that

$$\begin{cases}\left|\dfrac{1+e^{(\theta_j-\theta_i)\sqrt{-1}}}{2}\right|<1 \quad (\theta_j\neq\theta_i+2m\pi \ for\ all\ integer\ m),\\ \dfrac{1+e^{(\theta_j-\theta_i)\sqrt{-1}}}{2}=1 \quad (\theta_j=\theta_i+2m\pi \ for\ some\ integer\ m).\end{cases}$$




$$z=\frac{1+e^{\theta\sqrt{-1}}}{2}$$

Hence

$$\exists\lim_{n\to\infty}\left|\Delta_{\mathfrak{M}}^{n}(T)\right|=V^{*}\left(\exists\lim_{n\to\infty}\left|\Delta_{\mathfrak{M}}^{n}(DV|T|V^{*})\right|\right)V$$

$$=V^{*}\left(\exists\lim_{n\to\infty}\left[\left(\frac{1+e^{(\theta_j-\theta_i)\sqrt{-1}}}{2}\right)^{n}\right]\circ P\right)V:=P_0.$$

Moreover

$$\lim_{n\to\infty}\Delta_{\mathfrak{M}}^{n}(T)=UP_0=\Delta_{\mathfrak{M}}(UP_0)=\frac{P_0U+UP_0}{2}\Longrightarrow UP_0=P_0U\ (normal).$$

# Iteration (infinite dimensional case)

**Theorem 1.4**
**Let $\mathfrak{M}_f$ be an operator mean, s.t., $f'(1) \in (0, 1)$. Then there exists an operator in $B(\mathcal{H})$, s.t., the sequence $\{\Delta^n_{\mathfrak{M}_f}(\mathrm{T})\}$ does not converge in a week operator topology.**

Let $\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots) \in \ell^\infty$ and $W_\alpha$ be a weighted unilateral shift with a weight sequence $\alpha$, i.e.,

$$W_\alpha \boldsymbol{e_n} = \alpha_n \boldsymbol{e_{n+1}},$$

where $(\boldsymbol{e_0}, \boldsymbol{e_1}, \boldsymbol{e_2}, \dots)$ is a canonical base of $\ell^2$.

# Sketch of a proof.

**Step 1.** Let

$$W_{\alpha^{(1)}} := \Delta_{\mathfrak{M}_f}(W_\alpha).$$

Then $\alpha^{(1)} := (\alpha_0^{(1)}, \alpha_1^{(1)}, \ldots, \alpha_n^{(1)}, \ldots)$ and $\alpha_n^{(1)} = \mathcal{P}_f(\alpha_{n+1}, \alpha_n)$.

**Step 2.**

- Let

  $$\mathfrak{A} \cdots \textit{arithmetic mean}, \qquad \mathfrak{H} \cdots \textit{harmonic mean}$$

  with weight $f'(1)$.

- Let $W_\alpha$ be a weighted unilateral shift whose weights are either $a$ or $b$ $(a \neq b, a, b > 0)$.

- If there are only finitely many weights of $W_\alpha$ are equal to $a$.

- Then the sequence of the first weights of $\Delta_{\mathfrak{A}}^n(W_\alpha)$ and $\Delta_{\mathfrak{H}}^n(W_\alpha)$ are converge to $b$.

# Sketch of a proof.

**Step 3.** Let the sequence of the first weights of $\Delta_{\mathfrak{A}}^{n}(W_\alpha)$ and $\Delta_{\mathfrak{H}}^{n}(W_\alpha)$ are $\{\mathfrak{a}_n\}$ and $\{\mathfrak{h}_n\}$.

- Let $\alpha = (a, b, b, b, \ldots)$. Then there exists $n_1$, s.t.,

$$\left|\mathfrak{a}_{n_1} - b\right| < \frac{1}{2} \text{ and } \left|\mathfrak{h}_{n_1} - b\right| < \frac{1}{2}.$$

- Let $\alpha = (\overbrace{a, b, \ldots, b}^{n_1}, a, a, \ldots)$. Then there exists $n_2$, s.t.,

$$\left|\mathfrak{a}_{n_2} - a\right| < \frac{1}{2^2} \text{ and } \left|\mathfrak{h}_{n_2} - a\right| < \frac{1}{2^2}.$$

- Let $\alpha = (\underbrace{\overbrace{a, b, \ldots, b}^{n_1}, a, \ldots, a}_{n_2}, b, \ldots)$. Then there exists $n_3$, s.t.,

$$\left|\mathfrak{a}_{n_3} - b\right| < \frac{1}{2^3} \text{ and } \left|\mathfrak{h}_{n_3} - b\right| < \frac{1}{2^3}.$$

Repeating the procedure, we obtain that there exists a weighted unilateral shift $W_\alpha$ such that $\{\mathfrak{a}_n\}$ and $\{\mathfrak{h}_n\}$ do not converge.

# Sketch of a proof.

**Step 4.**

- Let $\mathfrak{M}_f$ be an arbitrary operator mean, s.t.,

$$[1 - f'(1) + f'(1)x^{-1}]^{-1} \leq f(x) \leq 1 - f'(1) + f'(1)x.$$

- Let the sequence of the first weights of $\Delta^n_{\mathfrak{M}_f}(W_\alpha)$ be $\{\mathfrak{m}_n\}$. Then by

$$\mathfrak{h}_n \leq \mathfrak{m}_n \leq \mathfrak{a}_n,$$

$\{\mathfrak{m}_n\}$ does not converge.