Limit of iterated induced Aluthge transformations of centered operators

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Induced Aluthge transformation

- For $f(t)$, define

$$\mathcal{P}_f(s,t) \coloneqq sf\left(\frac{t}{s}\right).$$

Definition (Induced Aluthge transformation).
Let $T \in B(H)$ be invertible with the polar decomposition $T = U|T|$. Let $|T| = \int_{\sigma(|T|)} s dE_s$ be the spectral decomposition. For each operator mean $\mathfrak{m}_f$ with a representing function f, the Induced Aluthge transformation $\Delta_{\mathfrak{m}_f}(T)$ respect to an operator mean $\mathfrak{m}_f$ is defined by

$$\Delta_{\mathfrak{m}_f}(T) := \int_{\sigma(|T|)} \int_{\sigma(|T|)} \mathcal{P}_f(s,t)\, dE_s U dE_t.$$

Define $\Delta^n_{\mathfrak{m}_f}(T) \coloneqq \Delta_{\mathfrak{m}_f}\left(\Delta^{n-1}_{\mathfrak{m}_f}(T)\right), \Delta^0_{\mathfrak{m}_f}(T) \coloneqq T$ for $n = 0,1,2,\ldots$

- Y., Linear Algebra Appl., **628** (2021) 1-28.

Induced Aluthge transformation

Example.

- **Mean transformation.** If $f(x) = \frac{1+x}{2}$ (arithmetic mean), Then
$$\Delta_{\mathfrak{m}_f}(T) = \frac{U|T| + |T|U}{2}.$$
- If $f(x) = \left(\frac{1+x^{\frac{1}{2}}}{2}\right)^2$ (power mean), then
$$\Delta_{\mathfrak{m}_f}(T) = \frac{1}{4}\left(U|T| + |T|U + 2|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}\right).$$
- **Aluthge transformation.** If $f(x) = \sqrt{x}$ (geometric mean), then
$$\Delta_{\mathfrak{m}_f}(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}.$$

◆ Lee, Lee, Yoon, J. Math. Anal. Appl., **410** (2014) 70-81.
◆ Aluthge, Integral Equations Operator Theory, **13** (1990) 307-315.

Convergency of iteration of $\{\Delta_{\mathbf{m}}^{n}(T)\}$

	$\dim \mathcal{H} < +\infty$	$\dim \mathcal{H} = +\infty$	
	No condition	Semi-hyponormal	No condition
Arithmetic mean case	○ [3]	○ [2]	× [3]
Geometric mean case (Aluthge transform)	○ [1]		× [3,4]
General case			× [3]

[1] Antezana Pujals Stojanoff, Adv. Math., **226** (2011), 1591-1620.
[2] Osaka Ramesh Udagawa Yamazaki, preprint.
[3] Cho Jung Lee, Integral Equ. Oper. Theory, **53** (2005) 321–329.
[4] Y., Linear Algebra Appl., **628** (2021) 1-28.

Centered operators

- $B(H)$: C^*-algebra of all bounded linear operators on a Hilbert space

Definition (Centered Operators).

Let $T = U|T|$ be the polar decomposition of $T \in B(H)$.

(i) T is called binormal, if $[|T|, |T^*|] = 0$.

(ii) T is called cnetered, if $\{U^{m*}|T|U^m, |T|, U^n|T|U^{n*}; n, m = 1, 2, \dots\}$ is commuting.

Example. Weighted shift and isometry.

- Morrel and Muhly, Studia Math., **51** (1974), 251-263.

Polar decomposition

Assume $T \in B(H)$ is invertible and $T = U|T|$ is the polar decomposition.

Theorem 1.
Let $\mathfrak{m}$ be an operator mean. If T is centered, then $\Delta_{\mathfrak{m}_f}(T)$ is invertible, centered and the polar decomposition is given as follows:

$$\Delta_{\mathfrak{m}}(T) = U\mathfrak{m}(U^*|T|U, |T|).$$

Moreover, for each non-negative integer n, $\Delta_{\mathfrak{m}}^n(T)$ is centered, and the polar decomposition of $\Delta_{\mathfrak{m}}^n(T)$ is

$$\Delta_{\mathfrak{m}}^n(T) = \mathrm{U}\mathfrak{M}_n(U, |T|),$$

where $\mathfrak{M}_0(U, |T|) = |T|$ and

$$\mathfrak{M}_n(U, |T|) = \mathfrak{m}\left(U^*\mathfrak{M}_{n-1}(U, |T|)U, \mathfrak{M}_{n-1}(U, |T|)\right).$$

Induced Aluthge transformation

Example. Assume T is invertible.

- **Mean transformation.** If $f(x)=\frac{1+x}{2}$, Then
$$\Delta^n_{\mathfrak{m}_f}(T)=U\left(\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}U^{*k}|T|U^k\right).$$
- If $f(x)=\left(\frac{1+x^r}{2}\right)^{\frac{1}{r}}$ and T is centered, then
$$\Delta^n_{\mathfrak{m}_f}(T)=U\left(\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}U^{*k}|T|^rU^k\right)^{\frac{1}{r}}.$$
- **Aluthge transformation.** If $f(x)=\sqrt{x}$ and T is centered, then
$$\Delta^n_{\mathfrak{m}_f}(T)=U\left(\prod_{k=0}^{n}U^{*k}|T|^{\binom{n}{k}}U^k\right)^{\frac{1}{2^n}}.$$

◆ Lee, Lee, Yoon, J. Math. Anal. Appl., **410** (2014) 70-81.

Semi-hyponormal (and centered) case

- T is semi-hyponormal if and only if $|T^*| \leq |T|$.

- If T is semi-hyponormal, then

$$L := \lim_{n\to\infty} U^{*n}|T|U^n$$

exists.

Theorem 2.
Let $T \in B(H)$ be an invertible centered operator and $\mathfrak{m}$ be a non-weighted operator mean, s.t., $\mathfrak{h}(A,B) \leq \mathfrak{m}(A,B) \leq \mathfrak{a}(A,B)$ for all positive invertible $A, B \in B(H)$, where $\mathfrak{h}$ and $\mathfrak{a}$ mean non-weighted harmonic and arithmetic means, respectively. If T is semi-hyponormal, then

$$\lim_{n\to\infty} \Delta_{\mathfrak{m}}^n(T) = UL.$$

Compact operators case

Theorem 3.

For $r \in [-1,1] \setminus \{0\}$, let $\mathfrak{m}_{f_r}$ be a non-weighted power mean, i.e., the representing function is $f_r(x) = \left(\frac{1+x^r}{2}\right)^{\frac{1}{r}}$. Assume that $T = U|T|$ is invertible and **centered**. If $T \in \mathcal{K}(H)$ or a n-by-n **matrix**, then **there exists** $\lim_{n\to\infty} \Delta^n_{\mathfrak{m}_{f_r}}(T)$.

Limit point

Theorem 4.

Let $T = U|T| \in \mathcal{M}_m$ be an invertible centered matrix.

If $U = V^* diag(\lambda_1, \dots, \lambda_m) \mathrm{V}$ such that V is unitary and $\lambda_i \neq \lambda_j$ for $i \neq j$.

Then for $f_r(x) = \left(\frac{1+x^r}{2}\right)^{\frac{1}{r}}$,

$$\lim_{n\to\infty} \Delta^n_{\mathfrak{m}_{f_r}}(T) = U \left[V^*(I \circ V|T|^r V^*)V\right]^{\frac{1}{r}}.$$

Example

Let $T = \begin{pmatrix} 0 & 0 & c \\ a & 0 & 0 \\ 0 & b & 0 \end{pmatrix}$ for $a, b, c > 0$ and $f_r(x) = \left(\frac{1+x^r}{2}\right)^{\frac{1}{r}}$. Then T is centered, and

$$L_r := \lim_{n\to\infty} \Delta^n_{\mathfrak{m}_{f_r}}(T) = \left(\frac{a^r + b^r + c^r}{3}\right)^{\frac{1}{r}} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Especially, if $r \to 0$, then

$$\lim_{n\to\infty} \Delta^n_{\mathfrak{m}_{f_r}}(T) = \sqrt[3]{abc} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Note that $\sigma(T) = \{\sqrt[3]{abc}, \sqrt[3]{abc}\omega, \sqrt[3]{abc}\omega^2\}$,

$$\sigma(L_r) = \left\{\left(\frac{a^r + b^r + c^r}{3}\right)^{\frac{1}{r}}, \left(\frac{a^r + b^r + c^r}{3}\right)^{\frac{1}{r}}\omega, \left(\frac{a^r + b^r + c^r}{3}\right)^{\frac{1}{r}}\omega^2\right\},$$

where ω is a complex number s.t. $\omega^3 = 1$.

Conclusion

$\Delta_{\mathfrak{m}}^{n}(T)$ の極限について。

	$\dim \mathcal{H} = +\infty$			$\dim \mathcal{H} < +\infty$
	No condition	Semi-hypo	compact	Matrices
Arithmetic mean	×	◎	◎	◎
Geometric mean (Aluthge 変換)	×	○	○	◎
Power mean	×	○	○	○
General case	×	○		
Limit point は具体的に求まる？		○ (◎:Arithmetic)		○ (◎:Arithmetic)
Limit point は一致する？		○		×

○ は centered という条件の下での収束。