Limit of iteration of the induced Aluthge transformations of centered operators

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Introduction (Aluthge transformation)

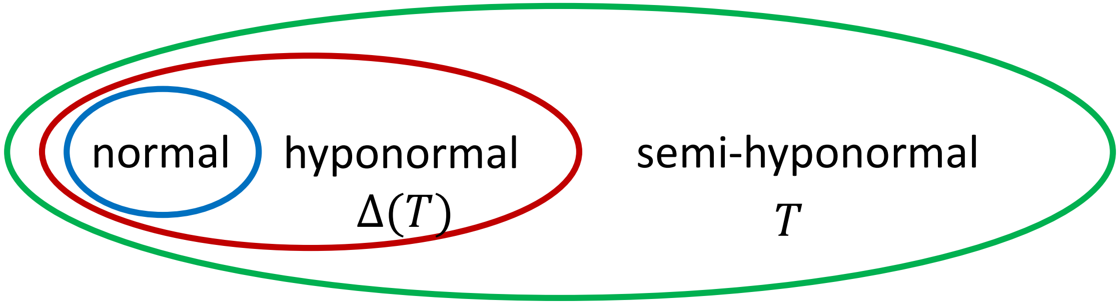
◆ $B(\mathcal{H})$: C^*-algebra of all bounded linear operators on a Hilbert space

Definition (Aluthge transformation).

Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition. Then the Aluthge transformation $\Delta(T)$ of T is defined as follows.

$$\Delta(T) := |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$$

- $\sigma(T) = \sigma(\Delta(T))$
- If T is semi-hyponormal (i.e., $|T^*| \leq |T|$),
  then $\Delta(T)$ is hyponormal (i.e., $|\Delta(T)^*|^2 \leq |\Delta(T)|^2$).



◆ Aluthge, Integral Equations Operator Theory, **13** (1990), 307-315.

Introduction (Aluthge transformation)

Basic properties

- $\Delta(T)$ has an invariant subspace iff T does so.
- If T is a $n \times n$ matrix, then iteration of the Aluthge transformation converges to a normal matrix N such that $\sigma(N) = \sigma(T)$.
- $\lim_{n\to\infty} \|\Delta^n(T)\| = r(T)$,
  where $\Delta^n(T)$ means n-th iterated of the Aluthge transformation.
- $co\sigma(T) = \bigcap_{n\in\mathbb{N}} \overline{W(\Delta^n(T))}$.

- Jung, Ko, Pearcy, IEOT, **37** (2000), 437-448.
- Antezana, Pujals, Stojanoff, Adv. Math., **226** (2011), 1591-1620.
- Ando, Y., Linear Algebra Appl., **375** (2003), 299-309.
- Y., Proc. Amer. Math. Soc., **130** (2002), 1131-1137.
- Ando, Linear and Multilinear Algebra, **52** (2004), 281-292.

Introduction (mean transformation)

Definition (mean transformation).
Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition. Then the mean transformation $\widehat{T}$ of T is defined as follows.

$$\widehat{T} := \frac{U|T| + |T|U}{2}$$

- $\sigma(T) \neq \sigma(\hat{T})$
- $\mathrm{Trace}(T) = \mathrm{Trace}(\hat{T})$
- If T is a $n \times n$ invertible matrix, then iteration of the mean transformation converges to a normal matrix N such that $\mathrm{Trace}(N) = \mathrm{Treace}(T)$ and $\mathrm{Trace}(|N|) = \mathrm{Treace}(|T|)$.

- Lee, Lee, Yoon, J. Math. Anal. Appl., **410** (2014) 70-81.
- Y., Linear Algebra Appl., **628** (2021) 1-28.

Introduction (Operator mean)

- $\mathcal{P}$: The set of all positive definite operators on a Hilbert space.

Operator mean.

Let $\mathfrak{M}: \mathcal{P}^2 \to \mathcal{P}$. If $\mathfrak{M}$ satisfies the following conditions, then $\mathfrak{M}$ is called an operator mean.

1. $\mathfrak{M}(A, B) \leq \mathfrak{M}(C, D)$ if $A \leq C$ and $B \leq D$,

2. $X^*\mathfrak{M}(A, B)X \leq \mathfrak{M}(X^*AX, X^*BX)$ for all bounded linear operator X,

3. $\mathfrak{M}$ is upper semi-continuous on $\mathcal{P}^2$,

4. $\mathfrak{M}(I, I) = I$.

- Kubo and Ando, Math. Ann., **246** (1980), 205-224.

Introduction

- $\mathcal{M}$: The set of all operator monotone functions on $(0,\infty)$.

Representing function.

Let $\mathfrak{M}$ be an operator mean. Then $\exists f \in \mathcal{M}$ such that $f(1) = 1$ and

$$\mathfrak{M}(A,B) = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}$$

for all $A, B \in \mathcal{P}$.

Examples. Let $\lambda \in [0,1]$.

- Arithmetic mean: $f(x) = 1 - \lambda + \lambda x$, $\mathfrak{M}_f(A,B) = (1-\lambda)A + \lambda B$.
- Geometric mean: $f(x) = x^{\lambda}$, $\mathfrak{M}_f(A,B) = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\lambda} A^{\frac{1}{2}}$.

- $\mathfrak{M}_f$: Operator mean with a representing function f.

- Kubo and Ando, Math. Ann., **246** (1980), 205-224.

Induced Aluthge transformation

- For $f(t)$, define

$$\mathcal{P}_f(s,t) \coloneqq sf\left(\frac{t}{s}\right).$$

Definition (Induced Aluthge transformation).
Let $T \in B(H)$ be invertible with the polar decomposition $T = U|T|$. Let $|T| = \int_{\sigma(|T|)} s dE_s$ be the spectral decomposition. For each operator mean $\mathfrak{m}_f$ with a representing function f, the Induced Aluthge transformation $\Delta_{\mathfrak{m}_f}(\mathrm{T})$ respect to an operator mean $\mathfrak{m}_f$ is defined by

$$\Delta_{\mathfrak{m}_f}(T) := \int_{\sigma(|T|)} \int_{\sigma(|T|)} \mathcal{P}_f(s,t)\, dE_s U dE_t.$$

Define $\Delta^n_{\mathfrak{m}_f}(T) \coloneqq \Delta_{\mathfrak{m}_f}\left(\Delta^{n-1}_{\mathfrak{m}_f}(T)\right), \Delta^0_{\mathfrak{m}_f}(T) \coloneqq T$ for $n = 0,1,2,\ldots$

- $\sigma(T) = \sigma(\Delta(T)) \neq \sigma\big(\Delta_{\mathfrak{m}}(T)\big)$ ($\mathfrak{m}$ is not a geometric mean),
- $\mathrm{Trace}(T) = \mathrm{Trace}(\Delta_{\mathfrak{m}}(T))$

◆ Y., Linear Algebra Appl., **628** (2021) 1-28.

Induced Aluthge transformation

Example.

- **Mean transformation.** If $f(x) = \frac{1+x}{2}$ (arithmetic mean), Then
$$\Delta_{\mathfrak{m}_f}(T) = \hat{T} = \frac{U|T| + |T|U}{2}.$$
- If $f(x) = \left(\frac{1+x^{\frac{1}{2}}}{2}\right)^2$ (power mean), then
$$\Delta_{\mathfrak{m}_f}(T) = \frac{1}{4}\left(U|T| + |T|U + 2|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}\right).$$
- **Aluthge transformation.** If $f(x) = \sqrt{x}$ (geometric mean), then
$$\Delta_{\mathfrak{m}_f}(T) = \Delta(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}.$$

◆ Lee, Lee, Yoon, J. Math. Anal. Appl., **410** (2014) 70-81.
◆ Aluthge, Integral Equations Operator Theory, **13** (1990) 307-315.

Convergency of iteration of $\{\Delta_{\mathrm{m}}^{n}(T)\}$

	$\dim \mathcal{H} < +\infty$	$\dim \mathcal{H} = +\infty$	
	No condition	Semi-hyponormal	No condition
Arithmetic mean case (Mean transform)	○ [4]	○ [2]	× [4]
Geometric mean case (Aluthge transform)	○ [1]		× [3,4]
Other mean case			× [4]

[1] Antezana Pujals Stojanoff, Adv. Math., **226** (2011), 1591-1620.
[2] Osaka Ramesh Udagawa Y., preprint.
[3] Cho Jung Lee, Integral Equ. Oper. Theory, **53** (2005) 321–329.
[4] Y., Linear Algebra Appl., **628** (2021) 1-28.

Centered operators

◆ $B(H)$: C^*-algebra of all bounded linear operators on a Hilbert space

Definition (Centered Operators).

Let $T = U|T|$ be the polar decomposition of $T \in B(H)$.

(i) T is called binormal, if $[|T|, |T^*|] = 0$.

(ii) T is called centered, if $\{U^{m*}|T|U^m, |T|, U^n|T|U^{n*}; n, m = 1, 2, \dots\}$ is commuting.

Example. Weighted shift and isometry.

$\{Normal\} \subset \{quasinormal\} \subset \{centered\} \subset \{binormal\} \not\subset \{subnormal\}$

◆ Morrel and Muhly, Studia Math., **51** (1974), 251-263.

Polar decomposition

Assume $T \in B(H)$ is invertible and $T = U|T|$ is the polar decomposition.

Theorem 1.
Let $\mathfrak{m}$ be an operator mean. If T is centered, then $\Delta_{\mathfrak{m}_f}(T)$ is invertible, centered and the polar decomposition is given as follows:

$$\Delta_{\mathfrak{m}}(T) = U\mathfrak{m}(U^*|T|U, |T|).$$

Moreover, for each non-negative integer n, $\Delta_{\mathfrak{m}}^n(T)$ is centered, and the polar decomposition of $\Delta_{\mathfrak{m}}^n(T)$ is

$$\Delta_{\mathfrak{m}}^n(T) = U\mathfrak{M}_n(U, |T|),$$

where $\mathfrak{M}_0(U, |T|) = |T|$ and

$$\mathfrak{M}_n(U, |T|) = \mathfrak{m}\big(U^*\mathfrak{M}_{n-1}(U, |T|)U, \mathfrak{M}_{n-1}(U, |T|)\big).$$

Induced Aluthge transformation

Example. Let $\lambda \in (0,1)$. Assume T is invertible.

- **Mean transformation.** If $f(x) = 1 - \lambda + \lambda x$, Then

$$\Delta_{\mathfrak{m}_f}^{n}(T) = U\left(\sum_{k=0}^{n} \binom{n}{k} (1-\lambda)^k \lambda^{n-k} U^{*k} |T| U^k\right).$$

- If $f(x) = (1 - \lambda + \lambda x^r)^{\frac{1}{r}}$ and T is centered, then

$$\Delta_{\mathfrak{m}_f}^{n}(T) = U\left(\sum_{k=0}^{n} \binom{n}{k} (1-\lambda)^k \lambda^{n-k} U^{*k} |T|^r U^k\right)^{\frac{1}{r}}.$$

- **Aluthge transformation.** If $f(x) = x^{\lambda}$ and T is centered, then

$$\Delta_{\mathfrak{m}_f}^{n}(T) = U\left(\prod_{k=0}^{n} U^{*k} |T|^{\binom{n}{k}(1-\lambda)^k \lambda^{n-k}} U^k\right).$$

◆ Chabbabi, Curto, Mbekhta, Proc. AMS., 147 (2019) 1119-1133.

Semi-hyponormal operators

- T is semi-hyponormal if and only if $U|T|U^* = |T^*| \leq |T|$.

Then

$$U|T|U^* \leq |T| \leq U^*|T|U \leq U^{*2}|T|U^2 \leq \cdots \leq U^{*n}|T|U^n \leq \cdots \leq \|T\|I$$

Hence if T is semi-hyponormal, then

$$\boldsymbol{L \coloneqq \lim_{n\to\infty} U^{*n}|T|U^n}$$

exists.

$$\{\boldsymbol{Normal}\} \subset \{quasinormal\} \subset \{\boldsymbol{subnormal}\} \subset \{\boldsymbol{hyponormal}\}$$
$$\subset \{\boldsymbol{semi-hyponormal}\} \subset \{w-hyponormal\} \subset \{class\ A\}$$
$$\subset \{\boldsymbol{paranormal}\}$$

Semi-hyponormal (and centered) case

- T is semi-hyponormal if and only if $|T^*| \leq |T|$.

- If T is semi-hyponormal, then
$$L \coloneqq \lim_{n\to\infty} U^{*n}|T|U^n$$
exists.

Theorem 2.

Let $T \in B(H)$ be an invertible centered operator and $\mathfrak{m}_f$ be an operator mean, s.t.,

$$\mathcal{P}_f(s,t) \coloneqq sf\left(\frac{t}{s}\right)$$

$$\left[(1-\lambda)a^{-1} + \lambda b^{-1}\right]^{-1} \leq af\left(\frac{b}{a}\right) \leq (1-\lambda)a + \lambda b \quad \text{for all } a, b > 0,$$

where $\lambda \coloneqq f'(1) \in (0,1)$. If T is semi-hyponormal, then

$$\lim_{n\to\infty} \Delta^n_{\mathfrak{m}_f}(T) = UL.$$

Note: The limit point UL is normal.

Outline of the proof

(1) Arithmetic mean case has been already shown.

(2) Because T is centered, $\Delta_{\mathfrak{m}}(T) = U\mathfrak{m}(U^*|T|U, |T|)$ is the polar decomposition. Moreover

$$\Delta_{\mathfrak{m}}^{n}(T) = U\mathfrak{M}_n(U, |T|).$$

Hence, **we prove $\lim_{n\to\infty} \mathfrak{M}_n(U, |T|) = L$.**

(3) Using (1), we can prove the harmonic mean case by using the following relation

$$\mathfrak{h}(a, b) = [(1-\lambda)a^{-1} + \lambda b^{-1}]^{-1} = \mathfrak{a}(a^{-1}, b^{-1})^{-1}.$$

(4) Using arithmetic-harmonic mean inequality

$$\left[(1-\lambda)a^{-1} + \lambda b^{-1}\right]^{-1} \leq af\left(\frac{b}{a}\right) \leq (1-\lambda)a + \lambda b,$$

we can prove Theorem 2.

◆ Osaka Ramesh Udagawa Y., preprint.

Matrices

Theorem 3.

Let $\mathfrak{m}_f$ be an operator mean with a representing function f satisfying $f(x) \le 1 - \lambda + \lambda x$ for all $x > 0$, where $\lambda := f'(1) \in (0, 1)$. Assume that $T = U|T|$ is a n-by-n invertible and centered matrix, then there exists

$$\lim_{n\to\infty} \Delta^n_{\mathfrak{m}_f}(T).$$

Note: The limit point is normal.

Outline of the proof

(1) Since T is centered, all $U^{*k}|T|U^k$ are simultaneous diagonalizable for all $k = 0,1,2,\dots$ We may assume

$$U^{*k}|T|U^k = diag\left(\lambda_{\sigma_k(1)}, \dots, \lambda_{\sigma_k(n)}\right),$$

where σ_k is a permutation on $\{1,2,\dots,n\}$.

(2) By (1), there exists $N \in \mathbb{N}$ such that $|T| = U^{*N}|T|U^N$.

(3) We can show that

$$\lim_{n\to\infty} \mathfrak{M}_n(U, |T|) = M\left(|T|, U^*|T|U, \dots, U^{*N-1}|T|U^{N-1}\right),$$

where M means the multivariate operator mean due to Pálfia.

◆ Pálfia, SIAM J. Matrix Anal. Appl. 32 (2011) 385–393.

Limit point

Theorem 4.

Let $T = U|T| \in \mathcal{M}_m$ be an invertible centered matrix.

If $U = V^* diag(\lambda_1, \ldots, \lambda_m) V$ such that V is unitary and $\lambda_i \neq \lambda_j$ for $i \neq j$.

Then for $f_r(x) = \left(\frac{1+x^r}{2}\right)^{\frac{1}{r}}$,

$$\lim_{n\to\infty} \Delta^n_{\mathfrak{m}_{f_r}}(T) = U\left[V^*(I \circ V|T|^r V^*)V\right]^{\frac{1}{r}},$$

where $\circ$ means the Hadamard product.

Outline of the proof

(1) Arithmetic mean case. **Scrutiny of the proof** that

$$\Delta_{\mathfrak{a}}^{n}(T) = U\left(\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} U^{*k}|T|U^k\right)$$

converges to a normal matrix, we obtain a concrete form of the limit point.

(2) Since T is centered, we have a formula of $\Delta_{\mathfrak{p}}^{n}(T)$

$$\Delta_{\mathfrak{p}}^{n}(T) = U\left(\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} U^{*k}|T|^r U^k\right)^{\frac{1}{r}}.$$

(3) By (1), we have a concrete form of the limit point of (2).

◆ Y., Linear Algebra Appl., **628** (2021) 1-28.

Example

Let $T = \begin{pmatrix} 0 & 0 & c \\ a & 0 & 0 \\ 0 & b & 0 \end{pmatrix}$ for $a, b, c > 0$ and $f_r(x) = \left(\frac{1+x^r}{2}\right)^{\frac{1}{r}}$. Then T is centered, and

$$L_r := \lim_{n\to\infty} \Delta^n_{\mathfrak{m}_{f_r}}(T) = \left(\frac{a^r + b^r + c^r}{3}\right)^{\frac{1}{r}} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Limit of mean transform ($r = 1$)

Especially, if $r \to 0$, i.e., $f_r(x) = \left(\frac{1+x^r}{2}\right)^{\frac{1}{r}} \to f_0(x) = \sqrt{x}$, then

$$\lim_{n\to\infty} \Delta^n_{\mathfrak{m}_{f_0}}(T) = \lim_{n\to\infty} \Delta^n(T) = \sqrt[3]{abc} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Limit of Aluthge transform

Note that $\sigma(T) = \left\{\sqrt[3]{abc}, \sqrt[3]{abc}\,\omega, \sqrt[3]{abc}\,\omega^2\right\}$ and

$$\sigma(L_r) = \left\{\left(\frac{a^r + b^r + c^r}{3}\right)^{\frac{1}{r}}, \left(\frac{a^r + b^r + c^r}{3}\right)^{\frac{1}{r}} \omega, \left(\frac{a^r + b^r + c^r}{3}\right)^{\frac{1}{r}} \omega^2\right\},$$

where ω is a complex number s.t. $\omega^3 = 1$.

$\sigma(T) = \sigma(\Delta(T)) \neq \sigma(\Delta_{\mathfrak{m}}(T))$ ($\mathfrak{m}$ is not a geometric mean), $\mathrm{Trace}(T) = \mathrm{Trace}(\Delta_{\mathfrak{m}}(T))$

Convergence of $\{\Delta_{\mathfrak{m}}^{n}(T)\}$

	$\dim \mathcal{H} = +\infty$		$\dim \mathcal{H} < +\infty$
	No condition	Semi-hypo	Matrices
Arithmetic mean (Mean transform)	×	◎	◎
Geometric mean (Aluthge transform)	×	○ (Th.2)	◎
Power mean	×	○ (Th.2)	○ (Th.3)
General case	×	○ (Th.2)	○ (Th.3)
Concrete form of the limit point		○ (Th.2) (◎:Arithmetic)	○: Power mean (Th. 4) (◎:Arithmetic)
Are the limit points of the above same?		○ (Th.2)	×

○: under the centered condition

Thanks!

Thank you very much for your attension!

Cảm ơn rất nhiều