Induced Aluthge sequence of compact operators

Takeaki Yamazaki (Toyo University)

2023 ICMTA-China-Korea-Japan

December 1–4, 2023

This is a joint work with Professor Hiroyuki Osaka.

This is a joint work with Professo

Contents

This is a joint work with Professo

contents

This is a joint work with Professo

Aluthge transform

- $\mathcal{B}(H)$: C^*-algebra of all bounded linear operators on a Hilbert space H.

Aluthge transform (Aluthge, IEOT 13 (1990), 307–315.)

Let $T = U|T| \in \mathcal{B}(H)$ be the polar decomposition. Then the Aluthge transform $\Delta(T)$ is defined by $\Delta(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$.

Properties of Aluthge transform.

1. $\sigma(T) = \sigma(\Delta(T))$. (Huruya, PAMS **125** (1997), 3617–3624.)
2. T has an invariant subspace if and only if $\Delta(T)$ does so.
   (Chō, Jung and Lee, IEOT **53** (2005), 321–329.)
3. If T is a n–by–n matrix, then $\{\Delta^n(T)\}$ converges to a normal matrix as $n \to \infty$.
   (Antezana, Pujals and Stojanoff, Adv. Math. **226** (2011), 1591–1620.)
4. $\lim_{n\to\infty} \|\Delta^n(T)\| = r(T)$. (Y. PAMS **130** (2002), 1131–1137.)
5. $\bigcap_{n\in\mathbb{N}} \overline{W(\Delta^n(T))} = \mathrm{co}\sigma(T)$. (Ando, LAMA **52** (2004), 281–292.)

This is a joint work with Professo

Mean transform

Mean transform (Lee, Lee and Yoon, JMAA 410 (2014), 70–81.)

Let $T = U|T| \in \mathcal{B}(H)$ be the polar decomposition. Then the mean transform $\hat{T}$ is defined by $\hat{T} = \frac{|T|U+U|T|}{2}$.

Properties of Mean transform.

1. If T is a matrix, then $\mathrm{Tr}(T) = \mathrm{Tr}(\hat{T})$.
2. Unfortunately, $\sigma(T) \neq \sigma(\hat{T})$, generally.
   (Lee, Lee and Yoon, JMAA **410** (2014), 70–81.)
3. If T is a n–by–n matrix, then $\{\hat{T}^{(n)}\}$ converges to a normal matrix as $n \to \infty$, where $\hat{T}^{(n)}$ means the n-th iteration of mean transform.
   (Y. LAA **628** (2021), 1–28.)

This is a joint work with Professo

Operator means

Operator mean (Kubo and Ando, Math. Ann. 246 (1979/80), 205–224.)

A binary operation $\mathfrak{m} : \mathcal{B}(H)_+ \times \mathcal{B}(H)_+ \to \mathcal{B}(H)_+$ is called an operator mean if the following four conditions hold:

1. If $A \leq C$ and $B \leq C$, then $\mathfrak{m}(A, B) \leq \mathfrak{m}(C, D)$.
2. $X^*\mathfrak{m}(A, B)X \leq \mathfrak{m}(X^*AX, X^*BX)$ for all $X \in \mathcal{B}(H)$.
3. If $A_n \searrow A$ and $B_n \searrow B$, then $\mathfrak{m}(A_n, B_n) \searrow \mathfrak{m}(A, B)$.
4. $\mathfrak{m}(I, I) = I$.

Theorem. (Kubo and Ando, Math. Ann. 246 (1979/80), 205–224.)

For any operator mean $\mathfrak{m}$, there exists a positive operator monotone function f defined on $(0, \infty)$ such that $f(1) = 1$ and

$$\mathfrak{m}(A, B) = A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}$$

Induced Aluthge transform

- Define $\mathcal{P}_f(s,t) := sf(\frac{t}{s})$.

Induced Aluthge transform (Y. LAA **628** (2021), 1–28.)

Let $T \in \mathcal{B}(H)$ be invertible with the polar decomposition $T = U|T|$, and let $|T| = \int_{\sigma(|T|)} s dE_s$ be the spectral decomposition. Then for any operator mean $\mathfrak{m}$ with the representing function f, the induced Aluthge transformation $\Delta_f(T)$ with respect to $\mathfrak{m}$ is defined as follows.

$$\Delta_f(T) := \int_{\sigma(|T|)} \int_{\sigma(|T|)} \mathcal{P}_f(s,t) dE_s U dE_t.$$

Define $\Delta_f^n(T) := \Delta_f(\Delta_f(T))$ and $\Delta_f^0(T) = T$. We call $\{\Delta_f^n(T)\}$ induced Aluthge sequence.

Properties of induced Aluthge transform.

1. If T is a matrix, then $\mathrm{Tr}(T) = \mathrm{Tr}(\Delta_f(T))$.
2. Unfortunately, $\sigma(T) \neq \sigma(\Delta_f(T))$ unless $f(t) = \sqrt{t}$.

This is a joint work with Professo

Example.

Let $T = U|T| \in \mathcal{B}(H)$ be invertible, and $\lambda \in [0,1]$.

(i) **Arithmetic mean.** Let $f_\lambda(x) = 1 - \lambda + \lambda x$. Then

$$\Delta_{f_\lambda}(T) = (1-\lambda)|T|U + \lambda U|T|.$$

Especially, $\Delta_{f_{1/2}}(T) = \hat{T}$ mean transform.

(ii) **Geometric mean.** Let $g_\lambda(x) = x^\lambda$. Then

$$\Delta_{g_\lambda}(T) = |T|^{1-\lambda}U|T|^\lambda.$$

We obtain $\Delta_{g_{1/2}} = \Delta$, the Aluthge transform.

(iii) **Power mean.** For $r \in [-1,1] \setminus \{0\}$, let $f_{r,\lambda}(x) = [1 - \lambda + \lambda x^r]^{\frac{1}{r}}$. Then

$$\Delta_{f_{\frac{1}{n},\lambda}}(T) = \left(\sum_{k=0}^{n} \binom{n}{k} (1-\lambda)^k \lambda^{n-k} |T|^{\frac{k}{n}} U |T|^{\frac{n-k}{n}} \right).$$

This is a joint work with Professo

Question.

For any $T \in \mathcal{B}(H)$ and an positive operator monotone function f on $(0, \infty)$ such that $f(1) = 1$, does $\{\Delta_f^n(T)\}$ converge?

Partial answers. Assume that $T \in \mathcal{B}(H)$ is invertible.

	No condition	Semi-hypo	Matrices
Arithmetic mean	$\times$[5]	$\bigcirc$[3]	$\bigcirc$[5]
Geometric mean	$\times$[2]	$\triangle$[4]	$\bigcirc$[1]
Power mean	$\times$[5]	$\triangle$[4]	$\triangle$[4]
General case	$\times$[5]	$\triangle$[4]	$\triangle$[4]

1. T is semi-hypo $\iff$ $|T^*| \leq |T|$.
2. $\triangle$: T is centered $\iff$ $U^{*n}|T|U^n, |T|, U^m|T|U^{*m}$ are commuting.

[1] Antezana, Pujals, Stojanoff, Adv. Math. **226** (2011), 1591–1620.

[2] Chō, Jung, Lee, IEOT **53** (2005), 321–329.

[3] Golla, Osaka, Udagawa, Y., Linear Algebra Appl. **678** (2023) 206–226.

[4] Osaka, Y., preprint.

[5] Y., LAA, **628** (2021), 1–28.

This is a joint work with Professo

Question.

How about the compact operators case?

Contents

This is a joint work with Professo

Iteration

Let $T = U|T|$ be the polar decomposition and invertible, and let $n = 0, 1, 2, ...$

Arithmetic mean case. (Chabbabi, Curto, Mbekhta, PAMS **147** (2019), 1119–1133.)

For $\lambda \in [0,1]$, let $f(x) = 1 - \lambda + \lambda x$. Then the polar decomposition of $\Delta_f^n(T)$ is given by

$$\Delta_f^n(T) = U\left[\sum_{k=0}^{n} \binom{n}{k} (1-\lambda)^k \lambda^{n-k} U^{*k}|T|U^k\right].$$

Power mean case. (Osaka, Y., preprint.)

Let T be a centered operator. For $\lambda \in [0,1]$ and $r \in [-1,1] \setminus \{0\}$, let $f(x) = [1 - \lambda + \lambda x^r]^{\frac{1}{r}}$. Then the polar decomposition of $\Delta_f^n(T)$ is given by

$$\Delta_f^n(T) = U\left[\sum_{k=0}^{n} \binom{n}{k} (1-\lambda)^k \lambda^{n-k} U^{*k}|T|^r U^k\right]^{\frac{1}{r}}.$$

This is a joint work with Professo

Ergodic theorem

Theorem 1.

Let $T \in \mathcal{B}(H)$ be a contraction, i.e., $\|T\| \le 1$ and $\lambda \in (0,1)$. Then, for every $v \in H$, the vector

$$L_n(\lambda)(v) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (1-\lambda)^k \lambda^{n-k} T^k v$$

converges to Pv as $n \to \infty$, where P is the orthogonal projection of H onto its subspace $\ker(T-I) = \{x \in H | Tx = x\}$.

$\lambda = \frac{1}{2}$ case has been shown by Dykema, Schultz, TAMS **361** (2009), 6583–6593.

Theorem 2.

Let $T \in \mathcal{B}(H)$ be a contraction, i.e., $\|T\| \le 1$ and $\lambda \in (0,1)$. Then, for every $v \in H$, there exists $\lim_{n\to\infty} L_n(\lambda)(v)$, and it does not depend on λ.

This is a joint work with Professo

Compact operators case.

Let $\mathcal{K}(H)$ and $\mathcal{F}(H)$ be the sets of all compact and finite rank operators in $\mathcal{B}(H)$, respectively.

Theorem 3. (Compact operators case)

For $\lambda \in (0,1)$ and $r \geq 0$, let $f_{r,\lambda}(x) = [1-\lambda+\lambda x^r]^{1/r}$. Assume that $T = U|T| \in \mathcal{B}(H)$ is centered. If $T \in \mathcal{K}(H)$ or $T \in \mathcal{F}(H)$, then there exists $\lim_{n\to\infty} \Delta^n_{f_{r,\lambda}}(T)$. Moreover, the limit point does not depend on λ.

- If $r = 1$, the centered condition of T is not needed.
- $f_{r,\lambda}(x) = [1-\lambda+\lambda x^r]^{1/r}$ is a representing function of the power mean.
- $\lim_{r\to 0} f_{r,\lambda}(x) = x^\lambda$, a representing function of the geometric mean.

This is a joint work with Professo

Sketch of the proof (finite rank operators case).

(1) Let $\mathcal{C}_2(H)$ be the Schatten class with a inner product $\langle A, B\rangle = \mathrm{Tr} B^* A$. Then $\mathcal{C}_2(H)$ is a Hilbert spaces, and $\mathcal{F}(H) \subset \mathcal{C}_2(H)$. Since the polar decomposition of $\Delta_f^n(T)$ is

$$\Delta_f^n(T) = U\left[\sum_{k=0}^{n}\binom{n}{k}(1-\lambda)^k\lambda^{n-k}U^{*k}|T|^rU^k\right]^{\frac{1}{r}},$$

we consider the positive part only. Especially, we consider $|\Delta_f^n(T)|^r$.

Let $\alpha(X) := U^*XU$ on $\mathcal{F}(H)$. Then $\alpha \in \mathcal{B}(\mathcal{F}(H))$ and $\alpha^n(X) := U^{*n}XU^n$.
Then by Theorem 1,

$$L_n(\lambda)(|T|^r) = \sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)(1-\lambda)^k\lambda^{n-k}\alpha^k(|T|^r) = \sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)(1-\lambda)^k\lambda^{n-k}U^{*k}|T|^rU^k$$

converges.

This is a joint work with Professo

Sketch of the proof (compact operators case).

Let $\mathcal{K}(H)_+$ (resp. $\mathcal{F}(H)_+$) be the set of all positive definite compact (resp. finite rank) operators, respectively.

Using the following fact and finite rank operators case, we can prove the compact operators case.

If $T \in \mathcal{K}(H)$, then $|T| \in \mathcal{K}(H)_+$ and there exists $\{P_m\} \subset \mathcal{F}(H)_+$ such that $\|P_m - |T|\| \to 0$ as $m \to \infty$. $\square$

Question.

Let $T \in \mathcal{K}(H)$. For any operator mean $\mathfrak{m}$, does $\{\Delta_{\mathfrak{m}}^n(T)\}$ converge as $n \to \infty$?

This is a joint work with Professo

Contents

This is a joint work with Professo

$\mathcal{AN}$ and $\mathcal{AM}$ operators

$\mathcal{AN}$ operators (Carvajal, Neves, IEOT 72 (2012), 179–195).

T: absolutely norm attaining operator or $T \in \mathcal{AN}(H)$. $\Longleftrightarrow$ There exists a unit vector $x \in M$ such that $\|T|_M\| = \|T|_M x\|$ for all closed subspace $M \subseteq H$.

$\mathcal{AM}$ operators ([1,2]).

T: absolutely minimal attaining operator or $T \in \mathcal{AM}(H)$. $\Leftrightarrow$ There exists a unit vector $x \in M$ such that $m(T|_M) = \|T|_M x\|$, where $m(T) = \inf\{\|Tx\| \mid x \in S_{H_1}\}$ for all closed subspace $M \subseteq H$.

[1] Bala, Ramesh, Banach J. Math. Anal. **14** (2020), 630–649.

[2] Carvajal, Neves, Bull. Braz. Math. Soc. (N.S.) **45** (2014), 293–312.

This is a joint work with Professo

$\mathcal{AN}$ and $\mathcal{AM}$ operators

1. Class of $\mathcal{AN}$ operators contains compact operators.
   (Carvajal, Neves, IEOT 72 (2012), 179–195.)
2. If T is isometry then $T \in \mathcal{AN}(H) \cap \mathcal{AM}(H)$.
3. If T is an orthogonal projection with a finite dimensional kernel, then $T \in \mathcal{AN}(H)$.
   (Carvajal, Neves, IEOT 72 (2012), 179–195.)
4. If T is a partial isometry with a finite dimensional kernel, then $T \in \mathcal{AN}(H)$.
   (Bala, Ramesh, Banach J. Math. Anal. 14 (2020), 630–649.)

This is a joint work with Professo

Theorem 4.

For $\lambda \in (0,1)$ and $r \in [-1,1] \setminus \{0\}$, let $f_{r,\lambda}(x) = [1-\lambda+\lambda x^r]^{1/r}$. Let $T = U|T| \in \mathcal{B}(H)$ be invertible and centered. If $T \in \mathcal{AN}(H)$ (resp. $T \in \mathcal{AM}(H)$). Then, there exists $\lim_{n\to\infty} \Delta^n_{f_{r,\lambda}}(T) \in \mathcal{AN}(H)$ (resp. $\mathcal{AM}(H)$). Moreover, the limit point does not depend on λ.

- If $r = 1$, the centered condition of T is not needed.
- $f_{r,\lambda}(x) = [1-\lambda+\lambda x^r]^{1/r}$ is a representing function of the power mean.
- $\lim_{r\to 0} f_{r,\lambda}(x) = x^\lambda$, a representing function of the geometric mean.

This is a joint work with Professo

Sketch of the proof ($\mathcal{AN}$ operators case).

Let $T \in \mathcal{AN}(H)$ with the polar decomposition $T = U|T|$. Then $|T| \in \mathcal{AN}(H)_+$. Moreover $|T|^r \in \mathcal{AN}(H)_+$ for $r \geq 0$ ($|T|^r \in \mathcal{AM}(H)_+$ for $r < 0$), and there exist $\alpha \geq 0$, $K \in \mathcal{K}(H)_+$ and $F \in \mathcal{F}(H)_+$ such that

$$|T|^r = \alpha I + K - F,$$

$KF = O$ and $0 \leq F \leq \alpha I$. (Naidu, Ramesh, Proc. Indian Acad. Sci. (Math. Sci.) 129 (2019) 54.)

Hence by Theorem 3,

$$\begin{aligned}
|\Delta_f^n(T)|^r &= \sum_{k=0}^{n} \binom{n}{k} (1-\lambda)^k \lambda^{n-k} U^{*k} |T|^r U^k \\
&= \sum_{k=0}^{n} \binom{n}{k} (1-\lambda)^k \lambda^{n-k} U^{*k} (\alpha I + K - F) U^k \\
&= \alpha I + \sum_{k=0}^{n} \binom{n}{k} (1-\lambda)^k \lambda^{n-k} U^{*k} K U^k - \sum_{k=0}^{n} \binom{n}{k} (1-\lambda)^k \lambda^{n-k} U^{*k} F U^k
\end{aligned}$$

converges.

This is a joint work with Professo

Sketch of the proof ($\mathcal{AM}$ operators case).

Let $T \in \mathcal{AM}(H)$ with the polar decomposition $T = U|T|$. Then $|T| \in \mathcal{AM}(H)_+$. Moreover $|T|^r \in \mathcal{AM}(H)_+$ for $r \geq 0$ ($|T|^r \in \mathcal{AN}(H)_+$ for $r < 0$), and there exist $\alpha \geq 0$, $K \in \mathcal{K}(H)_+$ and $F \in \mathcal{F}(H)_+$ such that

$$|T|^r = \alpha I - K + F,$$

$KF = O$ and $0 \leq K \leq \alpha I$.
(Ganesh, Ramesh, Sukumar, JMAA 428 (2015) 457–470.)

By the same way to the $\mathcal{AN}$ operators case, we can prove Theorem 4.

Contents

This is a joint work with Professo

An example in finite von Neumann algebra

Let (X, μ) be a probability space, ϕ be an invertible, measure preserving transformation of X.

Consider the crossed product algebra $M = L^\infty(X, \mu) \rtimes_{\alpha_\phi} \mathbb{Z}$.

- There is a unitary U such that $\alpha_\phi(f) = UfU^* = f \circ \phi$ for all $f \in L^\infty(X, \mu)$.
- Linear span $\{U^k f \mid k \in \mathbb{Z}, f \in L^\infty(X, \mu)\}$ is strongly dense in M.
- Any element in $\{U^k f \mid k \in \mathbb{Z}, f \in L^\infty(X, \mu)\}$ is centered.

Theorem 5.

Let $T = U|T| \in M$ such that $|T| \in L^\infty(X, \mu)$. For $r \in [-1, 1]\backslash\{0\}$, $\lambda \in (0, 1)$ and $f_{r,\lambda}(x) = [1 - \lambda + \lambda x^r]^{1/r}$,

$$\Delta^n_{f_{r,\lambda}}(T) \to UH \ (n \to \infty)$$

in the strongly operator topology, where $H = (E^\phi(|T|^r))^{1/r}$.

The proofs are similar to Dykema, Schultz, TAMS **361** (2009), 6583–6593.

This is a joint work with Professo

Conclusion.

Question.

For any $T \in \mathcal{B}(H)$ and an positive operator monotone function f on $(0, \infty)$ such that $f(1) = 1$, does $\{\Delta_f^n(T)\}$ converge?

	No condition	Semi-hypo	Matrices	Compact	$\mathcal{AN}$ and $\mathcal{AM}$
Arithmetic mean	$\times$[5]	○[3]	○[5]	○	○
Geometric mean	$\times$[2]	△[4]	○[1]	△	△
Power mean	$\times$[5]	△[4]	△[4]	△	△
General case	$\times$[5]	△[4]	△[4]	not yet	not yet

[1] Antezana, Pujals, Stojanoff, Adv. Math. **226** (2011), 1591–1620.

[2] Chō, Jung, Lee, IEOT **53** (2005), 321–329.

[3] Golla, Osaka, Udagawa, Y. Linear Algebra Appl. **678** (2023) 206–226.

[4] Osaka, Y., preprint.

[5] Y., LAA, **628** (2021), 1–28.

This is a joint work with Professo

Thanks!

Thank you very much for your attention!

This is a joint work with Professo