# The induced Aluthge sequence of compact operators

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# Aluthge transform

- $\mathcal{B}(H)$: $C^*$-algebra of all bounded linear operators on a Hilbert space $H$.

**Aluthge transform (Aluthge, IEOT 13 (1990), 307–315.)**

Let $T = U|T| \in \mathcal{B}(H)$ be the polar decomposition. Then the Aluthge transform $\Delta(T)$ is defined by $\Delta(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$.

**Properties of Aluthge transform.**

1. $\sigma(T) = \sigma(\Delta(T))$. (Huruya, PAMS **125** (1997), 3617–3624.)
2. $T$ has an invariant subspace if and only if $\Delta(T)$ does so.
   (Chō, Jung and Lee, IEOT **53** (2005), 321–329.)
3. If $T$ is a $n$–by–$n$ matrix, then $\{\Delta^n(T)\}$ converges to a normal matrix as $n \to \infty$.
   (Antezana, Pujals and Stojanoff, Adv. Math. **226** (2011), 1591–1620.)
4. $\lim_{n\to\infty} \|\Delta^n(T)\| = r(T)$. (Y. PAMS **130** (2002), 1131–1137.)
5. $\bigcap_{n\in\mathbb{N}} \overline{W(\Delta^n(T))} = \mathrm{co}\sigma(T)$. (Ando, LAMA **52** (2004), 281–292.)

# Mean transform

## Mean transform (Lee, Lee and Yoon, JMAA **410** (2014), 70–81.)

Let $T = U|T| \in \mathcal{B}(H)$ be the polar decomposition. Then the mean transform $\hat{T}$ is defined by $\hat{T} = \frac{|T|U+U|T|}{2}$.

**Properties of Mean transform.**

1. If $T$ is a matrix, then $\mathrm{Tr}(T) = \mathrm{Tr}(\hat{T})$.
2. Unfortunately, $\sigma(T) \neq \sigma(\Delta(T))$, generally.
   (Lee, Lee and Yoon, JMAA **410** (2014), 70–81.)
3. If $T$ is a $n$–by–$n$ matrix, then $\{\hat{T}^{(n)}(T)\}$ converges to a normal matrix as $n \to \infty$, where $\hat{T}^{(n)}$ means the $n$-th iteration of mean transform.
   (Y. LAA **628** (2021), 1–28.)

# Induced Aluthge transform

- Define $\mathcal{P}_f(s,t) := sf(\frac{t}{s})$.

**Induced Aluthge transform (Y. LAA 628 (2021), 1–28.)**

Let $T \in \mathcal{B}(H)$ be invertible with the polar decomposition $T = U|T|$, and let $|T| = \int_{\sigma(|T|)} s dE_s$ be the spectral decomposition. Then for any operator mean $\mathfrak{m}$ with the representing function $f$, the induced Aluthge transformation $\Delta_f(T)$ with respect to $\mathfrak{m}$ is defined as follows.

$$\Delta_f(T) := \int_{\sigma(|T|)} \int_{\sigma(|T|)} \mathcal{P}_f(s,t) dE_s U dE_t.$$

Define $\Delta_f^n(T) := \Delta_f(\Delta_f^{n-1}(T))$ and $\Delta_f^0(T) = T$. We call $\{\Delta_f^n(T)\}$ induced Aluthge sequence.

**Properties of induced Aluthge transform.**

1. If $T$ is a matrix, then $\mathrm{Tr}(T) = \mathrm{Tr}(\Delta_f(T))$.
2. Unfortunately, $\sigma(T) \neq \sigma(\Delta_f(T))$ unless $\mathfrak{m}$ is a geometric mean.

# Example.

Let $T = U|T| \in \mathcal{B}(H)$ be invertible, and $\lambda \in [0,1]$.

(i) **Arithmetic mean.** Let $f_\lambda(x) = 1 - \lambda + \lambda x$. Then

$$\Delta_{f_\lambda}(T) = (1-\lambda)|T|U + \lambda U|T|.$$

Especially, $\Delta_{f_{1/2}}(T) = \hat{T}$ mean transform.

(ii) **Geometric mean.** Let $g_\lambda(x) = x^\lambda$. Then

$$\Delta_{g_\lambda}(T) = |T|^{1-\lambda}U|T|^\lambda.$$

We obtain $\Delta_{g_{1/2}} = \Delta$, the Aluthge transform.

(iii) **Power mean.** For $r \in [-1,1] \setminus \{0\}$, let $f_{r,\lambda}(x) = [1-\lambda+\lambda x^r]^{\frac{1}{r}}$. Then

$$\Delta_{f_{\frac{1}{n},\lambda}}(T) = \left(\sum_{k=0}^{n} \binom{n}{k} (1-\lambda)^k \lambda^{n-k} |T|^{\frac{k}{n}} U |T|^{\frac{n-k}{n}}\right).$$

## Question.

For any $T \in \mathcal{B}(H)$ and an positive operator monotone function $f$ on $(0,\infty)$ such that $f(1) = 1$, **does $\{\Delta_f^n(T)\}$ converge?**

**Partial answers.** Assume that $T \in \mathcal{B}(H)$ is invertible.

| | No condition | Semi-hypo | Matrices |
|---|---|---|---|
| Arithmetic mean | $\times$[5] | ◯[3] | ◯[5] |
| Geometric mean | $\times$[2] | $\triangle$[4] | ◯[1] |
| Power mean | $\times$[5] | $\triangle$[4] | $\triangle$[4] |
| General case | $\times$[5] | $\triangle$[4] | $\triangle$[4] |

1. $T$ is semi-hypo $\iff$ $|T^*| \le |T|$.
2. $\triangle$: $T$ is centered $\iff$ $U^{*n}|T|U^n, |T|, U^m|T|U^{*m}$ are commuting.

[1] Antezana, Pujals, Stojanoff, Adv. Math. **226** (2011), 1591–1620.

[2] Chō, Jung, Lee, IEOT **53** (2005), 321–329.

[3] Golla, Osaka, Udagawa, Y., Linear Algebra Appl. **678** (2023) 206–226.

[4] Osaka, Y., preprint.

[5] Y., LAA, **628** (2021), 1–28.

# Question.

**How about the compact operators case?**

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# Iteration

Let $T = U|T|$ be the polar decomposition and invertible, and let $n = 0, 1, 2, \ldots$

**Arithmetic mean case. (Chabbabi, Curto, Mbekhta, PAMS 147 (2019), 1119–1133.)**

For $\lambda \in [0,1]$, let $f(x) = 1 - \lambda + \lambda x$. Then the polar decomposition of $\Delta_f^n(T)$ is given by

$$\Delta_f^n(T) = U\left[\sum_{k=0}^{n} \binom{n}{k} (1-\lambda)^k \lambda^{n-k} U^{*k}|T|U^k\right].$$

**Power mean case.(Osaka, Y., preprint.)**

Let $T$ be a centered operator. For $\lambda \in [0,1]$ and $r \in [-1,1] \setminus \{0\}$, let $f(x) = [1 - \lambda + \lambda x^r]^{\frac{1}{r}}$. Then the polar decomposition of $\Delta_f^n(T)$ is given by

$$\Delta_f^n(T) = U\left[\sum_{k=0}^{n} \binom{n}{k} (1-\lambda)^k \lambda^{n-k} U^{*k}|T|^r U^k\right]^{\frac{1}{r}}.$$

# Ergodic theorem

**Theorem 1.**

Let $T \in \mathcal{B}(H)$ be a contraction, i.e., $\|T\| \leq 1$ and $\lambda \in (0,1)$. Then, for every $v \in H$, the vector

$$L_n(\lambda)(v) = \sum_{k=0}^{n} \binom{n}{k} (1-\lambda)^k \lambda^{n-k} T^k v$$

converges to $Pv$ as $n \to \infty$, where $P$ is the orthogonal projection of $H$ onto its subspace $\ker(T-I) = \{x \in H | Tx = x\}$. The limit point does not depend on $\lambda$.

$\lambda = \frac{1}{2}$ case has been shown by Dykema, Schultz, TAMS **361** (2009), 6583–6593.

# Compact operators case.

Let $\mathcal{K}(H)$ and $\mathcal{F}(H)$ be the sets of all compact and finite rank operators in $\mathcal{B}(H)$, respectively.

**Theorem 2. (Compact operators case)**

For $\lambda \in (0,1)$ and $r \geq 0$, let $f_{r,\lambda}(x) = [1-\lambda+\lambda x^r]^{1/r}$. Assume that $T = U|T| \in \mathcal{B}(H)$ is centered. If $T \in \mathcal{K}(H)$ or $T \in \mathcal{F}(H)$, then there exists $\lim_{n\to\infty} \Delta^n_{f_{r,\lambda}}(T)$. Moreover, the limit point does not depend on $\lambda$.

- If $r = 1$, the centered condition of $T$ is not needed.
- $f_{r,\lambda}(x) = [1-\lambda+\lambda x^r]^{1/r}$ is a representing function of the power mean.
- $\lim_{r\to 0} f_{r,\lambda}(x) = x^\lambda$, a representing function of the geometric mean.

**Question.**

Let $T \in \mathcal{K}(H)$. For **any operator mean $\mathfrak{m}$,** does $\{\Delta^n_{\mathfrak{m}}(T)\}$ converge as $n \to \infty$?

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# $\mathcal{AN}$ and $\mathcal{AM}$ operators

**$\mathcal{AN}$ operators (Carvajal, Neves, IEOT 72 (2012), 179–195).**

$T$: absolutely norm attaining operator or $T \in \mathcal{AN}(H)$. $\Longleftrightarrow$ There exists a unit vector $x \in M$ such that $\|T|_M\| = \|T|_M x\|$ for all closed subspace $M \subseteq H$.

**$\mathcal{AM}$ operators ([1,2]).**

$T$: absolutely minimal attaining operator or $T \in \mathcal{AM}(H)$. $\Leftrightarrow$ There exists a unit vector $x \in M$ such that $m(T|_M) = \|T|_M x\|$, where $m(T) = \inf\{\|Tx\| \mid x \in S_{H_1}\}$ for all closed subspace $M \subseteq H$.


[1] Bala, Ramesh, Banach J. Math. Anal. **14** (2020), 630–649.

[2] Carvajal, Neves, Bull. Braz. Math. Soc. (N.S.) **45** (2014), 293–312.

# $\mathcal{AN}$ and $\mathcal{AM}$ operators

1. Class of $\mathcal{AN}$ operators contains compact operators.
   (Carvajal, Neves, IEOT 72 (2012), 179–195.)
2. If $T$ is isometry then $T \in \mathcal{AN}(H) \cap \mathcal{AM}(H)$.
3. If $T$ is an orthogonal projection with a finite dimensional kernel, then $T \in \mathcal{AN}(H)$.
   (Carvajal, Neves, IEOT 72 (2012), 179–195.)
4. If $T$ is a partial isometry with a finite dimensional kernel, then $T \in \mathcal{AN}(H)$.
   (Bala, Ramesh, Banach J. Math. Anal. 14 (2020), 630–649.)

$$\mathcal{K}(H) \subsetneq \mathcal{AN}(H) \subsetneq \overline{\mathcal{AN}(H)} \subsetneq \mathcal{B}(H)$$

$$\mathcal{K}(H) \subsetneq \mathcal{AM}(H) \subsetneq \overline{\mathcal{AM}(H)} \subsetneq \mathcal{B}(H)$$

## Theorem 3.

For $\lambda \in (0,1)$ and $r \in [-1,1] \setminus \{0\}$, let $f_{r,\lambda}(x) = [1-\lambda+\lambda x^r]^{1/r}$. Let $T = U|T| \in \mathcal{B}(H)$ be invertible and centered. If $T \in \mathcal{AN}(H)$ (resp. $T \in \mathcal{AM}(H)$). Then, there exists $\lim_{n\to\infty} \Delta^n_{f_{r,\lambda}}(T) \in \mathcal{AN}(H)$ (resp. $\mathcal{AM}(H)$). Moreover, the limit point does not depend on $\lambda$.

- If $r = 1$, the centered condition of $T$ is not needed.
- $f_{r,\lambda}(x) = [1-\lambda+\lambda x^r]^{1/r}$ is a representing function of the power mean.
- $\lim_{r\to 0} f_{r,\lambda}(x) = x^\lambda$, a representing function of the geometric mean.

# Conclusion.

## Question.

For any $T \in \mathcal{B}(H)$ and an positive operator monotone function $f$ on $(0,\infty)$ such that $f(1) = 1$, does $\{\Delta_f^n(T)\}$ converge?

| | No condition | Semi-hypo | Matrices | Compact | $\mathcal{AN}$ and $\mathcal{AM}$ |
|---|---|---|---|---|---|
| Arithmetic mean | $\times[5]$ | ◯[3] | ◯[5] | ◯ | ◯ |
| Geometric mean | $\times[2]$ | $\triangle[4]$ | ◯[1] | $\triangle$ | $\triangle$ |
| Power mean | $\times[5]$ | $\triangle[4]$ | $\triangle[4]$ | $\triangle$ | $\triangle$ |
| General case | $\times[5]$ | $\triangle[4]$ | $\triangle[4]$ | not yet | not yet |


[1] Antezana, Pujals, Stojanoff, Adv. Math. **226** (2011), 1591–1620.

[2] Chō, Jung, Lee, IEOT **53** (2005), 321–329.

[3] Golla, Osaka, Udagawa, Y. Linear Algebra Appl. **678** (2023) 206–226.

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