Ann. Funct. Anal. (2022) 13:27
https://doi.org/10.1007/s43034-022-00173-w

Tusi Mathematical Research Group

ORIGINAL PAPER

Furuta type inequalities via operator means and applications to Kadison's type inequalities

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Received: 27 October 2021 / Accepted: 4 February 2022

Abstract

We give a characterization of chaotic order via an arbitrary operator mean σ as follows. For $p, r > 0$,

$$\log A \geq \log B \quad \text{if and only if} \quad A^{-r\alpha} \, \sigma_h \, B^{p\alpha} \leq I,$$

for all $\alpha \geq 0$, where A and B are positive invertible operators, h is a normalized operator monotone function on $(0, \infty)$ satisfying $h(t^s) \leq h(t)^s$ for all $t > 0$, $s \geq 1$ and $h'(1) = \frac{r}{p+r}$. It is a generalization of the well-known characterization of chaotic order using operator geometric mean. We also obtain Furuta type inequalities via operator means. As applications of the result, we generalize an asymmetric Kadison's inequality as follows:

$$h_\alpha\left(\left|\phi(A^p)^\lambda \phi(A^q)^\mu\right|^2\right) \leq \phi(A^{2\alpha(p\lambda+q\mu)})$$

for all $p, q, \lambda, \mu \geq 0$ satisfying $2\alpha(p\lambda + q\mu) \leq p + 2q\mu$, $q \leq 2\alpha(p\lambda + q\mu) \leq 2q$, $0 \leq p \leq q$ and unital positive linear map ϕ.

Communicated by Yuki Seo.

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Published online: 23 February 2022

Keywords Operator monotone function · Operator mean · Furuta inequality · Positive unital linear map · Kadison's inequality

Mathematics Subject Classification 47A63 · 47B50 · 15A45

1 Introduction

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\ \cdot\ ,\ \cdot\ \rangle$ and $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. An operator A is said to be positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$, denoted by $A \geq 0$. For self-adjoint operators A and B, $A \geq B$ means that $A - B \geq 0$. An operator A is positive definite if $\langle Ax, x\rangle > 0$. Here, $\mathcal{P}$ and $\mathcal{S}$ represents the set of all positive definite operators and self-adjoint operators, respectively. A real valued function f defined on an interval $\mathcal{I} \subseteq \mathbb{R}$ is said to be operator monotone if $A \geq B$ implies $f(A) \geq f(B)$ for self-adjoint operators A, B whose spectra are contained in $\mathcal{I}$. A continuous function f defined on an interval $\mathcal{I} \subset \mathbb{R}$ is operator concave on $\mathcal{I}$ if $f((1-\lambda)A + \lambda B) \geq (1-\lambda)f(A) + \lambda f(B)$ for all real number $0 \leq \lambda \leq 1$ and for self-adjoint operators A, B whose spectra are contained in $\mathcal{I}$. It is well-known that a positive continuous function f on $(0, \infty)$ is operator monotone if and only if it is operator concave [3, 14].

Kubo and Ando [12] obtain that for a given positive operator monotone function f on $(0, \infty)$, one can define the binary operation σ_f on positive operators A and B as follows:

$$A\sigma_f B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}.$$

We call σ_f the operator connection associated with f. An operator monotone function f on $(0, \infty)$ is said to be normal if $f(1) = 1$. In what follows, by $\mathcal{O}^+$ we indicate the set of positive normalized operator monotone functions on $(0, \infty)$. If $f \in \mathcal{O}^+$ then $f'(1) = w \in [0, 1]$, and σ_f is called the (w-)weighted operator mean with a representing function f. The operator mean corresponding to the operator monotone function $f(x) = 1 - w + wx$, denoted by ∇_w, is called the weighted arithmetic mean. The operator mean corresponding to the operator monotone function $f(x) = [1 - w + wx^{-1}]^{-1}$, denoted by $!_w$, is called the weighted harmonic mean. When $f(x) = x^w$, the associated mean is denoted by $\sharp_w$ and is called the weighted geometric mean. We write $\nabla, \sharp$ and $!$ for $\nabla_{\frac{1}{2}}, \sharp_{\frac{1}{2}}$ and $!_{\frac{1}{2}}$, respectively.

Let A and B be positive definite operators. Fujii et al. [5] proved the following equivalence relation.

$$\log B \leq \log A \quad \Longleftrightarrow \quad A^{-r} \sharp_{\frac{r}{p+r}} B^p \leq I \quad \text{for all } p, r \geq 0.$$

It is known as the essential part of the Furuta inequality [6].

In Sect. 2, we shall generalize the above characterization of the chaotic order ($\log B \leq \log A$) as follows.

$$\log B \leq \log A \quad \Longleftrightarrow \quad A^{-r\alpha}\ \sigma_h\ B^{p\alpha} \leq I \quad \text{for all } \alpha \geq 0,$$

where $h \in \mathcal{O}^+$ such that $h(t^s) \le h(t)^s$ for all $s \ge 1, t > 0$, $h'(1) = \frac{r}{p+r}$ and the operator mean σ_h satisfies $!_{\frac{r}{p+r}} \le \sigma_h \le \nabla_{\frac{r}{p+r}}$.

Let $\mathcal{H}, \mathcal{K}$ be complex Hilbert spaces. A map $\phi : B(\mathcal{H}) \to B(\mathcal{K})$ is called a positive unital linear map if and only if ϕ is linear, $\phi(I_{\mathcal{H}}) = I_{\mathcal{K}}$ and $\phi(A) \ge 0$ for all $A \ge 0$, where $I_{\mathcal{H}}$ and $I_{\mathcal{K}}$ are identity operators on $\mathcal{H}$ and $\mathcal{K}$, respectively. Bourin and Ricard [4] show an interesting asymmetric Kadison's inequality for a positive operator A as follows.

$$|\phi(A^p)\phi(A^q)| \le \phi(A^{p+q}) \quad \text{for } 0 \le p \le q. \tag{1}$$

Furuta obtained a generalization of this inequality in [8], and then a further extension is given in [11, 19].

In Sect. 3, we shall improve the above Kadison's inequalities for a positive operator A as follows:

For each $\alpha \in [0, 1]$, *let* $h_\alpha \in \mathcal{O}^+$ *such that* $h(t^s) \le h(t)^s$ *for all* $s \ge 1, t > 0$, *and the operator mean* σ_h *satisfies* $! \le \sigma_h \le \nabla$. *Then*

$$h_\alpha\left(\left|\phi(A^p)^\lambda \phi(A^q)^\mu\right|^2\right) \le \phi(A^{2\alpha(p\lambda+q\mu)})$$

for $p, q, \lambda, \mu \ge 0$ *such that* $0 \le p \le q, 2\alpha(p\lambda + q\mu) \le p + 2q\mu, q \le 2\alpha(p\lambda + q\mu) \le 2q$ *and unital positive linear map* ϕ.

2 Characterizations of chaotic order via operator means

Recall that the famous Lie–Trotter formula is stating that

$$e^{A+B} = \lim_{p\to 0}\left(e^{\frac{pA}{2}} e^{pB} e^{\frac{pA}{2}}\right)^{\frac{1}{p}}$$

for $A, B \in \mathcal{S}$. To prove our main results, we need the operator-mean variant of the Lie–Trotter formula in the following lemma.

Lemma 2.1 [9, page 16] *Let* $h \in \mathcal{O}^+$ *such that* $h'(1) = w \in [0, 1]$ *and its associated operator mean* σ_h *satisfies* $!_w \le \sigma_h \le \nabla_w$. *Let* $A, B \in \mathcal{S}$. *Then*

$$e^{(1-w)A+wB} = \lim_{p\to 0}\left((e^{pA})\sigma_h(e^{pB})\right)^{\frac{1}{p}}.$$

Wada [16] gave generalizations of the Ando–Hiai inequality [2]. They are characterized by the power monotone increasing functions and power monotone decreasing functions as in Lemmas 2.2 and 2.3.

Lemma 2.2 (Ando–Hiai type inequality 1, [16]) *Let* $h \in \mathcal{O}^+$. *Then the following statements are equivalent*:

(i) $h(t^s) \le h(t)^s$ *for all* $t > 0, s \ge 1$;
(ii) $A\sigma_h B \le I \Longrightarrow A^s\sigma_h B^s \le I$ *for all* $A, B \in \mathcal{P}$ *and* $s \ge 1$.

Lemma 2.3 (Ando–Hiai type inequality 2, [16]) *Let* $g \in \mathcal{O}^+$. *Then the following statements are equivalent*:

(i) $g(t)^s \le g(t^s)$ *for all* $t > 0, s \ge 1$;
(ii) $A\sigma_g B \ge I \Longrightarrow A^s\sigma_g B^s \ge I$ *for all* $A, B \in \mathcal{P}$ *and* $s \ge 1$.

Recently, Wada and one of the author [17] (see also [15]) had proved a converse of Loewner–Heinz inequality in the view point of operator means as follows: Let $f, h \in \mathcal{O}^+$ with $h'(1) = w \in [0, 1]$. For $A, B \in \mathcal{S}$,

$$wB \le (1-w)A \quad \Longleftrightarrow \quad f(-\lambda A + I)\sigma_h f(\lambda B + I) \le I$$

for all sufficiently small $\lambda \ge 0$. The following Theorems 2.4 give more precise discussion of this relation.

Theorem 2.4 *Let* $f,\ h \in \mathcal{O}^+$ *with* $h'(1) = w \in [0, 1]$ *such that* $h(t^s) \le h(t)^s$ *for all* $s \ge 1, t > 0$ *and the operator mean* σ_h *satisfies* $!_w \le \sigma_h \le \nabla_w$. *Then for* $A, B \in \mathcal{S}$, *the following statements are equivalent*:

(i) $wB \le (1-w)A$;
(ii) $f(-\lambda A + I)\ \sigma_h f(\lambda B + I) \le I$ *for all sufficiently small* $\lambda \ge 0$;
(iii) $e^{-rA}\sigma_h e^{rB} \le I$ *for all* $r \ge 0$.

Proof (i) $\Longrightarrow$ (ii): Let us assume that $wB \le (1-w)A$. For sufficiently small $\lambda > 0$, we have $-\lambda A + I, \lambda B + I \in \mathcal{P}$ and

$$(1-w)(-\lambda A + I) + w(\lambda B + I) \le I.$$

Consequently, we have

$$\begin{aligned} f(-\lambda A + I)\ \sigma_h f(\lambda B + I) &\le f(-\lambda A + I)\ \nabla_w f(\lambda B + I) \\ &= (1-w)f(-\lambda A + I) + wf(\lambda B + I) \\ &\le f((1-w)(-\lambda A + I) + w(\lambda B + I)) \le I, \end{aligned}$$

where the second inequality follows from concavity of f and the last inequality follows from $f \in \mathcal{O}^+$.

(ii) $\Longrightarrow$ (iii): Let $0 < \lambda \le p$. By Lemma 2.2,

$$f(-\lambda A + I)^{\frac{p}{\lambda}}\ \sigma_h f(\lambda B + I)^{\frac{p}{\lambda}} \le I.$$

Letting $\lambda \to 0$, we get $\lim_{\lambda\to 0} f(\lambda A + I)^{1/\lambda} = e^{f'(1)A}$. Hence, we have

$$f(-\lambda A + I)^{\frac{p}{\lambda}}\sigma_h f(\lambda B + I)^{\frac{p}{\lambda}} \to e^{-f'(1)pA}\ \sigma_h\ e^{f'(1)pB}$$

as $\lambda \to 0$. By putting $r = f'(1)p \ge 0$, we have (iii).

(iii) $\implies$ (i):
We get

$$\left(e^{-rA}\sigma_h e^{rB}\right)^{\frac{1}{r}} \le I \quad \text{for all } r > 0.$$

By the assumption $!_w \le \sigma_h \le \nabla_w$ and Lemma 2.1, we have

$$\exp\left[-(1-w)A + wB\right] = \lim_{r\to 0}\left(e^{-rA}\sigma_h e^{rB}\right)^{\frac{1}{r}} \le I.$$

Therefore, we conclude $wB \le (1-w)A$. □

As a consequence of Theorem 2.4, we generalize a characterization of chaotic order.

Theorem 2.5 *Let $g, h \in \mathcal{O}^+$ such that $h(t^s) \le h(t)^s$ and $g(t)^s \le g(t^s)$ for all $s \ge 1$, $p, r, t > 0$, $h'(1) = g'(1) = \frac{r}{p+r}$ and the operator means σ_g and σ_h satisfy $!_{\frac{r}{p+r}} \le \sigma_h \le \nabla_{\frac{r}{p+r}}$ and $!_{\frac{r}{p+r}} \le \sigma_g \le \nabla_{\frac{r}{p+r}}$, respectively. Then for any $A, B \in \mathcal{P}$, the following statements are equivalent*:

(i) $\log B \le \log A$;
(ii) $A^{-r\alpha}\sigma_h B^{p\alpha} \le I$ *for all* $\alpha \ge 0$;
(iii) $A^{r\alpha}\sigma_g B^{-p\alpha} \ge I$ *for all* $\alpha \ge 0$.

Proof Proof of (i) $\implies$ (ii). The assumption $\log B \le \log A$ is equivalent to

$$\frac{r}{p+r}\log B^p \le \frac{p}{p+r}\log A^r,$$

so we have

$$A^{-r\alpha}\sigma_h B^{p\alpha} \le I$$

for all $\alpha \ge 0$ by applying Theorem 2.4.

Proof of (ii) $\implies$ (i). It is immediate from Theorem 2.4.

Proof of (i) $\iff$ (iii). Let $g_1(t) := g(t^{-1})^{-1}$. Then $g_1 \in \mathcal{O}^+$, $g_1'(1) = \frac{r}{p+r}$ and $g_1(t^s) \le g_1(t)^s$ holds for all $s \ge 1$ and $t > 0$. By using (i) $\iff$ (ii), we have

$$\log B \le \log A \iff A^{-r\alpha}\sigma_{g_1} B^{p\alpha} \le I \quad \text{for all } \alpha \ge 0.$$

Here we notice that

$$\begin{aligned} A^{-r\alpha}\sigma_{g_1} B^{p\alpha} &= A^{-\frac{r\alpha}{2}} g_1(A^{\frac{r\alpha}{2}} B^{p\alpha} A^{\frac{r\alpha}{2}})A^{-\frac{r\alpha}{2}} \\ &= A^{-\frac{r\alpha}{2}} g(A^{-\frac{r\alpha}{2}} B^{-p\alpha} A^{-\frac{r\alpha}{2}})^{-1}A^{-\frac{r\alpha}{2}} \\ &= \left[A^{\frac{r\alpha}{2}} g(A^{-\frac{r\alpha}{2}} B^{-p\alpha} A^{-\frac{r\alpha}{2}})A^{\frac{r\alpha}{2}}\right]^{-1} \\ &= \left[A^{r\alpha}\sigma_g B^{-p\alpha}\right]^{-1}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\log B \le \log A &\Longleftrightarrow A^{-r\alpha}\sigma_{g_1}B^{p\alpha} \le I \quad \text{for all } \alpha \ge 0\\ &\Longleftrightarrow A^{r\alpha}\sigma_g B^{-p\alpha} \ge I \quad \text{for all } \alpha \ge 0.\end{aligned}$$

The proof is completed. □

The following corollary is well known characterization of chaotic order which is obtained by putting $h(t) = t^{\frac{r}{p+r}}$ in Theorem 2.5.

Corollary 2.6 [1, 5, 7] *Let $A,\ B \in \mathcal{P}$. Then the following are equivalent.*

(i) $\log B \le \log A$;
(ii) $A^{-r} \sharp_{\frac{r}{p+r}} B^p \le I$ *for all* $p, r > 0$.

Proof Put $h(t) = t^{\frac{r}{p+r}}$ in Theorem 2.5. □

Using Theorem 2.5, we have Furuta type inequalities. Before introducing the results, we shall introduce the Furuta inequality [6] for the readers convenience.

Theorem 2.7 (Furuta inequality, [6]) *If $0 \le B \le A$, then for each $r \ge 0$,*

$$B^{\frac{p+r}{q}} \le (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \quad \text{and} \quad (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}} \le A^{\frac{p+r}{q}}$$

hold for $p \ge 0$ and $q \ge 1$ with $p + r \le (1 + r)q$.

We obtain the following Furuta type inequalities.

Theorem 2.8 (Furuta type inequality 1) *Let $A, B \in \mathcal{P}$ and $\{h_\alpha\}_{\alpha\in[0,1]} \subset \mathcal{O}^+$ such that for each $\alpha \in [0, 1]$, $h_\alpha(t^s) \le h_\alpha(t)^s$ for all $s \ge 1$, $t > 0$ and $h'_\alpha(1) = \alpha$ and the operator mean σ_h satisfies $!_\alpha \le \sigma_{h_\alpha} \le \nabla_\alpha$. Assume that $h_{\alpha\beta} = h_\alpha \circ h_\beta$ and $th_\alpha(t^{-1}) = h_{1-\alpha}(t)$ hold for all $\alpha, \beta \in [0, 1]$ and all $t > 0$.*

If $\log B \le \log A$, then

$$h_{\frac{1}{q}}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}) \le A^{\frac{r}{2}}h_{\frac{p+r-rq}{pq}}(B^p)A^{\frac{r}{2}}$$

holds for all $p, r \ge 0$ and $q \ge 1$ such that $rq \le p + r$.

Moreover, if $0 \le B \le A$, then

$$h_{\frac{1}{q}}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}) \le A^{\frac{p+r}{q}}$$

holds for all $p, r \ge 0$ and $q \ge 1$ such that $p + r \le (1 + r)q$.

Theorem 2.9 (Furuta type inequality 2) *Let $A, B \in \mathcal{P}$ and $\{g_\alpha\}_{\alpha\in[0,1]} \subset \mathcal{O}^+$ such that for each $\alpha \in [0, 1]$, $g_\alpha(t)^s \le g_\alpha(t^s)$ for all $s \ge 1$, $t > 0$ and $g'_\alpha(1) = \alpha$*

and the operator mean σ_g *satisfies* $!_\alpha \le \sigma_{g_\alpha} \le \nabla_\alpha$*. Assume that* $g_{\alpha\beta} = g_\alpha \circ g_\beta$ *and* $tg_\alpha(t^{-1}) = g_{1-\alpha}(t)$ *hold for all* $\alpha, \beta \in [0,1]$ *and all* $t > 0$.

If $\log B \le \log A$, *then*

$$B^{\frac{r}{2}} g_{\frac{p+r-rq}{pq}}(A^p) B^{\frac{r}{2}} \le g_{\frac{1}{q}}(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})$$

holds for all $p, r \ge 0$ *and* $q \ge 1$ *such that* $rq \le p + r$.

Moreover, if $0 \le B \le A$, *then*

$$B^{\frac{p+r}{q}} \le g_{\frac{1}{q}}(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})$$

holds for all $p, r \ge 0$ *and* $q \ge 1$ *such that* $p + r \le (1 + r)q$.

Generalizations of Furuta inequality have been considered in [13, 16]. However, the above theorems are new type of generalizations. It can be seemed as "mean theoretic generalization of Furuta inequality" [10] .

To prove Theorem 2.8, we prepare the following lemmas.

Lemma 2.10 [18] *Let* h *be a positive differential function defined on* $(0, \infty)$ *such that* $h(1) = 1$. *Then the following hold*:

(i) *if* $h(t^s) \le h(t)^s$ *holds for all* $t > 0$ *and* $s \ge 1$, *then* $h(t) \le t^{h'(1)}$;
(ii) *if* $h(t)^s \le h(t^s)$ *holds for all* $t > 0$ *and* $s \ge 1$, *then* $t^{h'(1)} \le h(t)$.

Proof Proof of (i). By the assumption $h(t^s) \le h(t)^s$, we have $h(t) \le h(t^{\frac{1}{s}})^s$ for all $s \ge 1$ and $t > 0$. Since $h(1) = 1$, we have

$$h(t) \le \lim_{s\to\infty} h(t^{\frac{1}{s}})^s = t^{h'(1)}.$$

(ii) can be proven by the same way. □

Lemma 2.11 *Let* $h \in \mathcal{O}^+$ *such that* $h'(1) = \alpha \in [0,1]$. *Then for any* $A, B \in \mathcal{P}$,

$$h(A^{\frac{1}{2}} B A^{\frac{1}{2}}) = A^{\frac{1}{2}} B^{\frac{1}{2}} \tilde{h}(B^{\frac{-1}{2}} A^{-1} B^{\frac{-1}{2}}) B^{\frac{1}{2}} A^{\frac{1}{2}},$$

where $\tilde{h}(t) := th(t^{-1})$ *is the transpose of* h.

We remark that if $h'(1) = \alpha \in [0,1]$, then $\tilde{h}'(1) = 1 - \alpha$.

Proof We can prove Lemma 2.11, immediately. By the definition of the transpose, we have

$$A^{-1} \sigma_h B = B \sigma_{\tilde{h}} A^{-1},$$

i.e.,

$$A^{-\frac{1}{2}}h(A^{\frac{1}{2}}BA^{\frac{1}{2}})A^{-\frac{1}{2}}=B^{\frac{1}{2}}\tilde{h}(B^{-\frac{1}{2}}A^{-1}B^{-\frac{1}{2}})B^{\frac{1}{2}}.$$

It is equivalent to the desired formula. □

Proof of Theorem 2.8 $\log B\le\log A$ ensures $\log A^{-1}\le\log B^{-1}$, and by Theorem 2.5,

$$h_{\frac{p}{p+r}}(B^{\frac{-p}{2}}A^{-r}B^{\frac{-p}{2}})\le B^{-p} \tag{2}$$

for all $p,r\ge 0$. Then we have

$$\begin{aligned}h_{\frac{1}{q}}(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})&=A^{\frac{r}{2}}B^{\frac{p}{2}}h_{\frac{q-1}{q}}(B^{\frac{-p}{2}}A^{-r}B^{\frac{-p}{2}})B^{\frac{p}{2}}A^{\frac{r}{2}}\quad\text{(Lemma 2.11)}\\&=A^{\frac{r}{2}}B^{\frac{p}{2}}h_{\frac{(q-1)(p+r)}{pq}}\circ h_{\frac{p}{p+r}}(B^{\frac{-p}{2}}A^{-r}B^{\frac{-p}{2}})B^{\frac{p}{2}}A^{\frac{r}{2}}\\&\le A^{\frac{r}{2}}B^{\frac{p}{2}}h_{\frac{(q-1)(p+r)}{pq}}(B^{-p})B^{\frac{p}{2}}A^{\frac{r}{2}}\quad(h_{\frac{(q-1)(p+r)}{pq}}\in\mathcal{O}^{+}\text{ and (2.1)})\\&=A^{\frac{r}{2}}h_{\frac{p+r-rq}{pq}}(B^{p})A^{\frac{r}{2}}\quad\text{(Lemma 2.11).}\end{aligned}$$

Therefore, we can obtain the first inequality in Theorem 2.8.

Next we shall prove the second inequality. Assume $0\le B\le A$. Since $h_{\alpha}(t^{s})\le h_{\alpha}(t)^{s}$ holds for all $s\ge 1$ and $t>0$, by Lemma 2.10(i), we have

$$h_{\frac{1}{q}}(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})\le A^{\frac{r}{2}}h_{\frac{p+r-rq}{pq}}(B^{p})A^{\frac{r}{2}}\le A^{\frac{r}{2}}B^{\frac{p+r-rq}{q}}A^{\frac{r}{2}}\le A^{\frac{p+r}{q}}, \tag{3}$$

where the last inequality follows from $B\le A$ and $rq\le p+r\le(1+r)q$.

Let $q'\ge q$. Then $\frac{q}{q'}\in[0,1]$. By (3) and Lemma 2.10(i), we have

$$h_{\frac{1}{q'}}\left(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}}\right)=h_{\frac{q}{q'}}\circ h_{\frac{1}{q}}\left(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}}\right)\le h_{\frac{q}{q'}}\left(A^{\frac{p+r}{q}}\right)\le A^{\frac{p+r}{q'}}$$

for $rq\le p+r\le(1+r)q$ and $q'\ge q\ge 1$. We notice that the condition $rq\le p+r\le(1+r)q$ is equivalent to

$$\frac{p+r}{1+r}\le q\le\frac{p+r}{r}.$$

Hence, we have

$$h_{\frac{1}{q'}}(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})\le A^{\frac{p+r}{q'}}$$

for $\frac{p+r}{1+r}\le q'$ and $q'\ge 1$., i.e., $p+r\le(1+r)q'$ and $q'\ge 1$. □

***Remark* 2.12** By Theorem 2.8, if $B\le A$, then

$$h_{\frac{1}{q}}(A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})\le A^{\frac{p+r}{q}} \tag{4}$$

holds for all $p,r\ge 0$ and $q\ge 1$ such that $p+r\le(1+r)q$. It looks like a generalization of the Furuta inequality. However, (4) follows from the Furuta inequality very easy as follows. Because $h_\alpha(t^s) \le h_\alpha(t)^s$ holds for all $s \ge 1$ and $t > 0$, we have $h_\alpha(t) \le t^\alpha$ by Lemma 2.10(i). Then using Furuta inequality [6],

$$h_{\frac{1}{q}}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}) \le (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}} \le A^{\frac{p+r}{q}}.$$

Proof of Theorem 2.9 Using Theorem 2.5, $\log B \le \log A$ ensures

$$g_{\frac{r}{p+r}}(A^{\frac{-r}{2}}B^{-p}A^{\frac{-r}{2}}) \ge A^{-r}$$

for all $p, r \ge 0$. The rest of proof is similar to the proof of Theorem 2.8. □

Corollary 2.13 (Furuta type inequality for power means I) *Let* $A, B \in \mathcal{P}$. *If* $\log B \le \log A$, *then for each* $\lambda \in [-1, 0)$,

$$\left[\frac{q-1}{q} + \frac{1}{q}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^\lambda\right]^{\frac{1}{\lambda}} \le A^{\frac{r}{2}}\left[\frac{(p+r)(q-1)}{pq} + \frac{p+r-rq}{pq}B^{p\lambda}\right]^{\frac{1}{\lambda}}A^{\frac{r}{2}}$$

holds for all $p, r \ge 0$ *and* $q \ge 1$ *such that such that* $rq \le p + r$.

Moreover, if $B \le A$, *then we have*

$$\left[\frac{q-1}{q}I + \frac{1}{q}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^\lambda\right]^{\frac{1}{\lambda}} \le A^{\frac{p+r}{q}}$$

for all $p, r \ge 0$ *and* $q \ge 1$ *such that* $p + r \le (1 + r)q$.

Proof Let $h_\alpha(t) := [1 - \alpha + \alpha t^\lambda]^{\frac{1}{\lambda}}$. Then for any $\lambda \in [-1, 0)$, $h_\alpha(t)$ satisfies all conditions in Theorem 2.8. □

Corollary 2.14 (Furuta type inequality for power means II) *Let* $A, B \in \mathcal{P}$. *If* $\log B \le \log A$, *then for each* $\lambda \in (0, 1]$,

$$\left[\frac{q-1}{q} + \frac{1}{q}(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^\lambda\right]^{\frac{1}{\lambda}} \ge B^{\frac{r}{2}}\left[\frac{(p+r)(q-1)}{pq} + \frac{p+r-rq}{pq}A^{p\lambda}\right]^{\frac{1}{\lambda}}B^{\frac{r}{2}}$$

holds for all $p, r \ge 0$ *and* $q \ge 1$ *such that* $rq \le p + r$.

Moreover, if $B \le A$, *then we have*

$$\left[\frac{q-1}{q}I + \frac{1}{q}(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^\lambda\right]^{\frac{1}{\lambda}} \ge B^{\frac{p+r}{q}}$$

for all $p, r \ge 0$ *and* $q \ge 1$ *such that* $p + r \le (1 + r)q$.

Remark In Theorem 2.5, we proved a characterization of chaotic order. Here we have a question. Under the assumption of Theorem 2.5, is it true that

$$\log B \le \log A \quad \Longleftrightarrow \quad A^{r\alpha}\sigma_h B^{-p\alpha} \ge I \quad \text{for all } \alpha \ge 0$$

and

$$\log B \le \log A \quad \Longleftrightarrow \quad A^{-r\alpha}\sigma_g B^{p\alpha} \le I \quad \text{for all } \alpha \ge 0?$$

It is not true as the following example.

Let $A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}^2 = \begin{pmatrix} 10 & 5 \\ 5 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$. Then since $A^{\frac{1}{2}} - B^{\frac{1}{2}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \ge 0$, we have $\log B \le \log A$.

Let $h(t) = \left[\frac{1+t^{-1}}{2}\right]^{-1}$. Then $h(t^s) \le h(t)^s$ holds for all $s \ge 1$, $t > 0$ and $h'(1) = \frac{1}{2}$. By Theorem 2.5, we have

$$A^{-\alpha p}\sigma_h B^{\alpha p} \le I \quad \text{for all } \alpha \ge 0.$$

On the other hand,

$$A\sigma_h B^{-1} = \left[\frac{A^{-1} + (B^{-1})^{-1}}{2}\right]^{-1} = \frac{5}{73}\begin{pmatrix} 7 & 1 \\ 1 & 21 \end{pmatrix}.$$

Since $A\sigma_h B^{-1} - I = \frac{1}{73}\begin{pmatrix} -38 & 5 \\ 5 & 32 \end{pmatrix} \ngeq 0$, we have $A\sigma_h B^{-1} \ngeq I$.

Let $g(t) = \frac{1+t}{2}$. Then $g(t)^s \le g(t^s)$ holds for all $s \ge 1$, $t > 0$ and $g'(1) = \frac{1}{2}$. By Theorem 2.5, we have

$$A^{\alpha p}\sigma_g B^{-\alpha p} \ge I \quad \text{for all } \alpha \ge 0.$$

On the other hand,

$$A^{-1}\sigma_g B = \frac{A^{-1} + B}{2} = \frac{1}{10}\begin{pmatrix} 21 & -1 \\ -1 & 7 \end{pmatrix}.$$

Since $I - A^{-1}\sigma_g B = \frac{1}{10}\begin{pmatrix} -11 & 1 \\ 1 & 3 \end{pmatrix} \ngeq 0$, we have $A^{-1}\sigma_g B \nleq I$.

Let $f(t) = t^{\frac{1}{2}}$. Then $f(t^s) = f(t)^s$ for all $s \ge 1$. Then by Theorem 2.8, we have

$$A\sigma_f B^{-1} \ge I \quad \text{and} \quad A^{-1}\sigma_f B \le I$$

hold.

3 Inequalities relating to the Choi–Davis inequality

In this section, we will use the chaotic inequality proved in Sect. 2 to prove inequalities related to Choi–Davis inequality as applications to Theorem 2.8.

A linear map ϕ is positive if $\phi(A) \geq 0$ whenever $A \geq 0$ and is said to be unital if $\phi(I) = I$. The inequality mentioned in the following lemma is called the Choi–Davis–Jensen inequality.

Lemma 3.1 [3] *Let Φ be a unital positive linear map and $A \geq 0$. Then*

$$\phi(A^p) \leq \phi(A)^p \quad \text{for } 0 \leq p \leq 1 \quad \text{and}$$
$$\phi(A^p) \geq \phi(A)^p \quad \text{for } 1 \leq p \leq 2.$$

Theorem 3.2 *Let $\{h_\alpha\}_{\alpha \in [0,1]} \subset \mathcal{O}^+$ such that for each $\alpha \in [0,1]$, $h_\alpha(t^s) \leq h_\alpha(t)^s$ for all $s \geq 1, t > 0$ and $h'_\alpha(1) = \alpha$ and the operator mean σ_h satisfies $!_\alpha \leq \sigma_{h_\alpha} \leq \nabla_\alpha$. Assume that $h_{\alpha\beta} = h_\alpha \circ h_\beta$ and $th_\alpha(t^{-1}) = h_{1-\alpha}(t)$ hold for all $\alpha, \beta \in [0,1]$ and all $t > 0$. Let $A \in \mathcal{P}$ and ϕ be a unital positive linear map. Then*

$$h_\alpha\left(|\phi(A^p)^\lambda \phi(A^q)^\mu|^2\right) \leq \phi(A^{2\alpha(p\lambda+q\mu)})$$

holds for all $p, q, \lambda, \mu \geq 0$ such that $2\alpha(p\lambda + q\mu) \leq p + 2q\mu$, $q \leq 2\alpha(p\lambda + q\mu) \leq 2q$ and $0 < p \leq q$.

Proof Let $0 < p \leq q$. Then by Lemma 3.1, $\phi(A^p) \leq \phi(A^q)^{\frac{p}{q}}$. Applying Theorem 2.8, we find that

$$h_\alpha(\phi(A^q)^{\frac{pt}{2q}} \phi(A^p)^s \phi(A^q)^{\frac{pt}{2q}}) \leq \phi(A^q)^{\frac{p\alpha(s+t)}{q}}$$

holds for $s, t \geq 0$ and $\alpha \in (0,1]$ such that $\alpha(s+t) \leq 1 + t$.

Put $s = 2\lambda \geq 0$ and $t = \frac{2q}{p}\mu \geq 0$. Then

$$h_\alpha\left(\phi(A^q)^\mu \phi(A^p)^{2\lambda} \phi(A^q)^\mu\right) \leq \phi(A^q)^{\frac{2\alpha(p\lambda+q\mu)}{q}}$$

holds for $p, q, \lambda, \mu \geq 0$, $\alpha \in (0,1]$ such that $2\alpha(p\lambda + q\mu) \leq p + 2q\mu$.

Moreover, if $q \leq 2\alpha(p\lambda + q\mu) \leq 2q$, then

$$\phi(A^q)^{\frac{2\alpha(p\lambda+q\mu)}{q}} \leq \phi(A^{2\alpha(p\lambda+q\mu)}).$$

The proof is completed. □

By putting $h_\alpha(t) = t^\alpha$, $\alpha = \frac{1}{2}$ and $\lambda = \mu = 1$, we have (1). Moreover, we have the following variations of asymmetric Kadison's type inequalities.

Corollary 3.3 *Let $A \in \mathcal{P}$ and ϕ be a unital positive linear map. Then*

$$\left|\phi(A^p)^{\frac{1}{p}} \phi(A^q)^{\frac{1}{q}}\right| \leq \phi(A^2).$$

holds for $q \in [1,2]$ and $0 < p \leq q$.

Proof Put $h(t) = t^\alpha$, $\alpha = \frac{1}{2}$ $\lambda = \frac{1}{p}$ and $\mu = \frac{1}{q}$ in Theorem 3.2. □

Corollary 3.4 *Let $A \in \mathcal{P}$ and ϕ be a unital positive linear map. Then*

$$|\phi(A^p)^q \phi(A^q)^p| \leq \phi(A^{2pq}).$$

holds for $p \in [\frac{1}{2}, 1]$ and $0 < p \leq q$.

Proof Put $h(t) = t^\alpha$, $\alpha = \frac{1}{2}$, $\lambda = q$ and $\mu = p$ in Theorem 3.2. □

The above corollaries are obtained by the main results in [19]. The following corollaries are Choi–Davis type inequalities induced by power mean. We notice that if $A \in \mathcal{P}$ is invertible, then there exists a positive real number $\varepsilon > 0$ such that $0 \leq \varepsilon I \leq A$. Therefore, for any unital positive linear map ϕ, $0 \leq \varepsilon I \leq \phi(A)$ holds, and hence $\phi(A)$ is invertible, too.

Corollary 3.5 *Let $A \in \mathcal{P}$ be invertible and ϕ be a unital positive linear map. Then for $\alpha \in [0, 1]$ and $-1 \leq r \leq 0$,*

$$\left[(1-\alpha)I + \alpha|\phi(A^p)^\lambda \phi(A^q)^\mu|^{2r}\right]^{\frac{1}{r}} \leq \phi(A^{2\alpha(p\lambda+q\mu)})$$

holds for all $p, q, \lambda, \mu \geq 0$ such that $2\alpha(p\lambda + q\mu) \leq p + 2q\mu$, $q \leq 2\alpha(p\lambda + q\mu) \leq 2q$ and $0 < p \leq q$.

Proof Put $h_\alpha(t) = [1 - \alpha + \alpha t^r]^{\frac{1}{r}}$ in Theorem 3.2 □

Corollary 3.6 *Let $A \in \mathcal{P}$ be invertible and ϕ be a unital positive linear map. Then*

$$4\left[I + |\phi(A^p)^\lambda \phi(A^q)^\mu|^{-1}\right]^{-2} \leq \phi(A^{p\lambda+q\mu})$$

holds for all $p, q, \lambda, \mu \geq 0$ such that $p\lambda \leq p + q\mu, q \leq p\lambda + q\mu \leq 2q$ and $0 < p \leq q$.

Proof Put $\alpha = \frac{1}{2}$ and $r = -\frac{1}{2}$ in Corollary 3.5. □

Acknowledgements The authors thanks to the anonymous referees for careful reading and various comments for improvement of this paper. The first author is thankful to SERB, India, for providing the grant (MTR/2018/001209) to carry out the research.

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