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STABILITY OF $\mathcal{AN}$-OPERATORS UNDER FUNCTIONAL CALCULUS

G. RAMESH[1,*,†], H. OSAKA[2,‡], Y. UDAGAWA[3] and T. YAMAZAKI[4]

[1]Department of Mathematics, Indian Institute of Technology Hyderabad, Kandi (V), Sangareddy, Telangana, India-502 285
e-mail: rameshg@iith.ac.in

[2]Department of Mathematical Sciences, Ritsumeikan University, Kusatsu, Shiga, 525-8577, Japan
e-mail: osaka@se.ritsumei.ac.jp

[3]Department of Advanced Practice of School Education, Naruto University of Education, 748, Nakajima, Takashima, Naruto-cho, Naruto-shi, 772-8502, Japan
e-mail: yudagawa@naruto-u.ac.jp

[4]Department of Electric and Computer Engineering, Toyo University, Kawagoe-Shi, Saitama, 350-8585, Japan
e-mail: t-yamazaki@toyo.jp

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Abstract. In this note we discuss absolutely norm attaining property ($\mathcal{AN}$-property in short) of the Jordan product and Lie-bracket. We propose a functional calculus for positive absolutely norm attaining operators and discuss the stability of the $\mathcal{AN}$-property under the functional calculus. As a consequence we discuss the operator mean of positive $\mathcal{AN}$-operators.

1. Introduction

In this note, we study functional calculus for absolutely norm attaining operators. More precisely, if T is an absolutely norm attaining positive operator and f is a strictly increasing continuous function, then when is $f(T)$ again an absolutely norm attaining operator? Recall that a bounded linear operator T defined on a Hilbert space H is called norm attaining if there exists $x \in H$ with $\|x\| = 1$ such that $\|Tx\| = \|T\|$. Further, T is said to be absolutely norm attaining or an $\mathcal{AN}$-operator if T attains its norm on every closed subspace of H. This class was initially proposed by Carvajal and

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Neves in [5] and further explored in [2,15,23–25]. This class comprises of the space of compact operators and the class of partial isometries with finite dimensional null spaces. It is not closed under sums, while it is closed with respect to the products of operators [15]. The sum of a compact operator and a partial isometry with finite dimensional null space need not be an $\mathcal{AN}$-operator, but it is in the operator norm closure of $\mathcal{AN}$-operators [26, Corollary 3.6]. Several characterizations of positive $\mathcal{AN}$-operators are studied in [23–25], while the study of self-adjoint, normal and paranormal $\mathcal{AN}$-operators is done in [2,23,25]. The hyperinvariant subspace problem is discussed in [3] and the maps preserving $\mathcal{AN}$-operators is discussed in [15]. As we have mentioned earlier that the set of all $\mathcal{AN}$-operators is closed under products, the powers of an $\mathcal{AN}$-operator is again an $\mathcal{AN}$-operator, but polynomials of an $\mathcal{AN}$-operator need not be an $\mathcal{AN}$-operator.

In this article we focus our attention on functions which preserve the $\mathcal{AN}$-property. We show that if T is an $\mathcal{AN}$ positive operator and f is a strictly increasing continuous function, then $f(T)$ is an $\mathcal{AN}$-operator. Later, we extend this to self-adjoint $\mathcal{AN}$-operators. We also study one more interesting problem related to $\mathcal{AN}$-operators, namely, under what conditions the Jordan product and Lie bracket of two $\mathcal{AN}$-operators is again an $\mathcal{AN}$-operator? We give a sufficient condition under which this question has a positive answer.

In most of our results, we use spectral characterizations of positive $\mathcal{AN}$-operators proved in [24] and in [23,25].

In the remaining part of this section we describe basic results which we need for proving our main results. In the second section we prove our main results; the Jordan product, Lie bracket of positive $\mathcal{AN}$-operators and the functional calculus for positive and self-adjoint $\mathcal{AN}$-operators and consequences of the functional calculus.

Notations and terminology. Throughout this note we denote infinite dimensional complex Hilbert spaces by H, H_1, H_2 etc. The unit sphere of a subspace M of H is defined by $S_M := \{x \in M : \|x\| = 1\}$ and the orthogonal complement $M^\perp$ of M in H is defined by

$$M^\perp = \big\{x \in H : \langle x, y\rangle = 0 \text{ for all } y \in M\big\}.$$

We denote the space of all bounded linear operators from H_1 into H_2 by $\mathcal{B}(H_1, H_2)$ and $\mathcal{B}(H, H)$ by $\mathcal{B}(H)$. The space of all finite rank operators and compact operators in $\mathcal{B}(H_1, H_2)$ will be denoted by $\mathcal{F}(H_1, H_2)$ and $\mathcal{K}(H_1, H_2)$, respectively. If $H_1 = H_2 = H$, these will be denoted by $\mathcal{F}(H)$ and $\mathcal{K}(H)$, respectively.

If $T \in \mathcal{B}(H_1, H_2)$, then the operator norm of T is defined by

$$\|T\| = \sup\{\|Tx\| : x \in S_{H_1}\}.$$

The null space and the range spaces of $T \in \mathcal{B}(H_1, H_2)$ are denoted by $N(T)$ and $R(T)$, respectively and the adjoint of T is denoted by T^*.

We say $T \in \mathcal{B}(H)$ to be *normal* if $T^*T = TT^*$ and *self-adjoint* if $T = T^*$. If $\langle Tx, x\rangle \geq 0$ for all $x \in H$, then T is called *positive*. If $\mathcal{A} \subseteq \mathcal{B}(H)$ is a nonempty subset, then the set of all positive operators and self-adjoint operators in $\mathcal{A}$ are denoted by $\mathcal{A}_+$ and $\mathcal{A}_{sa}$, respectively. Given $T \in \mathcal{B}(H)_+$, there exists a unique $S \in \mathcal{B}(H)_+$ such that $S^2 = T$. This operator S is called the *positive square root* of T and it is denoted by $T^{\frac{1}{2}}$.

If $S, T \in \mathcal{B}(H)_{sa}$ such that $S - T \geq 0$, then we denote this by $S \geq T$. If $P \in \mathcal{B}(H)$ is such that $P^2 = P$, then P is called a *projection*. In addition, if $N(P)$ and $R(P)$ are orthogonal to each other, then P is called an *orthogonal projection*. A projection P is orthogonal if and only if it is self-adjoint if and only if it is normal. We say two orthogonal projections P and Q to be mutually orthogonal if $R(P)$ and $R(Q)$ are orthogonal to each other. The orthogonal projection on a Hilbert space H whose range is M is denoted by P_M.

We say $V \in \mathcal{B}(H)$ to be an *isometry* if $\|Vx\| = \|x\|$ for each $x \in H$ and a *partial isometry* if $V|_{N(V)^\perp}$ is an isometry, that is, $\|Vx\| = \|x\|$ for all $x \in N(V)^\perp$. We say V to be unitary if $V \in \mathcal{B}(H)$ is an isometry and onto.

If $T \in \mathcal{B}(H)$, then $T^*T \in \mathcal{B}(H)$ is positive and $|T| := (T^*T)^{\frac{1}{2}}$ is called the *modulus* of T. The polar decomposition theorem asserts that if $T \in \mathcal{B}(H)$, then there exists a unique partial isometry $V \in \mathcal{B}(H)$ such that $T = V|T|$ and $N(V) = N(T)$.

A closed subspace M of H is said to be *invariant* under $T \in \mathcal{B}(H)$ if $TM \subseteq M$ and *reducing* if both M and $M^\perp$ are invariant under T.

For $T \in \mathcal{B}(H)$, the set

$$\rho(T) := \big\{\lambda \in \mathbb{C} : (T - \lambda I)^{-1} \text{ exists and } (T - \lambda I)^{-1} \in \mathcal{B}(H)\big\}$$

is called the *resolvent set* and the complement $\sigma(T) = \mathbb{C} \setminus \rho(T)$ is called the *spectrum* of T. It is well known that $\sigma(T)$ is a non-empty compact subset of $\mathbb{C}$. The point spectrum and the continuous spectrums of T are defined by

$$\sigma_p(T) := \big\{\lambda \in \mathbb{C} : T - \lambda I \text{ is not one-to-one}\big\}$$

$$\sigma_c(T) := \big\{\lambda \in \mathbb{C} : T - \lambda I \text{ is one-to-one, } \overline{R(T)} = H, \\ \text{but } T^{-1} \text{ is not continuous}\big\},$$

respectively. Note that $\sigma_p(T), \sigma_c(T) \subseteq \sigma(T)$.

If $T \in \mathcal{B}(H)_{sa}$, then the discrete spectrum $\sigma_d(T)$ of T is defined by

$$\sigma_d(T) := \big\{\lambda \in \mathbb{C} : \lambda \text{ is an isolated spectral value of } T \\ \text{with } \dim N(T - \lambda I) < \infty\big\}.$$

The essential spectrum $\sigma_{\text{ess}}(T)$ is the complement of $\sigma_d(T)$ in $\sigma(T)$. Note that if H is finite dimensional, then $\sigma(T) = \sigma_d(T)$ and $\sigma_{\text{ess}}(T) = \emptyset$. If H is infinite dimensional, then $\sigma_{\text{ess}}(T)$ is a non-empty closed subset of $\sigma(T)$, hence compact.

If $T \in \mathcal{B}(H)$ and $\pi\colon \mathcal{B}(H) \to \mathcal{B}(H)/\mathcal{K}(H)$ is the canonical quotient map, then we can define the essential spectrum of T by $\sigma_{\text{ess}}(T) := \sigma(\pi(T))$. In case if $T = T^*$, the above two definitions of the essential spectra we have given coincide. We refer to [29, Lemma 2, Corollary 3] and [17, Problem 181]. Also, see [7, Page 359, Proposition 4.6].

Let H be infinite dimensional and $T \in \mathcal{B}(H)$. The quantity

$$m_e(T) := \inf \left\{\lambda : \lambda \in \sigma_{\text{ess}}(|T|)\right\}$$

is called the essential minimum modulus of T. We refer to [4] for more details on this concept. Another important quantity of an operator is the minimum modulus which is defined as follows.

Definition 1.1 [14]. Let $T \in \mathcal{B}(H)$ and $T \neq 0$. Then

$$m(T) := \inf \left\{\|Tx\| : x \in S_H\right\}$$

is called the minimum modulus of T.

It is readily seen that $m(T) > 0$ if and only if T is bounded below, that is, there exists $m > 0$ such that $\|Tx\| \geq m\|x\|$ for all $x \in H$.

The following formula for the minimum modulus is known:

$$m(T) = \inf\{\lambda : \lambda \in \sigma(|T|)\} = \min\{\lambda : \lambda \in \sigma(|T|)\}.$$

In particular, when $T \geq 0$, then we have $m(T) = d(0, \sigma(T))$.

To describe a decomposition of an operator, the operator matrix is a useful tool. Here we give details of it.

Let $H = H_1 \oplus H_2$ and $T \in \mathcal{B}(H)$. Let $P_j\colon H \to H$ be an orthogonal projection onto H_j for $j = 1, 2$. Then

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

where $T_{ij}\colon H_j \to H_i$ is the operator given by $T_{ij} = P_i T|_{H_j}$. In particular, $T(H_1) \subseteq H_1$ if and only if $T_{21} = 0$. Also, H_1 reduces T if and only if $T_{12} = 0$ and $T_{21} = 0$ (for details see [30, Page 287] or [7, Page 39, Proposition 3.7]).

For all the above mentioned basics of operator theory we refer to [7,17, 20,28,30].

Next, we discuss norm attaining operators.

Definition 1.2 [5]. Let $T \in \mathcal{B}(H_1, H_2)$. Then

(1) T is said to be a norm attaining or $\mathcal{N}$-operator if there exist a unit vector $x \in H_1$ such that $\|Tx\| = \|T\|$;

(2) T is said to be an absolutely norm attaining or $\mathcal{AN}$-operator if $T|_M$, the restriction of T to M, is norm attaining for every non-zero closed subspace of H_1.

We denote the class of norm attaining operators between H_1 and H_2 by $\mathcal{N}(H_1, H_2)$ and $\mathcal{N}(H, H)$ by $\mathcal{N}(H)$. It is known that $\mathcal{N}(H_1, H_2)$ is dense in $\mathcal{B}(H_1, H_2)$ with respect to the operator norm of $\mathcal{B}(H_1, H_2)$. We refer to [12] for a simple proof of this fact. We denote the class of absolutely norm attaining operators between H_1 and H_2 by $\mathcal{AN}(H_1, H_2)$ and if $H_1 = H_2 = H$, then $\mathcal{AN}(H_1, H_2)$ will be written as $\mathcal{AN}(H)$. The class $\mathcal{AN}(H_1, H_2)$ contains $\mathcal{K}(H_1, H_2)$, the space of all compact operators in $\mathcal{B}(H_1, H_2)$, and the class of partial isometries with finite dimensional null space. For more information on absolutely norm attaining operators, we refer to [2,5,15,23–25]. Notice that $T \in \mathcal{AN}(H_1, H_2)$ if and only if $|T| \in \mathcal{AN}(H_1)$ if and only if $T^*T = |T|^2 \in \mathcal{AN}(H_1)$ [23, Corollary 2.11].

If $W \in \mathcal{B}(H)$ is an isometry, then $W \in \mathcal{AN}(H)$ by [5, Proposition 3.2].

Theorem 1.3 [24, Theorem 5.1]. *Let H be a Hilbert space of arbitrary dimension and $T \in \mathcal{B}(H)_+$. Then $T \in \mathcal{AN}(H)$ if and only if there exist $\alpha \geq 0$, a positive compact operator K, and a self-adjoint finite rank operator F such that $T = \alpha I + K + F$.*

The following result is more useful while computing the powers of $\mathcal{AN}$-operators.

Theorem 1.4 [23, Theorem 2.5]. *Let H be an infinite dimensional Hilbert space and $T \in \mathcal{B}(H)$. Then the following statements are equivalent:*

(1) $T \in \mathcal{AN}(H)_+$,

(2) *there exists a unique triple (K, F, α), where*

  (a) $K \in \mathcal{K}(H)_+$, $\alpha \geq 0$,

  (b) $F \in \mathcal{F}(H)_+$ *and* $0 \leq F \leq \alpha I$

such that $T = \alpha I + K - F$ and $KF = 0$.

From Theorem 1.3 we know that the positive set $\mathcal{AN}(H)_+$ of $\mathcal{AN}(H)$ is a positive cone. As pointed out in [24, Example 4.12] $\mathcal{AN}(H)$ is not stable under the sums. We give a simple example in the next section.

2. Functional calculus

It is well known that the product of two $\mathcal{AN}$-operators is again an $\mathcal{AN}$-operator (see [15, Corollary 2.7] for details). In this section first we look at the Jordan product of positive $\mathcal{AN}$-operators. We show that under some condition the Jordan product of positive $\mathcal{AN}$-operators is again an $\mathcal{AN}$-operator. But in general, the class of $\mathcal{AN}$-operators is not closed under the Jordan product. We illustrate this with several examples. Finally we discuss the continuous functional calculus for positive $\mathcal{AN}$-operators.

Proposition 2.1. *Let $W\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the unilateral right shift operator. Then*

(1) *W and W^* belong to $\mathcal{AN}(\ell^2(\mathbb{N}))$,*
(2) *$W + W^*$ does not belong to $\mathcal{AN}(\ell^2(\mathbb{N}))$.*

Proof. (1) Since W is an isometry, $W \in \mathcal{AN}(\ell^2(\mathbb{N}))$. Since

$$\dim R(I - WW^*) = 1, \quad WW^* = I - (I - WW^*) \in \mathcal{AN}(\ell^2(\mathbb{N})$$

by Theorem 1.3. Hence, $W^* \in \mathcal{AN}(\ell^2(\mathbb{N}))$.

(2) Suppose that $W + W^*$ belongs to $\mathcal{AN}(\ell^2(\mathbb{N}))$. Then, so is $(W + W^*)^2$, because $\mathcal{AN}(\ell^2)$ is closed under the product, that is,

$$I + W^2 + WW^* + W^*W^* = \alpha I + K + F$$

for some positive number α, a positive compact operator K, and a self-adjoint finite rank operator F. Let $\pi\colon \mathcal{B}(\ell^2(\mathbb{N})) \to \mathcal{B}(\ell^2(\mathbb{N}))/\mathcal{K}(\ell^2(\mathbb{N}))$ be the canonical quotient map. Then,

$$\pi((W + W^*)^2) = \pi(\alpha I + K + F) = \alpha I.$$

On the contrary,

$$\pi((W+W^*)^2) = \pi(I+WW+WW^*+W^*W^*) = 2I+\pi(W)^2+(\pi(W)^*)^2.$$

Hence, $\pi(W)^2 + (\pi(W)^*)^2 \in \mathbb{C}I$. Since $C^*(\pi(W), \pi(W)^*) \cong C(\mathbb{T})$ (see [8, Proposition 7.12 and Theorem 7.23]) and $\pi(W)$ can be identified with the canonical generator z of $C(\mathbb{T})$, the above observation implies that $z^2 + \overline{z}^2 \in \mathbb{C}I$. This is impossible. Therefore, $W + W^* \notin \mathcal{AN}(\ell^2(\mathbb{N}))$. □

Using the above result, we show that $\mathcal{AN}(H)_+$ is not stable under the symmetric bracket $\circ$, where $T \circ S = \frac{TS+ST}{2}$.

Lemma 2.2. *Let U be a unitary in a unital commutative C^*-algebra $C(\sigma(U))$. If $\sigma(U)$ contains some arc in the circle $\mathbb{T}$, then $U^2 + (U^*)^2 \notin \mathbb{C}I$.*

Proof. With a simple calculation the proof can be obtained. □

Lemma 2.3. *Let $T \in \mathcal{AN}(H)$ such that $T^* \in \mathcal{AN}(H)$, and let $\pi\colon \mathcal{B}(H) \to \mathcal{B}(H)/\mathcal{K}(H)$ be the canonical quotient map. Then, there is a positive number γ such that $\gamma\pi(T)$ is a unitary.*

Proof. Since $T\in\mathcal{AN}(H)$, $T^*T=\alpha I+K_1+F_1$ and $TT^*=\beta I+K_2+F_2$ for some positive numbers α, β, positive compact operators K_1, K_2, and finite rank self-adjoint operators F_1, F_2 by Theorem 1.3. Hence, $\pi(T)^*\pi(T)=\alpha I$ and $\pi(T)\pi(T)^*=\beta I$. From the norm estimate, we have $\alpha=\beta$. Hence, if $\gamma=\frac{1}{\sqrt{\alpha}}$, then $\gamma\pi(T)$ is unitary. $\square$

From Lemma 2.3, $T\in\mathcal{AN}(H)$ with $T^*\in\mathcal{AN}(H)$, we may assume that $\pi(T)$ is unitary.

Proposition 2.4. *Let $S,T\in\mathcal{AN}(H)_{sa}$. Suppose that $\sigma(\pi(ST))$ contains some arc in $\mathbb{T}$, then, $ST+TS\notin\mathcal{AN}(H)_{sa}$.*

Proof. Since $(ST)^*=TS\in\mathcal{AN}(H)$, we may assume that $\pi(ST)$ is unitary from Lemma 2.3.

Suppose that $ST+TS\in\mathcal{AN}(H)$. Then by Theorem 1.3,

$$(ST+TS)^*(ST+TS)=(ST+TS)^2=\alpha I+K+F,$$

for some positive number α, positive compact operator K, and finite-rank self-adjoint operator F. Then,

$$\pi((ST+TS)^2)=\pi(\alpha I+K+F),$$
$$\pi(ST)^2+\pi(TS)^2+2I=\alpha I,$$
$$\pi(ST)^2+(\pi(ST)^*)^2=(\alpha-2)I\in\mathbb{C}I.$$

This is impossible by Lemma 2.2. Hence, $ST+TS\notin\mathcal{AN}(H)_{sa}$. $\square$

We consider the converse of Lemma 2.3. For this purpose we need the definition of absolutely minimum attaining operator.

Definition 2.5 [6]. Let $T\in\mathcal{B}(H_1,H_2)$. Then T is said to be

(1) minimum attaining if there exists $x\in H_1$ with $\|x\|=1$ such that $\|Tx\|=m(T)$;

(2) absolutely minimum attaining if for every non-zero closed subspace M of H_1, the operator $T|_M\colon M\to H_2$ is minimum attaining.

We refer to [1,3,6,13] for more details of this class. We denote the absolutely minimum attaining operators in $\mathcal{B}(H_1,H_2)$ by $\mathcal{AM}(H_1,H_2)$ and if $H_1=H_2=H$, then $\mathcal{AM}(H_1,H_2)$ will be denoted by $\mathcal{AM}(H)$.

Proposition 2.6. *Let $\pi\colon\mathcal{B}(H)\to\mathcal{B}(H)/\mathcal{K}(H)$ be the canonical quotient map. If $\pi(T)$ is proportional to an isometry, that is, there exists a non-zero $\beta\in\mathbb{C}$ such that $\pi(T)^*\pi(T)=\beta I$, then there are two cases.*

(1) $T\in\mathcal{AN}(H)$,

(2) $T=u(|T_1|\oplus|T_2|)$, *where* $H=H_1\oplus H_2$, $T_1\in\mathcal{AN}(H_1)$, $T_2\in\mathcal{AM}(H_2)$ *and* $T=u|T|$ *is the polar decomposition of* T. *In this case* $T\in\overline{\mathcal{AN}(H)}$.

PROOF. Since $\pi(T) = \alpha W$ for some isometry W, we have $\pi(T^*T) = \beta I$ for $\beta = |\alpha|^2$. Hence, $T^*T = \beta I + K$ for some compact operator K. Let $K = K_+ - K_-$ be the Hahn decomposition such that K_+ and K_- are positive operators, and let P be the orthogonal projection from H onto $\overline{R(K_+)}$. Set $H_1 = R(P)$, $H_2 = R(I - P)$.

If K_- $(= F)$ is finite rank, then $T^*T = \beta I + K_+ - F \in \mathcal{AN}(H)$. Hence, $T^*T \in \mathcal{AN}(H)$, that is, $T \in \mathcal{AN}(H)$.

If $\dim R(K_-) = \infty$,

$$T_1 = \beta P + K_+ \in \mathcal{AN}(H) \quad \text{and} \quad T_2 = \beta(I - P) - K_2 \in \mathcal{AM}(H_2)$$

by [13, Theorem 5.9]. Hence, $T^*T = T_1 \oplus T_2$, and $T = u(|T_1| \oplus |T_2|)$. Since $\mathcal{AM}(H) \subset \overline{\mathcal{AM}(H)} = \overline{\mathcal{AN}(H)}$ by [26, Theorem 6.10] we conclude that $T \in \overline{\mathcal{AN}(H)}$. □

The following example implies that $\mathcal{AN}(H)$ is not a Jordan algebra with respect to

$$S \circ T := \frac{ST + TS}{2}.$$

If $S, T \in \mathcal{B}(H)_{sa}$, then $S \circ T$ need not be an $\mathcal{AN}$-operator. Here we give an example to illustrate this.

EXAMPLE 2.7. Let W be the unilateral shift operator on $\ell^2(\mathbb{N})$ and let $S = \begin{pmatrix} 0 & W \\ W^* & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & W^* \\ W & 0 \end{pmatrix}$. Then, S and T are $\mathcal{AN}$ operators on $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$. We have

$$\begin{aligned} ST + TS &= \begin{pmatrix} 0 & W \\ W^* & 0 \end{pmatrix} \begin{pmatrix} 0 & W^* \\ W & 0 \end{pmatrix} + \begin{pmatrix} 0 & W^* \\ W & 0 \end{pmatrix} \begin{pmatrix} 0 & W \\ W^* & 0 \end{pmatrix} \\ &= \begin{pmatrix} W^2 + W^{*2} & 0 \\ 0 & W^2 + W^{*2} \end{pmatrix}. \end{aligned}$$

By the argument as in Proposition 2.1, $W^2 + W^{*2}$ is not an $\mathcal{AN}$-operator. Hence we can conclude that $S \circ T$ is not an $\mathcal{AN}$-operator.

REMARK 2.8. Following the same argument as in Example 2.7, we know that $W^2 - (W^*)^2 \notin \mathcal{AN}(H)$. By a simple calculation, we get

$$ST - TS = \begin{pmatrix} W^2 - W^{*2} & 0 \\ 0 & W^{*2} - W^2 \end{pmatrix}.$$

Hence, $ST - TS \notin \mathcal{AN}(H)$.

QUESTION 1. Let S and T in $\mathcal{AN}(H)_+$ and assume that $ST + TS$ is positive. Is it true that $ST + TS \in \mathcal{AN}(H)$?

The Lie bracket in $\mathcal{AN}(H)_+$ keeps the $\mathcal{AN}$-property. Note that this is not true in $\mathcal{AN}(H)_{sa}$ (see Remark 2.8).

PROPOSITION 2.9. *Let S and T in $\mathcal{AN}(H)_+$. Then $ST - TS \in \mathcal{AN}(H)$.*

PROOF. Since $S, T \in \mathcal{AN}(H)_+$, there exist positive numbers α, β, positive compact operators K_1, K_2, and self-adjoint finite rank operators such that $S = \alpha I_H + K_1 + F_1$ and $T = \beta I_H + K_2 + F_2$. Then

$$\begin{aligned} ST - TS &= K_1K_2 + K_1F_2 + F_1K_2 + F_1F_2 \\ &- K_2K_1 - K_2F_1 - F_2K_1 - K_2F_1 - F_1F_2 \in \mathcal{K}(H). \end{aligned}$$

Hence, $ST - TS \in \mathcal{AN}(H)$. □

Next, we give a sufficient condition for the Jordan product of two positive $\mathcal{AN}$-operators to be again an $\mathcal{AN}$-operator.

PROPOSITION 2.10. *Let $S, T \in \mathcal{AN}(H)_+$ such that $(T - m_e(T)I)^+$ and $(S - m_e(S)I)^+$ commute and $ST + TS$ is positive, then $ST + TS \in \mathcal{AN}(H)_+$.*

PROOF. Let $S = \alpha_1 I + K_1 - F_1$ and $T = \alpha_2 I + K_2 - F_2$, where $\alpha_i \geq 0$, $K_i \in \mathcal{K}(H)_+$ and $F_i \in \mathcal{F}(H)_+$ with $K_iF_i = 0$ for $i = 1, 2$. Then we have

$$ST + TS = 2\alpha_1\alpha_2 I + K + F,$$

where

$$K = 2\alpha_1 K_2 + 2\alpha_2 K_1 + (K_1K_2 + K_2K_1)$$

and

$$\begin{aligned} F = F_1F_2 + F_2F_1 &- (K_1F_2 + F_2K_1) - (F_1K_2 + K_2F_1) \\ &- 2\alpha_1 F_2 - 2\alpha_2 F_1 \in \mathcal{F}(H)_{sa}. \end{aligned}$$

It is easy to see that $K_1 = (S - m_e(S)I)^+$ and $K_2 = (T - m_e(T)I)^+$ (see [26, Theorem 4.2]). By the assumption, we have that $K_1K_2 = K_2K_1$ by which we can get $K_1K_2 \geq 0$ and hence we can conclude that $K \geq 0$. Hence $ST + TS \in \mathcal{AN}(H)_+$ by Theorem 1.3. □

REMARK 2.11. If $S, T \in \mathcal{K}(H)$, then $m_e(S) = 0 = m_e(T)$. Hence, the assumption on the commutativity of $(T - m_e(T))^+$ and $(S - m_e(S)I)^+$ reduces to the commutativity of S and T. But, when $m_e(T) > 0$ or $m_e(S) > 0$, the operators S and T need not commute.

In the following theorem we discuss functional calculus for $\mathcal{AN}(H)_+$.

THEOREM 2.12. *Let $T \in \mathcal{AN}(H)_+$ with $T = \alpha I + K - F$ for $\alpha \geq 0$, $K \in \mathcal{K}(H)_+$ and $F \in \mathcal{F}(H)_+$ such that $KF = 0$ and $0 \leq F \leq \alpha I$, and let f be a continuous strictly increasing function from $[0, \infty)$ into $[0, \infty)$. Then*

$$f(T) = f(\alpha)I + K' - F' \in \mathcal{AN}(H)_+, \tag{1}$$

where

$$K' := f(\alpha P_M + K) - f(\alpha)P_M \in \mathcal{K}(H)_+,$$
$$F' := f(\alpha)P_{N(F)^\perp} - f(\alpha P_{N(F)^\perp} - F) \in \mathcal{F}(H)_+$$

with $H = N(F)^\perp \oplus M \oplus (N(K) \cap N(F))$ *and* $M \supseteq N(K)^\perp$*. Moreover,* $0 \leq F' \leq f(\alpha)I$ *and* $K'F' = 0$ *hold.*

The operators K' and F' satisfying Equation (1) are unique by Theorem 1.4.

PROOF. Let $T = \alpha I + K - F$ for $\alpha \geq 0$, $K \in \mathcal{K}(H)_+$, $F \in \mathcal{F}(H)_+$ satisfying $0 \leq F \leq \alpha I$ and $KF = 0$. Since $KF = FK = 0$, we have $N(F)^\perp \subseteq N(K)$ and we can decompose H using some closed subspace M as follows (see [16]).

$$H = N(F)^\perp \oplus M \oplus (N(K) \cap N(F)),$$

where $\dim N(F)^\perp < +\infty$. Here we put $M_1 = N(F)^\perp$, $M_2 = M \supseteq N(K)^\perp$ and $M_3 = N(K) \cap N(F)$. Moreover K and F can be diagonalized simultaneously because $KF = FK = 0$. We can put

$$F = \begin{pmatrix} F_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & K_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

on $H = M_1 \oplus M_2 \oplus M_3$, where $K_1 \geq 0$ and $0 \leq F_1 \leq \alpha I_{M_1}$. Then we have

$$T = \alpha I + K - F = \begin{pmatrix} \alpha I_{M_1} - F_1 & 0 & 0 \\ 0 & \alpha I_{M_2} + K_1 & 0 \\ 0 & 0 & \alpha I_{M_3} \end{pmatrix}$$

and

$$\begin{aligned} f(T) &= \begin{pmatrix} f(\alpha I_{M_1} - F_1) & 0 & 0 \\ 0 & f(\alpha I_{M_2} + K_1) & 0 \\ 0 & 0 & f(\alpha) I_{M_3} \end{pmatrix} \\ &= f(\alpha)I + \big(0 \oplus \{f(\alpha I_{M_2} + K_1) - f(\alpha)I_{M_2}\} \oplus 0\big) \\ &\quad - \big(\{f(\alpha)I_{M_1} - f(\alpha I_{M_1} - F_1)\} \oplus 0 \oplus 0\big) \\ &= f(\alpha)I + \{f(\alpha P_{M_2} + K) - f(\alpha)P_{M_2}\} - \{f(\alpha)P_{M_1} - f(\alpha P_{M_1} - F)\} \end{aligned}$$

on $H = M_1 \oplus M_2 \oplus M_3$.

Put $K' = f(\alpha P_{M_2} + K) - f(\alpha)P_{M_2}$ and $F' = f(\alpha)P_{M_1} - f(\alpha P_{M_1} - F)$. We shall show that $K' \in \mathcal{K}(H)_+$ and $F' \in \mathcal{F}(H)_+$ such that $F'K' = 0$ and $0 \leq F' \leq f(\alpha)I$ as follows.

(1) $F' \in \mathcal{F}(H)$ is obvious, since $\dim M_1 < +\infty$.

(2) Since f is a positive increasing function and $F_1 \geq 0$,

$$f(\alpha I_{M_1} - F_1) \leq f(\alpha) I_{M_1}$$

holds, and then

$$0 \leq f(\alpha) I_{M_1} - f(\alpha I_{M_1} - F_1) \leq f(\alpha) I_{M_1},$$

that is, $0 \leq F' \leq f(\alpha) I$ holds.

(3) Since f is an increasing function and $K_1 \geq 0$, we have $K' \geq 0$.

(4) Let $K_1 = \sum_{n=0}^{\infty} s_n P_n$ be the spectral decomposition of K_1, where $\{s_n\}$ is a decreasing sequence which converges to 0. Then

$$f(\alpha I_{M_2} + K_1) - f(\alpha) I_{M_2} = \sum_{n=0}^{\infty} \{f(\alpha + s_n) - f(\alpha)\} P_n.$$

Since f is a continuous strictly increasing function, a sequence

$$\{f(\alpha + s_n) - f(\alpha)\}$$

is a decreasing sequence which converges to 0. Hence $K' \in \mathcal{K}(H)_+$.

(5) $K'F' = 0$ is obtained by the structures of K' and F'.

Therefore $f(T) = f(\alpha)I + K' - F' \in \mathcal{AN}(H)_+$, moreover, K' and F' are uniquely determined by Theorem 1.4. □

Remark 2.13. Let f be a continuous, strictly increasing function from $[0, \infty)$ into $[0, \infty)$. From Theorem 2.12 we know that $f(\overline{\mathcal{AN}(H)_+}) \subset \overline{\mathcal{AN}(H)_+}$.

Remark 2.14. We can give another proof of Theorem 2.12.

By [23, Proposition 2.16], we represent T as the block operator matrix with respect to the decomposition $H = N(K) \oplus N(K)^{\perp}$ by

$$T = \begin{pmatrix} \alpha I_{N(K)} - F_0 & 0 \\ 0 & K_0 + \alpha I_{N(K)^{\perp}} \end{pmatrix},$$

where $F_0 = F|_{N(K)}$ and $K_0 = K|_{N(K)^{\perp}}$. Let

$$F_0 = \sum_{j=1}^{k} \delta_j \tilde{P}_j \quad \text{and} \quad K_0 = \sum_{n=1}^{\infty} s_m P_m$$

be the spectral representations of F_0 and K_0, respectively, where $\delta_j \geq 0$ for $j = 1, 2, \ldots, k$ and $s_m \geq 0$ and $s_m \to 0$ as $m \to \infty$. Note that

$$\sum_{j=1}^{k} \tilde{P}_j + \sum_{n=1}^{\infty} P_n = I.$$

Hence we can write

$$T = \begin{pmatrix} \sum_{j=1}^{n} (\alpha - \delta_j)\tilde{P}_j & 0 \\ 0 & \sum_{j=1}^{\infty} (s_j + \alpha)P_j \end{pmatrix}.$$

Moreover, we have

$$f(T) = \begin{pmatrix} \sum_{j=1}^{n} f(\alpha - \delta_j)\tilde{P}_j & 0 \\ 0 & \sum_{j=1}^{\infty} f(s_j + \alpha)P_j \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{n} \big(f(\alpha - \delta_j) - f(\alpha)\big)\tilde{P}_j & 0 \\ 0 & \sum_{j=1}^{\infty} \big(f(s_j + \alpha) - f(\alpha)\big)P_j \end{pmatrix} + f(\alpha)I.$$

Let

$$F_1 = \sum_{j=1}^{n} \big(f(\alpha) - f(\alpha - \delta_j)\big)\tilde{P}_j \quad \text{and} \quad K_1 = \sum_{j=1}^{\infty} \big(f(s_j + \alpha) - f(\alpha)\big)P_j\,.$$

It is clear that F_1 is a positive finite rank operator and as f is increasing, $f(s_j + \alpha) - f(\alpha)$ converges to 0 as $n \to \infty$, we can conclude that K_1 is positive and compact. If we write $F = \left(\begin{smallmatrix} F_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $K = \left(\begin{smallmatrix} 0 & 0 \\ 0 & K_1 \end{smallmatrix}\right)$, then F is a finite rank positive operator and K is a positive compact operator and $f(T) = f(\alpha)I + K - F$. It is easy to see that $KF = 0$ and as f is increasing we have $F \le f(\alpha)I$. Hence by Theorem 1.4, $f(T) \in \mathcal{AN}(H)$ and the representation of $f(T)$ is unique.

Using continuous functional calculus we can prove the following.

Remark 2.15. Let $T \in \mathcal{AN}(H)_+$ with $T = \alpha I + K - F$ for $\alpha \ge 0$, $K \in \mathcal{K}(H)_+$ and $F \in \mathcal{F}(H)_+$ such that $KF = 0$ and $0 \le F \le \alpha I$, and let f be a continuous strictly increasing function from $[0, \infty)$ into $[0, \infty)$. Then

(1) $\sigma_{\text{ess}}(f(T)) = f(\alpha) = f(\sigma_{\text{ess}}(T))$
(2) $\sigma_d(f(T)) = f(\sigma_d(T))$.

The functional calculus described in Theorem 2.12 cannot be extended to self-adjoint $\mathcal{AN}$-operators.

Example 2.16. Let $T = T^* \in \mathcal{AN}(H)$. Then by [2, Corollary 2.2], there exists $\alpha \ge 0$ such that $\sigma_{\text{ess}}(T) \subseteq \{-\alpha, \alpha\}$ and $\sigma(T) \cap (-\alpha, \alpha)$ is a finite set. Assume that $\sigma_{\text{ess}}(T) = \{-\alpha, \alpha\}$. Now define $f(t) = t + \alpha + \|T\|$

for all $t \in \mathbb{R}$. Then f is strictly increasing, continuous and $f(T) \geq 0$. Also, $\sigma_{\text{ess}}(f(T)) = \{\|T\|, 2\alpha + \|T\|\}$. Hence by [25, Theorem 2.4], we can conclude that $f(T) \notin \mathcal{AN}(H)$.

It is easy to find a self-adjoint operator T with $\sigma_{\text{ess}}(T) = \{-\alpha, \alpha\}$ for some $\alpha > 0$. To this end, let $A \in \mathcal{AN}(H)_+$ with $\sigma_{\text{ess}}(A) = \{\alpha\}$ for some $\alpha > 0$. Define $T = A \oplus (-A)$. Then $T = T^*$ and $T^*T = T^2 = A^2 \oplus A^2 \in \mathcal{AN}(H \oplus H)_+$. Hence $T \in \mathcal{AN}(H \oplus H)_{sa}$ and $\sigma_{\text{ess}}(T) = \{-\alpha, \alpha\}$.

Proposition 2.17. *Let $T = T^* \in \mathcal{AN}(H)$ with $\sigma_{\text{ess}}(T) \subseteq \{-\alpha, \alpha\}$ for some $\alpha \geq 0$. If $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function such that $f(-\alpha) = -f(\alpha)$, then $f(T) \in \mathcal{AN}(H)$.*

Proof. By the functional calculus, we get $f(T)$ is self-adjoint and $f(\sigma(T)) = \sigma(f(T))$. Since by Remark 2.15,

$$\sigma_{\text{ess}}(f(T)) = f(\sigma_{\text{ess}}(T)) \subset f(\{-\alpha, \alpha\}) = \{f(-\alpha), f(\alpha)\} = \{-f(\alpha), f(\alpha)\},$$

by [2, Corollary 2.2], we can conclude that $f(T) \in \mathcal{AN}(H)$. □

Remark 2.18. Note that in Example 2.16, we have

$$\sigma_{\text{ess}}(f(T)) = f(\sigma_{\text{ess}}(T)) \subseteq f(\{-\alpha, \alpha\}) \subseteq \{f(-\alpha), f(\alpha)\} = \{\|T\|, 2\alpha + \|T\|\}.$$

It is easy to verify that $f(-\alpha) \neq -f(\alpha)$ for any $\alpha \in \mathbb{R}$.

Next, we will discuss the stability of $\mathcal{AN}(H)_+$ under the perspective functions.

Definition 2.19. For a real valued continuous function f on $(0, \infty)$, a two-variable operator function P_f defined by

$$P_f(A, B) = B^{\frac{1}{2}} f(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}) B^{\frac{1}{2}} \quad (\text{for invertible } A, B \geq 0)$$

is called the operator perspective of f.

It is firstly considered in [10] for commuting matrices and then the above concrete form for non commutative matrices is given in [9]. Properties of operator perspective are obtained in many papers, for example, [11,18,19].

Note that when f is operator monotone with $f(1) = 1$, the operator perspective $P_f(B, A) = A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}$ (for invertible $A, B \geq 0$) is called the operator mean in the sense of Kubo–Ando [19].

Proposition 2.20. *Let f be a positive, continuous, strictly increasing function on $(0, \infty)$. Then, for any invertible $S, T \in \mathcal{AN}(H)_+$, $P_f(S, T) \in \overline{\mathcal{AN}(H)}$.*

Proof. Note that since $T \in \mathcal{AN}(H)_+$, $\sigma_{\text{ess}}(T)$ is singleton. Hence, $\sigma_{\text{ess}}(T^{-1})$ is also singleton, and $T^{-1} \in \overline{\mathcal{AN}(H)}$ by [26, Theorem 4.5].

Since $T^{-\frac{1}{2}}ST^{-\frac{1}{2}} \in \overline{\mathcal{AN}(H)}$, we have $f(T^{-\frac{1}{2}}ST^{-\frac{1}{2}}) \in \overline{\mathcal{AN}(H)}$ by Remark 2.13. Hence, $P_f(S,T) = T^{\frac{1}{2}}f(T^{-\frac{1}{2}}ST^{-\frac{1}{2}})T^{\frac{1}{2}} \in \overline{\mathcal{AN}(H)}$ by [26]. □

COROLLARY 2.21. *Let σ be an operator mean. Then, for any invertible $S,T \in \mathcal{AN}(H)_+$, we have $S\sigma T \in \overline{\mathcal{AN}(H)}$.*

PROOF. Let f be the corresponding operator monotone function to σ $(=\sigma_f)$. Then, $S\sigma_f T = P_f(T,S) \in \overline{\mathcal{AN}(H)}$ by Proposition 2.20. □

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