Linear Algebra and its Applications 678 (2023) 206–226

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Stability of $\mathcal{AN}$-property for the induced Aluthge transformations

Ramesh Golla [a], Hiroyuki Osaka [b], Yoichi Udagawa [c], Takeaki Yamazaki [d,*]

[a] *Department of Mathematics, I. I. T. Hyderabad Kandi (V), Sangareddy, Telangana, 502 285, India*
[b] *Department of Mathematical Sciences, Ritsumeikan University, Kusatsu, Shiga, 525-8577, Japan*
[c] *Department of Advanced Practice of School Education, Naruto University of Education, 748, Nakajima, Takashima, Naruto-cho, Naruto-shi, Tokushima, 772-8502, Japan*
[d] *Department of Electrical, Electronic and Communications Engineering, Toyo University, Kawagoe-Shi, Saitama, 350-8585, Japan*

A R T I C L E I N F O

Article history:
Received 6 December 2022
Received in revised form 20 August 2023
Accepted 25 August 2023
Available online 30 August 2023
Submitted by P. Semrl

MSC:
47A10
47A64

Keywords:
Norm attaining operator
Absolutely norm attaining operator
Aluthge transform
Induced Aluthge transform
Mean transform

A B S T R A C T

We shall show that if a bounded linear operator T on a complex Hilbert space H satisfies $\mathcal{AN}$-property, then the induced Aluthge transformations with respect to some means of T satisfy $\mathcal{AN}$-property, too. Moreover, we shall discuss a limit point of mean transformation. Chabbabi, Curto and Mbekhta pointed out that a sequence of iterated mean transform of an operator with a special condition converges to a normal operator in the strong operator topology without proof. We shall give a proof of iterated mean transformation of a semi-hyponormal operator converges. Moreover, we shall show that the limit point satisfies $\mathcal{AN}$-property if T does so.

* Corresponding author.
E-mail addresses: rameshg@iith.ac.in (R. Golla), osaka@se.ritsumei.ac.jp (H. Osaka), yudagawa@naruto-u.ac.jp (Y. Udagawa), t-yamazaki@toyo.jp (T. Yamazaki).

https://doi.org/10.1016/j.laa.2023.08.016
0024-3795/

1. Introduction

In this note, we study the stability of absolutely norm attaining property ($\mathcal{AN}$-property, for short) for the induced Aluthge transformation. Especially, we focus on the iteration of the mean transformation which is a special case of induced Aluthge transformation. We shall show that if an operator is a semi-hyponormal operator satisfying $\mathcal{AN}$-property, then the limit of iteration of mean transformation of this operator satisfies $\mathcal{AN}$-property, again.

1.1. Notations and terminologies

Throughout this note, we consider complex Hilbert spaces which will be denoted by H, H_1, H_2, etc. All the Hilbert spaces are assumed to be infinite dimensional. The inner product and the induced norm on H are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. The unit sphere of a subspace M of H is defined by $S_M := \{x \in M : \|x\| = 1\}$. The orthogonal complement of M in H is denoted by $M^\perp$.

We denote the space of all bounded linear operators from H_1 into H_2 by $\mathcal{B}(H_1, H_2)$ and $\mathcal{B}(H, H)$ by $\mathcal{B}(H)$. We denote I by the identity operator on $\mathcal{B}(H)$. The null space and the range spaces of T are denoted by $\ker(T)$ and $R(T)$, respectively. The adjoint of $T \in \mathcal{B}(H)$ is denoted by T^*.

We say $T \in \mathcal{B}(H)$ to be *normal* if $T^*T = TT^*$, *self-adjoint* if $T = T^*$. If $\langle Tx, x\rangle \geq 0$ for all $x \in H$, then T is called *positive* or *positive semi definite*. If $\mathcal{A} \subseteq \mathcal{B}(H)$ is a nonempty subset, then the set of all positive elements in $\mathcal{A}$ is denoted by $\mathcal{A}_+$. Given $T \in \mathcal{B}(H)_+$, there exists a unique $S \in \mathcal{B}(H)_+$ such that $S^2 = T$. This operator S is called the *positive square root* of T and it is denoted by $T^{\frac{1}{2}}$.

If $S, T \in \mathcal{B}(H)$ are self-adjoint and $S - T \geq 0$, then we write this by $S \geq T$. If $P \in \mathcal{B}(H)$ is such that $P^2 = P$, then P is called a *projection*. In addition, if, $\ker(P)$ and $R(P)$ are orthogonal to each other, then P is called an *orthogonal projection*.

A projection P is orthogonal if and only if it is self-adjoint if and only if it is normal. We say two orthogonal projections P and Q to be mutually orthogonal if $R(P)$ and $R(Q)$ are orthogonal to each other. The orthogonal projection on a Hilbert space H whose range is M is denoted by P_M.

An operator $V \in \mathcal{B}(H)$ is said to be an *isometry* if $\|Vx\| = \|x\|$ for all $x \in H$ and a *partial isometry* if $V|_{\ker(V)^\perp}$ is an isometry. That is, $\|Vx\| = \|x\|$ for all $x \in \ker(V)^\perp$. We say V to be unitary if $V \in \mathcal{B}(H)$ is isometry and onto.

If $T \in \mathcal{B}(H)$, then $T^*T \in \mathcal{B}(H)$ is positive and $|T| := (T^*T)^{\frac{1}{2}}$ is called the *modulus* of T. In fact, for any $T \in \mathcal{B}(H)$, there exists a unique partial isometry $V \in \mathcal{B}(H)$ such that $T = V|T|$ and $\ker(V) = \ker(T)$. This factorization is called the *polar decomposition* of T.

The space of all finite rank operators between H_1 and H_2 is denoted by $\mathcal{F}(H_1, H_2)$ and we write $\mathcal{F}(H, H) = \mathcal{F}(H)$. We denote the space of all compact operators between H_1 and H_2 by $\mathcal{K}(H_1, H_2)$. In case if $H_1 = H_2 = H$, then $\mathcal{K}(H_1, H_2)$ is denoted by $\mathcal{K}(H)$.

An operator $T \in \mathcal{B}(H)$ is said to be

(i) *quasinormal* if T commutes with T^*T, that is $T(T^*T) = (T^*T)T$,
(ii) *hyponormal* if $T^*T \geq TT^*$, equivalently, $\|Tx\| \geq \|T^*x\|$ holds for all $x \in H$,
(iii) *semi-hyponormal* if $|T| \geq |T^*|$.

It is to be mentioned that

$$Normal \subset Quasinormal \subset Hyponormal \subset Semi-hyponormal.$$

These inclusions are strict. We refer to [11] for more details about all these classes of operators.

For $T \in \mathcal{B}(H)$, the set

$$\rho(T) := \{\lambda \in \mathbb{C} : \text{ There exists } (T - \lambda I)^{-1} \in \mathcal{B}(H)\}$$

is called the *resolvent set* and the complement $\sigma(T) = \mathbb{C} \setminus \rho(T)$ is called the *spectrum* of T. It is well-known that $\sigma(T)$ is a non-empty compact subset of $\mathbb{C}$.

An operator T is *Fredholm* if and only if $R(T)$ is closed, $\dim \ker(T) < +\infty$ and $\operatorname{codim} R(T) < +\infty$. The essential spectrum $\sigma_{ess}(T)$ is defined as follows:

$$\sigma_{ess}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\}.$$

1.2. $\mathcal{AN}$-operators

Next, we discuss norm attaining operators. An operator $T : H_1 \to H_2$ is said to be a *norm attaining* or $\mathcal{N}$-operator if there exists a unit vector $x \in H_1$ such that $\|Tx\| = \|T\|$ [6]. We denote the class of norm attaining operators between H_1 and H_2 by $\mathcal{N}(H_1, H_2)$ and $\mathcal{N}(H, H)$ by $\mathcal{N}(H)$. It is known that $\mathcal{N}(H_1, H_2)$ is dense in $\mathcal{B}(H_1, H_2)$ with respect to the operator norm of $\mathcal{B}(H_1, H_2)$. We refer [10] for a simple proof of this fact.

Definition 1.1 *([22,23]).* An operator $T : H_1 \to H_2$ is said to be an *absolutely norm attaining* or an $\mathcal{AN}$-operator if $T|_M$, the restriction of T to M, is norm attaining for every non-zero closed subspace M of H_1. If T is an $\mathcal{AN}$-operator, we often say that T has $\mathcal{AN}$-property.

We denote the class of absolutely norm attaining operators between H_1 and H_2 by $\mathcal{AN}(H_1, H_2)$ and $\mathcal{AN}(H, H)$ by $\mathcal{AN}(H)$. We note that $\mathcal{K}(H) \subsetneq \mathcal{AN}(H) \subsetneq \mathcal{N}(H)$ [6].

This class was initiated by Carvajal and Neves in [6] and further studied in [4,12, 21,22,24]. This class includes the space of compact operators and the class of partial isometries with finite-dimensional null spaces. It is not closed under sum, while it closed with respect to the product of operators [12]. The sum of a compact operator and a partial isometry with finite-dimensional null space need not be an $\mathcal{AN}$-operator, but it is in the operator norm closure of $\mathcal{AN}$-operators [25]. Characterizations of positive $\mathcal{AN}$-operators are described in [21,22,24], while the study of self-adjoint, normal and paranormal $\mathcal{AN}$-operators is done in [4,21,24]. The hyperinvariant subspace problem for this class is studied in [5]. The maps preserving $\mathcal{AN}$-operators are discussed in [12]. The power of an $\mathcal{AN}$-operator satisfies $\mathcal{AN}$-property, too, but polynomials of an $\mathcal{AN}$-operator need not be an $\mathcal{AN}$-operator, as the sum need not be closed under product.

We recall the following characterizations of positive $\mathcal{AN}$-operators which are very important in our study.

Theorem 1.2 *([22]). An operator $T \in \mathcal{B}(H)_+$ is an $\mathcal{AN}$-operator if and only if there exists a non-negative number α, $K \in \mathcal{K}(H)_+$ and self-adjoint $F \in \mathcal{F}(H)$ such that*

$$T = \alpha I + K + F.$$

Theorem 1.3 *([23]). An operator $T \in \mathcal{B}(H)_+$ is an $\mathcal{AN}$-operator if and only if there exists a non-negative number α, $K \in \mathcal{K}(H)_+$ and $F \in \mathcal{F}(H)_+$ such that $FK = 0$, $0 \leq F \leq \alpha I$ and*

$$T = \alpha I + K - F. \tag{1}$$

Moreover, (α, K, F) is uniquely determined.

Although, a representation in Theorem 1.2 is not determined uniquely, we shall often use these characterizations in suitable cases.

1.3. Operator mean

An operator mean is a binary operation on positive semi-definite operators. It was defined by Kubo-Ando as follows. Let $\mathcal{B}(H)_{++}$ be the set of positive invertible operators.

Definition 1.4 *(Operator mean, [18]).* Let $m : \mathcal{B}(H)_+ \times \mathcal{B}(H)_+ \to \mathcal{B}(H)_+$ be a binary operation. If m satisfies the following four conditions, then m is called an *operator mean*.

(i) If $A \leq C$ and $B \leq D$, then $m(A, B) \leq m(C, D)$,
(ii) $X^*\ m(A, B)X \leq m(X^*AX, X^*BX)$ for all $X \in \mathcal{B}(H)$,
(iii) $A_n \searrow A$ and $B_n \searrow B$ imply $m(A_n, B_n) \searrow m(A, B)$ in the strong operator topology,
(iv) $m(I, I) = I$.

To get a concrete formula of an operator mean, the next theorem is very important. Let f be a real-valued function defined on an interval $J \subseteq (0, \infty)$. Then f is said to be *operator monotone* if $A \leq B$ for self-adjoint operators $A, B \in \mathcal{B}(H)$ whose spectra are contained in J, then $f(A) \leq f(B)$, where $f(A)$ and $f(B)$ are defined by the functional calculus.

Theorem 1.5. [18] *Let m be an operator mean. Then there exists an operator monotone function f on $(0,\infty)$ such that $f(1)=1$ and*

$$m(A,B)=A^{\frac{1}{2}}f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$$

for all $A\in\mathcal{B}(H)_{++}$ and $B\in\mathcal{B}(H)_{+}$.

If $A\in\mathcal{B}(H)_{+}$, we can obtain $m(A,B)=s-\lim_{\varepsilon\searrow 0}m(A+\varepsilon I,B)$ because $A+\varepsilon I\in\mathcal{B}(H)_{++}$ for $\varepsilon>0$ and Definition 1.4 (iii). The function f is called a *representing function* of an operator mean m. Throughout this paper, we denote m_f by an operator mean with a representing function f.

1.4. Aluthge transformation

The well-known polar decomposition of any operator $T=u|T|\in\mathcal{B}(H)$, where $u\in\mathcal{B}(H)$ is a partial isometry satisfying $\ker(u)=\ker(T)$. From such a decomposition, the Aluthge transform of T is defined as follows:

Definition 1.6 *(Aluthge transformation, [1]).* Let $T=u|T|\in\mathcal{B}(H)$ be the polar decomposition of T. The *Aluthge transformation* $\Delta(T)$ of T is defined as follows:

$$\Delta(T)=|T|^{\frac{1}{2}}u|T|^{\frac{1}{2}}.$$

Aluthge transform has nice properties. Especially, the following properties are well-known: (i) $\sigma(\Delta(T))=\sigma(T)$ [16]. (ii) $\Delta(T)$ has a non-trivial invariant subspace if and only if so does T [17]. (iii) If T is semi-hyponormal (i.e. $|T^*|\leq|T|$), then $\Delta(T)$ is hyponormal [1]. More generally, for any scalar $\lambda\in[0,1]$, the λ-Aluthge transform of T is defined by $\Delta_\lambda(T):=|T|^{\lambda}u|T|^{1-\lambda}$ in [16]. Many authors are interested in the iteration of Aluthge transform. This theme has been firstly considered in [17], and then it has been shown that the norm of iteration of Aluthge transform of any operator converges to its spectral radius [28]. Lastly, the iteration of the Aluthge transform of any n–by–n matrix converges to a normal matrix [2,3,14]. This problem has been extended to von-Neumann algebra and Lie group [9,15].

In recent years, a similar mapping $\hat{T}$ is discussed. For an operator $T=u|T|\in\mathcal{B}(H)$, $\hat{T}:=\frac{|T|u+u|T|}{2}$ is called the *mean transform* of T [19]. There are a lot of papers on the mean transforms, especially, iteration of mean transforms is discussed in [7]. Exactly, it is known that

(i) if T is a n-by-n matrix, then the iteration of mean transform of T converges to a normal matrix in [29],
(ii) let $T=u|T|$ be the polar decomposition such that $\ker(T^*)\subseteq\ker(T)$. If there exists a limit $u^{*n}|T|u^n$ as $n\to\infty$ in the strong operator topology, then the mean transformation converges to a quasi-normal operator in [7] without proof.

We notice that there is a counterexample of an operator T such that the iteration of mean transformation converges in [29].

Recently, one of the authors defined the induced Aluthge transform in the viewpoint of the axiom of operator means, which is defined by using double operator integrals in [29]. It interpolates means and Aluthge transformations when $|T|$ is invertible.

Definition 1.7 *(Induced Aluthge transformation, [29]).* Let $T = u|T| \in B(H)$ with the spectral decomposition $|T| = \int_{\sigma(T)} s dE_s$. For an operator mean m_f with a representing function f, the *induced Aluthge transformation* $\Delta_{m_f}(T)$ of T with respect to m_f is defined as follows.

(i) If $|T|$ is invertible, then

$$\Delta_{m_f}(T) = \int\limits_{\sigma(|T|)} \int\limits_{\sigma(|T|)} P_f(s,t) dE_s u dE_t,$$

where $P_f(s,t) = sf(\frac{t}{s})$ for $s,t \in (0,\infty)$.

(ii) If $|T|$ is not invertible, and if there exists an isometry V such that $T_\varepsilon = V(|T| + \varepsilon I_H)$ is the polar decomposition for all $\varepsilon > 0$ and $s - \lim_{\varepsilon \downarrow 0} T_\varepsilon = T$, then

$$\Delta_{m_f}(T) = s - \lim_{\varepsilon \downarrow 0} \Delta_{m_f}(T_\varepsilon),$$

in the strong operator topology.

Example 1.8. [29, Example 2] Let $T \in \mathcal{B}(H)$ such that $|T|$ is invertible with the polar decomposition $T = u|T|$.

(i) Let $\lambda \in [0,1]$ and $f_\lambda(t) = 1 - \lambda + \lambda t$ for $t \in [0,\infty)$, i.e., the corresponding operator mean m_{f_λ} is called the λ-weighted arithmetic mean. The induced Aluthge transform $\Delta_{m_{f_\lambda}}$ is

$$\Delta_{m_{f_\lambda}}(T) = (1-\lambda)|T|u + \lambda u|T|.$$

Especially, if T is invertible, then $\Delta_{m_{f_{1/2}}}(T) = \hat{T}$, the mean transformation of T.

(ii) Let $\lambda \in [0,1]$ and $g_\lambda(t) = t^\lambda$ for $t \in [0,\infty)$, i.e., the corresponding operator mean m_{g_λ} is called the λ-weighted geometric mean. The induced Aluthge transform $\Delta_{m_{g_\lambda}}$ is

$$\Delta_{m_{g_\lambda}} = |T|^{1-\lambda} u |T|^\lambda.$$

When $\lambda = \frac{1}{2}$, we know that $\Delta_{m_{g_{1/2}}} = \Delta$, the Aluthge transform.

The following example is the induced Aluthge transformations with respect to the power mean: Let $f_r(t) = (\frac{1+t^r}{2})^{\frac{1}{r}}$. If $r \in [-1,1]$, then f_r is a representing function of the operator power mean. For a natural number n and invertible $T \in \mathcal{B}(H)$, the induced Aluthge transformation with respect to the power mean with the parameter $r = \frac{1}{n}$ is given as follows.

$$\begin{aligned}
\Delta_{m_f}(T) := \Delta_{m_{f_{1/n}}}(T) &= \int\limits_{\sigma(|T|)} \int\limits_{\sigma(|T|)} P_{f_{1/n}}(s,t) dE_s u dE_t \\
&= \int\limits_{\sigma(|T|)} \int\limits_{\sigma(|T|)} \left(\frac{s^{\frac{1}{n}} + t^{\frac{1}{n}}}{2} \right)^n dE_s u dE_t \qquad (2) \\
&= \int\limits_{\sigma(|T|)} \int\limits_{\sigma(|T|)} \frac{1}{2^n} \left(\sum_{i=0}^{n} \binom{n}{i} s^{\frac{n-i}{n}} t^{\frac{i}{n}} \right) dE_s u dE_t \\
&= \frac{1}{2^n} \left(\sum_{i=0}^{n} \binom{n}{i} |T|^{\frac{n-i}{n}} u |T|^{\frac{i}{n}} \right),
\end{aligned}$$

where we used Example 1.8 at the last equality.

In this paper, we shall discuss the stability of $\mathcal{AN}$-property under the induced Aluthge transforms. Moreover, we shall consider the iteration of the mean transform (i.e., induced Aluthge transform with respect to the arithmetic mean) of semi-hyponormal operators. We give a concrete limit point of the sequence, and we shall show that $\mathcal{AN}$-property is stable under the limit operation.

This paper is organized as follows: In Section 2, we shall show the stability of $\mathcal{AN}$-property under the induced Aluthge transforms with respect to the arithmetic and geometric means. Then we shall discuss the iteration of mean transforms of a semi-hyponormal operator in Section 3. In Section 4, we shall show some operator inequalities related to semi-hyponormal operators.

2. Stability of $\mathcal{AN}$-property under the induced Aluthge transform

In this section, we shall discuss the following question, and give partial answers.

Question 1. *Let $T \in \mathcal{AN}(H)$ and f be an operator monotone function on $(0,\infty)$ with $f(1) = 1$. Does $\Delta_{m_f}(T) \in \mathcal{AN}(H)$ hold?*

Firstly, we shall consider the general case.

Theorem 2.1. *Let f be a non-negative positive operator monotone function on $[0,\infty)$. Suppose that $T \in \mathcal{AN}(H)$ with invertible $|T|$. Then, $\Delta_{m_f}(T) \in \overline{\mathcal{AN}(H)}$, where $\overline{M}$ means the closure of a subset $M \subseteq \mathcal{B}(H)$ in the operator norm.*

Proof. Let $\pi\colon B(H) \to B(H)/K(H)$ be the canonical quotient map. Suppose that $T \in \mathcal{AN}(H)$ with invertible $|T|$. From [29, Theorem 4.1]

$$\Delta_{m_f}(T) = \int_0^1 \left(\int_0^\infty e^{-x(1-\lambda)|T|^{-1}} u e^{-x\lambda|T|^{-1}} dx \right) d\mu(\lambda),$$

where $T = u|T|$ is the polar decomposition and μ is a probability measure. Since $T \in \mathcal{AN}(H)$, $\pi(|T|) = \alpha I$ for some positive number by Theorems 1.2 and 1.3, and $\pi(u)$ is isometry.

If $\alpha = 0$, then $T \in \mathcal{K}(H)$, and hence $\Delta_{m_f}(T) \in \mathcal{K}(H) \subset \mathcal{AN}(H)$. If $\alpha \neq 0$, then

$$\pi(\Delta_{m_f}(T)) = \int_0^1 \left(\int_0^\infty e^{-\frac{1}{\alpha}x} \pi(u) dx \right) d\mu(\lambda) = \alpha\pi(u).$$

Since $\pi(u)$ is isometry, we conclude that $\Delta_{m_f}(T) \in \overline{\mathcal{AN}(H)}$ by [26, Proposition 2.6]. □

It is known that $\mathcal{AN}(H) \subsetneq \overline{\mathcal{AN}(H)}$ [25]. Next, we shall introduce some special cases of operator monotone functions for which $\Delta_{m_f}(T) \in \mathcal{AN}(H)$ if $T \in \mathcal{AN}(H)$, i.e., more precise results than Theorem 2.1.

The following is useful to analyze the stability of $\mathcal{AN}$-property.

Lemma 2.2 *([26]). Let f be a continuous, strictly increasing function from $[0,\infty)$ into $[0,\infty)$. Then, for any $T \in \mathcal{AN}(H)_+$, $f(T) \in \mathcal{AN}(H)$.*

Fortunately, $\mathcal{AN}(H)$ is stable under the generalized Aluthge transform $\Delta_{m_{g_\lambda}}$.

Theorem 2.3. *For $\lambda \in [0,1]$, let $g_\lambda(t) = t^\lambda$. If $T \in \mathcal{AN}(H)$, then $\Delta_{m_{g_\lambda}}(T) \in \mathcal{AN}(H)$. Especially, if $T \in \mathcal{AN}(H)$, then $\Delta(T) \in \mathcal{AN}(H)$.*

Proof. Since $T \in \mathcal{AN}(H)$, then $|T| \in \mathcal{AN}(H)$ by [22, Lemma 6.2]. Hence, by Theorem 1.2, there exist a non-negative number α, $K \in \mathcal{K}(H)_+$ and a self-adjoint operator $F \in \mathcal{F}$ such that $|T| = \alpha I + K + F$.

If $\alpha = 0$, $|T|$ is a compact, and so is $\Delta_{m_{g_\lambda}}(T)$. Hence $\Delta_{m_{g_\lambda}}(T) \in \mathcal{K}(H) \subset \mathcal{AN}(H)$.

If $\alpha \neq 0$, then $\ker(T)$ is finite dimensional by [21, Proposition 2.8 (2)] (It is easily obtained by Theorem 1.3 and $\ker(T) = \ker(|T|)$). Since $\ker(u) = \ker(T)$ is finite dimensional, $P_{\ker(u)}$ is finite rank and

$$u^*u = P_{\ker(u)^\perp} = I - P_{\ker(u)}.$$

Again, by Theorem 1.2, we have $u \in \mathcal{AN}(H)$ by [22, Lemm 6.2].

Since $|T|^{1-\lambda}, |T|^\lambda \in \mathcal{AN}(H)$ by Lemma 2.2 and by [12, Corollary 2.3] we conclude that $\Delta_{m_{g_\lambda}}(T) = |T|^{1-\lambda} u |T|^\lambda \in \mathcal{AN}(H)$.

Therefore, we get the conclusion. $\square$

Next, we shall discuss the induced Aluthge transform with respect to the arithmetic mean case.

Theorem 2.4. *For $\lambda \in [0,1]$, let $f_\lambda(t) = 1 - \lambda + \lambda t$, and let $T \in \mathcal{AN}(H)$ with the polar decomposition $T = u|T|$. Then $\Delta_{m_{f_\lambda}}(T) = (1-\lambda)|T|u + \lambda u|T| \in \mathcal{AN}(H)$.*

Before proving Theorem 2.4, we shall prepare the following lemma.

Lemma 2.5. *Let $T \in \mathcal{AN}(H)$ with the polar decomposition $T = u|T|$ and let α be a non-negative number, $K \in \mathcal{K}(H)_+$ and $F \in \mathcal{F}(H)_+$ such that $|T| = \alpha I + K - F$ and $KF = 0$. Then, $u^*uK = Ku^*u = K$.*

Proof. Since, $|T| = \alpha I + K - F$ and $KF = 0$ (i.e., $\overline{R(F)} \subseteq \ker(K)$), we can represent $|T|$ as follows:

$$|T| = \begin{pmatrix} \alpha I_{\ker(K)} - F_1 & 0 \\ 0 & K_1 + \alpha I_{\ker(K)^\perp} \end{pmatrix}, \tag{3}$$

where $K = 0 \oplus K_1$ and $F = F_1 \oplus 0$ on $H = \ker(K) \oplus \ker(K)^\perp$ by [21, Proposition 2.16]. Since $K \geq 0$ and $K_1 + \alpha I_{\ker(K)^\perp}$ is positive invertible, (3) implies $\ker(T) \subset \ker(K)$. Hence there exists a projection P such that

$$u^*u = \begin{pmatrix} P & 0 \\ 0 & I_{\ker(K)^\perp} \end{pmatrix}.$$

Then

$$u^*uK = \begin{pmatrix} P & 0 \\ 0 & I_{ker(K)^\perp} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & K_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & K_1 \end{pmatrix} = K$$

and $Ku^*u = K$ holds. Therefore $u^*uK = Ku^*u = K \geq 0$. $\square$

Proof of Theorem 2.4. Since $T \in \mathcal{AN}(H)$, $|T| \in \mathcal{AN}(H)$ and there exist a non-negative number α, $K \in \mathcal{K}(H)_+$ and $F \in \mathcal{F}(H)_+$ such that $|T| = \alpha I + K - F$ and $KF = 0$ by Theorem 1.3.

(i) If $\alpha = 0$, then $|T|$ is a compact operator. Hence $\Delta_{m_{f_\lambda}}(T)$ is a compact operator. Therefore $\Delta_{m_{f_\lambda}}(T) \in \mathcal{K}(H) \subset \mathcal{AN}(H)$.

(ii) We shall consider $\alpha \neq 0$. In this case, $\dim\ker(T) < +\infty$ holds. Then

$$\begin{aligned} \Delta_{m_{f_\lambda}}(T) &= \lambda u|T| + (1-\lambda)|T|u \\ &= \lambda u(\alpha I + K - F) + (1-\lambda)(\alpha I + K - F)u \\ &= \alpha u + (\lambda uK + (1-\lambda)Ku) - (\lambda uF + (1-\lambda)Fu). \end{aligned}$$

Then we have

$$
\begin{aligned}
&|\Delta_{m_{f_\lambda}}(T)|^2 \\
&= \{\alpha u + (\lambda u K + (1-\lambda)Ku) - (\lambda u F + (1-\lambda)Fu)\}^* \\
&\qquad \times\{\alpha u + (\lambda u K + (1-\lambda)Ku) - (\lambda u F + (1-\lambda)Fu)\} \\
&= \alpha^2 u^* u + K' + F',
\end{aligned}
$$

where

$$K' = \alpha(\lambda u^* u K + \lambda K u^* u + 2(1-\lambda)u^* K u) + |\lambda u K + (1-\lambda)Ku|^2$$

and

$$
\begin{aligned}
F' = -(&\alpha(\lambda u^* u F + \lambda F u^* u + 2(1-\lambda)u^* F u) \\
&- (\lambda u K + (1-\lambda)Ku)^* (\lambda u F + (1-\lambda F u) \\
&- (\lambda u F + (1-\lambda)Fu)^* (\lambda u K + (1-\lambda)Ku) \\
&+ |\lambda u F + (1-\lambda)Fu|^2.
\end{aligned}
$$

We note that K' is a compact operator, and F' is a finite rank self-adjoint operator.

From Lemma 2.5 we have $uu^*K = Kuu^* = K$. Hence, K' is positive.

Note that since $|T| \in \mathcal{AN}(H)$, $I - u^*u = P_{\ker(u)}$ is finite rank (see the proof of the Theorem 2.3). Then we have

$$
\begin{aligned}
|\Delta_{m_{f_\lambda}}(T)|^2 &= \alpha^2 u^* u + K' + F' \\
&= \alpha^2 I + K' + (F' - \alpha^2(I - u^* u)) \in \mathcal{AN}(H).
\end{aligned}
$$

Hence, $|\Delta_{m_{f_\lambda}}(T)| \in \mathcal{AN}(H)$ by Theorem 1.2, Lemma 2.2. Therefore $\Delta_{m_{f_\lambda}}(T) \in \mathcal{AN}(H)$ by [22, Lemma 6.2]. $\square$

In this section, we showed the stability of $\mathcal{AN}$-property of induced Aluthge transformations with respect to arithmetic and geometric mean cases. The following question arises.

Question 2. *Let f be an operator monotone function with $f(1) = 1$ such that $t^\lambda \leq f \leq (1-\lambda) + \lambda t$ for $\lambda \in [0,1]$. Is there an example $T \in \mathcal{AN}(H)$ such that $\Delta_{m_f}(T) \notin \mathcal{AN}(H)$?*

3. Iteration

In this section, we shall give a partial answer of the following question in the case of mean transformations of semi-hyponormal operators.

Question 3. *Let $T \in \mathcal{AN}(H)$ and f be an operator monotone function on $(0,\infty)$ with $f(1)=1$. Does $\lim_{n\to\infty} \Delta^n_{m_f}(T)$ exist?*

It is known that the iteration of Aluthge and mean transformations of any n–by–n matrix T converge to a normal matrix [3,29]. However, if T is a bounded linear operator, then the iteration of Aluthge transform does not converge, in general [8,29]. In [7], the authors pointed out that if there exists a limit of $u^{*n}|T|u^n$ as $n \to \infty$ in the strong operator topology, then the iteration of mean transform converges to the same limit point without proof.

In this section, we shall show that the iteration of mean transformations of semi-hyponormal operators converges, and show the stability of $\mathcal{AN}$-property for its limit point. At the beginning of this section, we shall introduce basic properties of the polar decomposition.

Lemma 3.1. *Let $T \in \mathcal{B}(H)$ with the polar decomposition $T = u|T|$. Suppose that $uu^*|T| = |T|$. Then*

(i) $|T| \le |T^*| \Leftrightarrow u^*|T|u \le |T|$,
(ii) $|T^*| \le |T| \Leftrightarrow |T| \le u^*|T|u$.

Proof. Proof of (i).

$$\begin{aligned} |T| \le |T^*| &\Longrightarrow uu^*|T|uu^* \le u|T|u^* \\ &\Longrightarrow u^*(uu^*|T|uu^*)u \le u^*(u|T|u^*)u \\ &\Longrightarrow u^*|T|u \le |T|. \end{aligned}$$

Conversely,

$$u^*|T|u \le |T| \Longrightarrow uu^*|T|uu^* \le u|T|u^* = |T^*| \Longrightarrow |T| \le |T^*|.$$

(ii) can be proven by the same way. $\square$

Remark 3.2.

(i) In Lemma 3.1, if we drop the condition $uu^*|T| = |T|$, then only ($\Longrightarrow$) holds. Indeed, let $T = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$ on $H \oplus H$. Then the polar decomposition of T is

$$T = u|T| = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that $uu^*|T| = 0 \neq |T|$ and $u^*|T|u = 0 \le |T|$, but

$$|T| = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \not\leq \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = |T^*|.$$

(ii) It is well-known that $uu^*|T| = |T| \Leftrightarrow \ker(T^*) \subseteq \ker(T)$.

Let $T = u|T|$ be the polar decomposition. Since $u^*u|T| = |T|u^*u = |T|$, a semi-hyponormal operator T satisfies

$$|T^*| = u|T|u^* \leq |T| \leq u^*|T|u \leq {u^*}^2|T|u^2 \leq \cdots \leq {u^*}^n|T|u^n \leq \cdots \leq \|T\|I. \tag{4}$$

Hence there exists $L := \lim_{n\to\infty} {u^*}^n|T|u^n$ in the strong operator topology. The operator L is called the *polar symbol* ([27]).

Theorem 3.3. *Let T be a semi-hyponormal operator with the polar decomposition $T = u|T|$, and let $\hat{T} = \frac{u|T|+|T|u}{2}$ (mean transform). Then*

$$|\hat{T}^*| \leq |T| \leq |\hat{T}|$$

holds. Hence $\hat{T}$ is semi-hyponormal.

Theorem 3.3 with the kernel condition $\ker(T^*) = \ker(T)$ was shown in [7, Theorem 2.17]. Theorem 3.3 does not require any kernel condition.

Proof. Since $\hat{T} = \frac{u|T|+|T|u}{2} = \left(\frac{u|T|u^*+|T|}{2}\right)u$, (4) and Hansen's inequality [13], we have

$$\begin{aligned}
|\hat{T}| &= \left[u^*\left(\frac{u|T|u^* + |T|}{2}\right)^2 u\right]^{\frac{1}{2}} \\
&\geq u^*\left(\frac{u|T|u^* + |T|}{2}\right)u \quad \text{(by Hansen's inequality [13])} \\
&= \frac{|T| + u^*|T|u}{2} \\
&\geq |T| \quad (\text{by } u^*|T|u \geq |T|) \\
&\geq \frac{u|T|u^* + |T|}{2} \quad (\text{by } |T| \geq |T^*| = u|T|u^*) \\
&\geq \left[\left(\frac{u|T|u^* + |T|}{2}\right)uu^*\left(\frac{u|T|u^* + |T|}{2}\right)\right]^{\frac{1}{2}} = |\hat{T}^*|.
\end{aligned}$$

The proof is completed. □

For a non-negative integer n, let $\hat{T}^{(n)} := \widehat{\hat{T}^{(n-1)}}$ and $\hat{T}^{(0)} := T$. By Theorem 3.3 and the fact $\|\hat{T}\| \leq \|T\|$, we obtain that if T is semi-hyponormal, then $\hat{T}^{(n)}$ is also semi-hyponormal for all $n = 1, 2, \ldots$, and

$$|T| \le |\hat{T}| \le |\hat{T}^{(2)}| \le \cdots |\hat{T}^{(n)}| \le \cdots \le \|T\| I.$$

Hence, there exists $s - \lim_{n\to\infty} |\hat{T}^{(n)}|$. We notice that if $\ker(T) = \ker(T^*)$, for $f_{1/2}(t) = \frac{1+t}{2}$, the induced Aluthge transformation $\Delta_{m_{f_{1/2}}}(T)$ can be defined, and $\hat{T} = \Delta_{m_{f_{1/2}}}(T)$ holds [29]. So we shall use the symbol $\Delta_{m_{f_{1/2}}}(T)$ instead of $\hat{T}$ for future discussions. The next result gives us a concrete form of the limit point of $\{\Delta^n_{m_{f_{1/2}}}(T)\}$ for a semi-hyponormal operator T.

Theorem 3.4. *Let $f_{1/2}(t) = \frac{1+t}{2}$ and $T \in \mathcal{B}(H)$ be a semi-hyponormal operator with the polar decomposition $T = u|T|$. If $\ker(T^*) = \ker(T)$, then*

$$s - \lim_{n\to\infty} \Delta^n_{m_f}(T) = uL$$

in the strong topology, where

$$L = s - \lim_{n\to\infty} u^{*n}|T|u^n.$$

Moreover, uL is a normal operator and $\sigma(T) = \sigma(uL)$.

Proof. The polar decomposition of $\Delta^n_{m_{f_{1/2}}}(T)$ is shown in [7, Theorem 2.15]. Exactly, the polar decomposition of $\Delta^n_{m_{f_{1/2}}}(T)$ is

$$\Delta^n_{m_{f_{1/2}}}(T) = u\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} u^{*k}|T|u^k\right].$$

By (4), we have

$$|\Delta^n_{m_{f_{1/2}}}(T)| = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} u^{*k}|T|u^k \le u^{*n}|T|u^n.$$

Here, we shall prove that $|\Delta^n_{m_f}(T)|$ converges to L as $n \to \infty$. For a natural number n, define $S_n := u^{*n}|T|u^n$ and $S_0 := |T|$. For any positive real number $\varepsilon > 0$, there exists a natural number n_0 such that

$$\|(S_n - L)x\| < \varepsilon \quad \text{and} \quad \frac{n_0 n^{n_0-1}}{2^n} < \varepsilon$$

hold for all unit vectors x and all natural numbers $n \ge n_0$. Let

$$M = \max\{\|S_0 - L\|, ..., \|S_{n_0-1} - L\|\}.$$

Then for any unit vector $x \in H$ and $n \ge 2n_0$, we have

$$
\begin{aligned}
\|(|\Delta^n_{m_{f_{1/2}}}(T)| - L)x\| &\leq \frac{1}{2^n}\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\|(S_k - L)x\| \\
&< \frac{1}{2^n}\sum_{k=0}^{n_0-1}\begin{pmatrix} n \\ k \end{pmatrix} M + \frac{1}{2^n}\sum_{k=n_0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\varepsilon \\
&\leq \frac{n_0}{2^n}\begin{pmatrix} n \\ n_0 - 1 \end{pmatrix} M + \frac{1}{2^n}\sum_{k=n_0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\varepsilon \\
&= \frac{n_0}{2^n}\frac{n(n-1)\cdots(n-n_0+2)}{(n_0-1)!}M + \varepsilon \\
&\leq \frac{n_0 n^{n_0-1}}{2^n}M + \varepsilon < \varepsilon(M+1).
\end{aligned}
$$

Hence $|\Delta_{m_{f_{1/2}}}(T)|$ converges to L in the strong operator topology. The normality of uL is shown as follows. Since $\ker(T) = \ker(T^*)$, we have $u^*u = uu^*$. For any natural number n, we have

$$
\begin{aligned}
u\left(u^{n*}|T|u^n\right) &= u^*uu^{n-1^*}|T|u^{n-1}u = \left(u^{n-1^*}|T|u^{n-1}\right)u, \text{ and} \\
u\left(u^{n*}|T|u^n\right)u^* &= u^*uu^{n-1^*}|T|u^{n-1}u^*u = u^{n-1^*}|T|u^{n-1}.
\end{aligned}
$$

Hence $uL = Lu$ and $uLu^* = L$ hold, i.e., uL is quasinormal. Moreover, since uL is the polar decomposition,

$$|uL| = L = uLu^* = |(uL)^*|,$$

and uL is normal. Lastly,

$$\sigma(uu^{*n}|T|u^n) \setminus \{0\} = \sigma(u|T|) \setminus \{0\} = \sigma(T) \setminus \{0\}$$

holds. By (4), we have $\ker(T) = \ker(u^{*n}|T|u^n) = \ker(uu^{*n}|T|u^n)$. Then $\sigma(uu^{*n}|T|u^n) = \sigma(T)$, and hence $\sigma(T) = \sigma(uL)$. □

If $T \in \mathcal{AN}(H)$, then $\Delta_{m_{f_{1/2}}}(T) \in \mathcal{AN}(H)$ by Theorem 2.4. Hence if $\ker(T) = \ker(T^*)$, then, we have $s - \lim_{n\to\infty}\Delta^n_{m_{f_{1/2}}}(T) \in \overline{\mathcal{AN}(H)}$. The following theorem shows a more exact consequence.

Theorem 3.5. *If $T \in \mathcal{AN}(H)$ is a semi-hyponormal operator such that* $\ker(T) = \ker(T^*)$, *then* $\lim_{n\to\infty}\Delta^n_{m_{f_{1/2}}}(T) \in \mathcal{AN}(H)$.

Proof. By Theorem 3.4, $|\Delta^n_{m_f}(T)| \to L = \lim_{n\to\infty} u^{*n}|T|u^n$ in the strong operator topology, where u is the polar part of the polar decomposition of T.

Since $T \in \mathcal{AN}(H)$, $|T| = \alpha I + K - F$ for some positive number α, $K \in \mathcal{K}(H)_+$, and $F \in \mathcal{F}(H)_+$ such that $0 \le K \le \alpha I$ and $KF = 0$ by Theorem 1.3.

If $\alpha = 0$, then $|T| \in \mathcal{K}(H)$, and hence $\Delta_{m_{f_{1/2}}}(T) \in \mathcal{K}(H)$. Therefore $s-\lim_{n\to\infty} \Delta_{m_{f_{1/2}}}(T) \in \mathcal{K}(H) \subset \mathcal{AN}(H)$. So we have to discuss only $\alpha \neq 0$ case. In this case $\dim \ker(T) < +\infty$ holds, and $I - u^*u$ is a finite rank operator. Therefore

$$u^*|T|u = \alpha u^*u + u^*Ku - u^*Fu = \alpha I + u^*Ku - \{\alpha(I - u^*u) + u^*Fu\} \in \mathcal{AN}(H)$$

by Theorem 1.2. If $\ker(T) = \ker(T^*)$, then $u^*u = P_{\ker(T)^\perp} = P_{\ker(T^*)^\perp} = uu^*$. Hence $u^{2^*}u^2 = u^*uu^*u = u^*u$ holds, and

$$\begin{aligned} u^{*2}|T|u^2 &= \alpha u^*u + u^{*2}Ku^2 - u^*\{\alpha(I - u^*u) + u^*Fu\}u \\ &= \alpha I + u^{*2}Ku^2 - \{\alpha(I - u^{2^*}u^2) + u^{*2}Fu^2\} \\ &= \alpha I + u^{*2}Ku^2 - \{\alpha(I - u^*u) + u^{*2}Fu^2\}. \end{aligned}$$

Therefore $u^{*2}|T|u^2 \in \mathcal{AN}(H)$ by Theorem 1.2, again. By the same calculation, we can obtain that for each natural number n,

$$u^{*n}|T|u^n = \alpha I + u^{*n}Ku^n - \{\alpha(I - u^*u) + u^{*n}Fu^n\}, \tag{5}$$

and $u^{*n}Ku^n \in \mathcal{K}(H)_+$, $\alpha(I - u^*u) + u^{*n}Fu^n \in \mathcal{F}(H)_+$ such that

$$\begin{aligned} \dim R(\alpha(I - u^*u) + u^{*n}Fu^n) &\le \dim R(\alpha(I - u^*u)) + \dim R(u^{*n}Fu^n) \\ &\le \dim R(I - u^*u) + \dim R(F) < +\infty. \end{aligned}$$

Hence $u^{*n}|T|u^n \in \mathcal{AN}(H)$ for all natural number n by Theorem 1.2.

Next, we shall prove existence of limits of the sequences $\{u^{*n}Ku^n\}$ and $\{\alpha(I - u^*u) + u^{*n}Fu^n\}$ as $n \to \infty$. By (5), we have

$$u^{*n}|T|u^n = \alpha u^*u + u^{*n}(K - F)u^n.$$

Then there exists $D := L - \alpha u^*u = s - \lim_{n\to\infty} u^{*n}(K - F)u^n$. Let $D_n := u^{*n}(K - F)u^n$. By $KF = 0$, Lemma 2.5 and $\ker(T) = \ker(T^*)$, we have

$$\begin{aligned} |D_n|^2 &= u^{*n}(K - F)u^n u^{*n}(K - F)u^n \\ &= u^{*n}(K - F)u^*u(K - F)u^n \\ &= u^{*n}(K^2 + Fu^*uF)u^n. \end{aligned}$$

Moreover, by using $KF = 0$, again, we have $u^{*n}(K + F)u^n \ge 0$ and

$$\begin{aligned}\{u^{*n}(K+F)u^n\}^2 &= u^{*n}(K+F)u^n u^{*n}(K+F)u^n \\ &= u^{*n}(K+F)u^*u(K+F)u^n \\ &= u^{*n}(K^2+Fu^*uF)u^n = |D_n|^2.\end{aligned}$$

Hence $|D_n| = u^{*n}(K+F)u^n$. Let

$$K_n := u^{*n}Ku^n = \frac{1}{2}(|D_n| + D_n) \in \mathcal{K}(H)_+$$

and

$$F_n := \alpha(I-u^*u) + u^{*n}Fu^n = \alpha(I-u^*u) + \frac{1}{2}(|D_n| - D_n) \in \mathcal{F}(H)_+.$$

Moreover, there exists $K_\infty := s - \lim_{n\to\infty} K_n \in \mathcal{K}(H)_+$ and $F_\infty := s - \lim_{n\to\infty} F_n \in \mathcal{K}(H)_+$. We notice that for each natural number n,

$$\dim R(F_n) \le \dim R(I-u^*u) + \dim R(F) < +\infty$$

and hence $F_\infty \in \mathcal{F}(H)_+$. Therefore by Theorem 1.2,

$$\lim_{n\to\infty} |\Delta^n_{m_f}(T)| = \alpha I + K_\infty - F_\infty \in \mathcal{AN}(H), \tag{6}$$

and thus $\lim_{n\to\infty} \Delta^n_{m_f}(T) \in \mathcal{AN}(H)$ by [22, Lemma 6.2].

We notice that $K_\infty F_\infty = 0$ and $0 \le F_\infty \le \alpha I$ as follows:

By Lemma 2.5,

$$\begin{aligned}K_nF_n &= u^{*n}Ku^n\{\alpha(I-u^*u) + u^{*n}Fu^n\} \\ &= \alpha u^{*n}Ku^n(I-u^*u) + u^{*n}Ku^nu^{*n}Fu^n \\ &= u^{*n}Ku^*uFu^n \\ &= u^{*n}KFu^n = 0.\end{aligned}$$

Moreover, since $0 \le F \le \alpha I$, we have

$$\begin{aligned}0 &\le \alpha(I-u^*u) + u^{*n}Fu^n \\ &\le \alpha(I-u^*u) + \alpha u^{*n}u^n = \alpha I.\end{aligned}$$

Therefore, (6) is uniquely determined by Theorem 1.3. $\square$

4. Inequalities

In the last section, we shall show some related operator inequalities.

Theorem 4.1. *Let $T \in \mathcal{B}(H)$ with the polar decomposition $T = u|T|$. Then*

$$\Delta(T)^*\Delta(T) \leq T^*T \Leftrightarrow u^*|T|u \leq |T|.$$

Proof. Note that

$$\Delta(T)^*\Delta(T) \leq T^*T \Leftrightarrow |T|^{\frac{1}{2}}u^*|T|u|T|^{\frac{1}{2}} \leq |T|^2.$$

Hence, we only have to show the implication ($\Rightarrow$). For any $y \in H$ and $z \in \ker(T) = (|T|H)^{\perp}$, we have

$$\begin{aligned}\langle u^*|T|u(|T|y+z), |T|y+z\rangle &= \langle u^*|T|u(|T|y), |T|y\rangle \\ &= \langle |T|^{\frac{1}{2}}u^*|T|u|T|^{\frac{1}{2}}(|T|^{\frac{1}{2}}y), |T|^{\frac{1}{2}}y\rangle \\ &\leq \langle |T|^2(|T|^{\frac{1}{2}}y), |T|^{\frac{1}{2}}y\rangle \\ &= \langle |T|(|T|y), |T|y\rangle \\ &= \langle |T|(|T|y+z), |T|y+z\rangle,\end{aligned}$$

we have $u^*|T|u \leq |T|$. □

Corollary 4.2. *Let $T \in \mathcal{B}(H)$ with the polar decomposition $T = u|T|$. Suppose that $uu^*|T| = |T|$. If $|T| \leq |T^*|$, then $\Delta(T)^*\Delta(T) \leq T^*T$.*

Proof. By Lemma 3.1 and Theorem 4.1, we can prove it. □

The following theorem is similar inequality for the induced Aluthge transformation with respect to the power mean.

Theorem 4.3. *For a natural number n, let $f(t) = (\frac{1+t^{1/n}}{2})$, and $T = u|T| \in \mathcal{B}(H)$ be the polar decomposition. If $|T|^2 \leq |T^*|^2$. Then*

$$\Delta_{m_f}(T)^*\Delta_{m_f}(T) \leq T^*T.$$

To prove Theorem 4.3, we shall prepare a lemma.

Lemma 4.4. *Let $A, B \in \mathcal{B}(H)_+$. If $B \leq A$, then*

$$B^\lambda A^\mu + A^\mu B^\lambda \leq 2A^{\lambda+\mu}$$

holds for $0 \leq \mu \leq \lambda \leq 1 - \mu$.

Proof. By the conditions of λ and μ, we have $0 \leq \lambda - \mu \leq \lambda + \mu \leq 1$. For any $x \in H$, we have

$$\begin{aligned}
\langle (B^\lambda A^\mu + A^\mu B^\lambda)x, x\rangle &\leq \left|\langle (B^\lambda A^\mu + A^\mu B^\lambda)x, x\rangle\right| \\
&\leq \left|\langle B^\lambda A^\mu x, x\rangle\right| + \left|\langle A^\mu B^\lambda x, x\rangle\right| \\
&= \left|\langle B^{\frac{\lambda-\mu}{2}} A^\mu x, B^{\frac{\lambda+\mu}{2}} x\rangle\right| + \left|\langle B^{\frac{\lambda+\mu}{2}} x, B^{\frac{\lambda-\mu}{2}} A^\mu x\rangle\right| \\
&\leq 2\left\|B^{\frac{\lambda-\mu}{2}} A^\mu x\right\| \left\|B^{\frac{\lambda+\mu}{2}} x\right\| \text{ (by Cauchy-Schwarz inequality)} \\
&= 2\langle A^\mu B^{\lambda-\mu} A^\mu x, x\rangle^{\frac{1}{2}} \langle B^{\lambda+\mu} x, x\rangle^{\frac{1}{2}} \\
&\leq 2\langle A^{\lambda+\mu} x, x\rangle^{\frac{1}{2}} \langle A^{\lambda+\mu} x, x\rangle^{\frac{1}{2}} \text{ (by Loewner-Heinz inequality [20])} \\
&= 2\langle A^{\lambda+\mu} x, x\rangle.
\end{aligned}$$

Hence we get the desired inequality. □

The converse assertion in Lemma 4.4 does not hold as in the following example.

Example 4.5. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then for any $\lambda, \mu > 0$, $0 = B^\lambda A^\mu + A^\mu B^\lambda \leq 2A^{\lambda+\mu}$. However $A - B \not\geq 0$.

Proof of Theorem 4.3. First of all, $u^*|T|^\alpha u \leq |T|^\alpha$ holds for all $\alpha \in [0, 1]$, since $|T|^\alpha \leq |T^*|^\alpha = u|T|^\alpha u^*$ by Loewner-Heinz inequality [20].

Recall that by (2), we have

$$\Delta_{m_f}(T) = \frac{1}{2^n}\left(\sum_{i=0}^{n} \binom{n}{i} |T|^{\frac{n-i}{n}} u|T|^{\frac{i}{n}}\right).$$

Hence

$$\begin{aligned}
&\Delta_{m_f}(T)^*\Delta_{m_f}(T) \\
&= \frac{1}{2^{2n}} \sum_{i=0}^{n}\sum_{j=0}^{n} \binom{n}{i}\binom{n}{j} |T|^{\frac{i}{n}} u^* |T|^{\frac{2n-(i+j)}{n}} u|T|^{\frac{j}{n}} \\
&= \frac{1}{2^{2n}} \Bigg[\sum_{0\leq i<j\leq n} \binom{n}{i}\binom{n}{j} (|T|^{\frac{i}{n}} u^*|T|^{\frac{2n-(i+j)}{n}} u|T|^{\frac{j}{n}} + |T|^{\frac{j}{n}} u^*|T|^{\frac{2n-(i+j)}{n}} u|T|^{\frac{i}{n}}) \\
&\qquad\qquad + \sum_{i=0}^{n} \binom{n}{j}^2 |T|^{\frac{i}{n}} u^*|T|^{\frac{2n-2i}{n}} u|T|^{\frac{i}{n}}\Bigg] \\
&= \frac{1}{2^{2n}} \Bigg[\sum_{0\leq i<j\leq n} \binom{n}{i}\binom{n}{j} |T|^{\frac{i}{n}} (u^*|T|^{\frac{2n-(i+j)}{n}} u|T|^{\frac{j-i}{n}} + |T|^{\frac{j-i}{n}} u^*|T|^{\frac{2n-(i+j)}{n}} u)|T|^{\frac{i}{n}} \\
&\qquad\qquad + \sum_{i=0}^{n} \binom{n}{j}^2 |T|^{\frac{i}{n}} u^*|T|^{\frac{2n-2i}{n}} u|T|^{\frac{i}{n}}\Bigg]
\end{aligned}$$

$$\leq \frac{1}{2^{2n}}\left[\sum_{0\leq i<j\leq n}\binom{n}{i}\binom{n}{j}|T|^{\frac{i}{n}}(2|T|^{\frac{2n-2i}{n}})|T|^{\frac{i}{n}}+\sum_{i=0}^{n}\binom{n}{j}^2|T|^2\right]$$

$$\text{(by } 0\leq \tfrac{j-i}{2n}\leq \tfrac{2n-(i+j)}{2n}\leq 1-\tfrac{j-i}{2n}\text{, Lemma 4.4 and } u^*|T|^{\frac{2n-2i}{n}}u\leq |T|^{\frac{2n-2i}{n}}\text{)}$$

$$=\frac{1}{2^{2n}}\sum_{i,j=0}^{n}\binom{n}{i}\binom{n}{j}|T|^2=T^*T. \quad \square$$

5. Questions

In this section, we list some questions for future discussions.

Let f be an operator monotone function such that $f(1)=1$ and $T\in\mathcal{B}(H)$.

Question 4. *If $|T^*|^2\leq|T|^2$, then does $|T|^2\leq\Delta_{m_f}(T)^*\Delta_{m_f}(T)$ hold for any induced Aluthge transformation?*

Question 5. *If T is hyponormal, then is $\Delta_{m_f}(T)$ a hyponormal operator for any induced Aluthge transformation?*

The above questions are true for the Aluthge transformation [1]. We know that if T is quasi-normal, then $\Delta_{m_f}(T)=T$ [29].

Question 6. *Is it true that $\Delta_{m_f}(T)=T$ implies quasi-normality of T?*

Question 7. *Let f be an operator monotone function such that $f(1)=1$. Then, $\Delta_{m_f}(\mathcal{AN}(H))\subseteq\mathcal{AN}(H)$?*

Question 8. *If $T\in\mathcal{AN}(H)$. Then, does $s-\lim\Delta^n_{m_f}(T)$ exist? Moreover, does the limit point belong to $\mathcal{AN}(H)$?*

Question 9. *Determine a concrete form of $L=s-\lim_{n\to\infty}u^{*n}|T|u^n$ without using "lim" if T is semi-hyponormal?*

Question 10. *Let $T\in B(H)$ with the polar decomposition $T=u|T|$ and f an operator monotone function such that $f(1)=1$. Suppose that $uu^*|T|=|T|$. Is it true that*

$$\Delta_{m_f}(T)^*\Delta_{m_f}(T)\leq\Delta_{m_f}(T^*T)=T^*T?$$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

The first author's research is supported by SERB grant No. MTR/2019/001307, Govt. of India. The second author's research is supported by KAKENHI Grant No. JP20K03644.

References

[1] A. Aluthge, On p-hyponormal operators for $0 < p < 1$, Integral Equ. Oper. Theory 13 (1990) 307–315.
[2] T. Ando, T. Yamazaki, The iterated Aluthge transforms of a 2–by–2 matrix converge, Linear Algebra Appl. 375 (2003) 299–309.
[3] J. Antezana, E.R. Pujals, D. Stojanoff, The iterated Aluthge transforms of a matrix converge, Adv. Math. 226 (2011) 1591–1620.
[4] N. Bala, G. Ramesh, A representation of hyponormal absolutely norm attaining operators, Bull. Sci. Math. 171 (2021) 103020.
[5] N. Bala, G. Ramesh, Hyperinvariant subspace for absolutely norm attaining and absolutely minimum attaining operators, Preprint, https://arxiv.org/abs/2002.09167, 2022.
[6] X. Carvajal, W. Neves, Operators that achieve the norm, Integral Equ. Oper. Theory 72 (2012) 179–195.
[7] F. Chabbabi, R.E. Curto, M. Mbekhta, The mean transform and the mean limit of an operator, Proc. Am. Math. Soc. 147 (2019) 1119–1133.
[8] M. Chō, I.B. Jung, W.Y. Lee, On Aluthge transforms of p-hyponormal operators, Integral Equ. Oper. Theory 53 (2005) 321–329.
[9] K. Dykema, H. Schultz, Brown measure and iterates of the Aluthge transform for some operators arising from measurable actions, Trans. Am. Math. Soc. 361 (2009) 6583–6593.
[10] P. Enflo, J. Kover, L. Smithies, Denseness for norm attaining operator–valued functions, Linear Algebra Appl. 338 (2001) 139–144.
[11] T. Furuta, Invitation to Linear Operators. From Matrices to Bounded Linear Operators on a Hilbert Space, Taylor & Francis Group, London, 2001, x+255 pp.
[12] R. Golla, H. Osaka, Linear maps preserving $\mathcal{AN}$-operators, Bull. Korean Math. Soc. 57 (2020) 831–838.
[13] F. Hansen, An operator inequality, Math. Ann. 246 (1980) 249–250.
[14] H. Huang, T.-Y. Tam, On the convergence of Aluthge sequence, Oper. Matrices 1 (2007) 121–141.
[15] H. Huang, T.-Y. Tam, Aluthge iteration in semisimple Lie group, Linear Algebra Appl. 432 (2010) 3250–3257.
[16] T. Huruya, A note on p-hyponormal operators, Proc. Am. Math. Soc. 125 (1997) 3617–3624.
[17] I.B. Jung, E. Ko, C. Pearcy, Aluthge transforms of operators, Integral Equ. Oper. Theory 37 (2000) 437–448.
[18] F. Kubo, T. Ando, Means of positive linear operators, Math. Ann. 246 (1979/80) 205–224.
[19] S.H. Lee, W.Y. Lee, J. Yoon, The mean transform of bounded linear operators, J. Math. Anal. Appl. 410 (2014) 70–81.
[20] K. Löwner, Über monotone Matrixfunktionen, Math. Z. 38 (1934) 177–216.
[21] D.V. Naidu, G. Ramesh, On absolutely norm attaining operators, Proc. Indian Acad. Sci. Math. Sci. 129 (2019) 54.
[22] S.K. Pandey, V.I. Paulsen, A spectral characterization of $\mathcal{AN}$ operators, J. Aust. Math. Soc. 102 (2017) 369–391.
[23] G. Ramesh, Structure theorem for AN-operators, J. Aust. Math. Soc. 96 (2014) 386–395.
[24] G. Ramesh, Absolutely norm attaining paranormal operators, J. Math. Anal. Appl. 465 (2018) 547–556.

[25] G. Ramesh, S.S. Sequeira, On the closure of absolutely norm attaining operators, Linear Multilinear Algebra (2023), in press, https://doi.org/10.1080/03081087.2022.2126426.
[26] G. Ramesh, H. Osaka, Y. Udagawa, T. Yamazaki, Functional calculus for $\mathcal{AN}$-operators, Anal. Math. (2023), in press.
[27] D. Xia, Spectral Theory of Hyponormal Operators, Operator Theory: Advances and Applications, vol. 10, Birkhäuser Verlag, Basel, 1983, xiv+241 pp.
[28] T. Yamazaki, An expression of spectral radius via Aluthge transformation, Proc. Am. Math. Soc. 130 (2002) 1131–1137.
[29] T. Yamazaki, The induced Aluthge transformations, Linear Algebra Appl. 628 (2021) 1–28.