Set-theoretic principles which imply that the continuum is fairly large

Sakaé Fuchino (渕野 昌)

fuchino@diamond.kobe-u.ac.jp

2019 年 11 月 10 日 (於 3. PPICTA, 成都)

2020 年 06 月 30 日 (00:46 JST) 版

These slides are (going to be) downloadable as:

http://fuchino.ddo.jp/slides/chengdu2019-11-slides-pf.pdf

The results connected to the following slides ...

are to be found in the joint papers with André Ottenbereit Maschio Rodriques and Hiroshi Sakai:

[1] Sakaé Fuchino, André Ottenbereit Maschio Rodriques and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, I, submitted.
http://fuchino.ddo.jp/papers/SDLS-x.pdf

[2] Sakaé Fuchino, André Ottenbereit Maschio Rodriques and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, II — reflection down to the continuum, pre-preprint. http://fuchino.ddo.jp/papers/SDLS-II-x.pdf

[3],[4] Sakaé Fuchino, André Ottenbereit Maschio Rodriques and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, III, IV, in preparation.

Are there reasonable axioms ...

which imply that the size of the continuum is
very large ?

- "reasonable" possibly in line with Gödel's program: see e.g.

  [4] Joan Bagaria, Natural axioms of set theory and the continuum problem, In: Proceedings of the 12-th International Congress of Logic, Methodology, and Philosophy of Science, King's College London (2005), 43-64.

A diagonal reflection principle ...

on the stationarity of subsets of $\mathcal{P}_\kappa(\lambda)$

▷ The Diagonal Reflection Principles on the stationarity of $[\lambda]^{\aleph_0}$ were introduced by Sean Cox. We consider the following $\mathcal{P}_\kappa(\lambda)$ version of the principle:

- For sets X, Y with $X \subseteq Y$, we denote

$$\mathcal{P}_X(Y) = \mathcal{P}_{|X|}(Y) = [Y]^{<|X|} \; .$$

- Let κ be a regular cardinal and $\lambda \geq \kappa$.

$(*)^{int+PKL}_{<\kappa,\lambda}$: *For any countable expansion $\mathfrak{A}$ of the structure $\langle \mathcal{H}(\lambda), \kappa, \in \rangle$ and any family $\langle S_a : a \in \mathcal{H}(\lambda) \rangle$ s.t. S_a is a stationary subset of $\mathcal{P}_\kappa(\mathcal{H}(\lambda))$ for all $a \in \mathcal{H}(\lambda)$, there are stationarily many $M \in \mathcal{P}_\kappa(\mathcal{H}(\lambda))$ s.t. $|\kappa \cap M|$ is regular, $\mathfrak{A} \restriction M \prec \mathfrak{A}$ and $S_a \cap \mathcal{P}_{\kappa\cap M}(M) \cap M$ is stationary in $\mathcal{P}_{\kappa\cap M}(M)$ for all $a \in M$.*

A diagonal reflection principle ...

on the stationarity of subsets of $\mathcal{P}_\kappa(\lambda)$

- ▷ The Diagonal Reflection Principles on the stationarity of $[\lambda]^{\aleph_0}$ were introduced by Sean Cox. We consider the following $\mathcal{P}_\kappa(\lambda)$ version of the principle:

- For sets X, Y with $X \subseteq Y$, we denote

$$\mathcal{P}_X(Y) = \mathcal{P}_{|X|}(Y) = [Y]^{<|X|}\ .$$

- Let κ be a regular cardinal and $\lambda \geq \kappa$.

$(*)^{int+PKL}_{<\kappa,\lambda}$: *For any countable expansion $\mathfrak{A}$ of the structure $\langle \mathcal{H}(\lambda), \kappa, \in \rangle$ and any family $\langle S_a : a \in \mathcal{H}(\lambda) \rangle$ s.t. S_a is a stationary subset of $\mathcal{P}_\kappa(\mathcal{H}(\lambda))$ for all $a \in \mathcal{H}(\lambda)$, there are stationarily many $M \in \mathcal{P}_\kappa(\mathcal{H}(\lambda))$ s.t. $|\kappa \cap M|$ is regular, $\mathfrak{A} \restriction M \prec \mathfrak{A}$ and $S_a \cap \mathcal{P}_{\kappa\cap M}(M) \cap M$ is stationary in $\mathcal{P}_{\kappa\cap M}(M)$ for all $a \in M$.*

$(*)^{int+PKL}_{<\kappa,\lambda}$ implies that κ is large

Proposition 1. *For a regular cardinal κ, $(*)^{int+PKL}_{<\kappa,\lambda}$ implies that κ is weakly Mahlo, weakly hyper-Mahlo, etc.*

- Though the original definition of $(*)^{int+PKL}_{<\kappa,\lambda}$ looks rather technical an arbitrary, it can be characterized in terms of a strong downward Löwenheim-Skolem theorem for a generalized logic.

$(*)^{int+PKL}_{<\kappa,\lambda}$ as a downward Löwenheim-Skolem Theorem

- A structure $\mathfrak{A}$ with a designated unary predicate $\underset{\sqcup}{K}$ is a *PKL*-structure if $|\underset{\sqcup}{K}^{\mathfrak{A}}|$ is regular uncountable.
- A logic $\mathcal{L}^{PKL}_{stat}$ is a monadic second-order logic with built-in unary predicate $\underset{\sqcup}{K}(\cdot)$ and with the unique second-order quantifier *stat*. The second-order variables run over the elements of $\mathcal{P}_{\underset{\sqcup}{K}^{\mathfrak{A}}}(A)$ for a *PKL*-structure $\mathfrak{A} = \langle A, ...\rangle$.
- the internal interpretation of a $\mathcal{L}^{PKL}_{stat}$-formula $\varphi = \varphi(x_0, ..., X_0, ...)$ in a *PKL*-structure $\mathfrak{A}$ with $a_0, ... \in A$ and $U_0, ... \in \mathcal{P}_{\underset{\sqcup}{K}^{\mathfrak{A}}}(A)$ is defined similarly to the usual second order logic with the crucial step in the recursive definition:

$$\mathfrak{A} \models^{int} stat\, X\, \varphi(a_0, ..., U_0, ..., X) \;\Leftrightarrow$$
$$\{U \in \mathcal{P}_{\underset{\sqcup}{K}^{\mathfrak{A}}}(A) \cap A \,:\, \mathfrak{A} \models^{int} \varphi(a_0, ..., U_0, ..., U)\} \text{ is stationary in } \mathcal{P}_{\underset{\sqcup}{K}^{\mathfrak{A}}}(A)$$

$(*)^{int+PKL}_{<\kappa,\lambda}$ as a downward Löwenheim-Skolem Theorem (2/2)

- For *PKL*-structures $\mathfrak{A}$, $\mathfrak{B}$ with $\mathfrak{B} \subseteq \mathfrak{A}$, $\mathfrak{B} = \langle B, ...\rangle$,

  $\mathfrak{B} \prec^{int}_{\mathcal{L}^{PKL}_{stat}} \mathfrak{A} \;\Leftrightarrow \mathfrak{B} \models^{int} \varphi(b_0, ..., U_0, ...)$ *if and only if* $\mathfrak{A} \models^{int} \varphi(b_0, ..., U_0, ...)$ *for all* $\mathcal{L}^{PKL}_{stat}$*-formulas* φ *in the language of the structures with* $\varphi = \varphi(x_0, ..., X_0, ...)$, $b_0, ... \in B$ *and* $U_0, ... \in \mathcal{P}_{\underline{K}^{\mathfrak{B}}}(B) \cap B$.

- The following Strong Downward Löwenheim-Skolem Theorem characterizes the principle $(*)^{int+PKL}_{<\kappa,\lambda}$:

$\mathrm{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, <\kappa)$: *For any PKL-structure* $\mathfrak{A} = \langle A, \underline{K}^{\mathfrak{A}}, ...\rangle$ *of countable signature with* $|A| \geq \kappa$ *and* $|\underline{K}^{\mathfrak{A}}| = \kappa$, *there are stationarily many* $M \in [A]^{<\kappa}$ *s.t.* $\mathfrak{A} \restriction M$ *is a PKL-structure and* $\mathfrak{A} \restriction M \prec^{int}_{\mathcal{L}^{PKL}_{stat}} \mathfrak{A}$.

Theorem 2. *For a regular cardinal* $\kappa > \aleph_1$, *t.f.a.e.*
(a) $(*)^{int+PKL}_{<\kappa,\lambda}$ *for all regular* $\lambda \geq \kappa$. *(b)* $\mathrm{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, <\kappa)$.

Generic supercompactness and $\mathsf{SDLS}_+^{int}(\mathcal{L}_{stat}^{PKL}, <\kappa)$

▷ For a class $\mathcal{P}$ of forcing p.o.s, a cardinal κ is said to be generically supercompact by $\mathcal{P}$, if for any regular $\lambda \geq \kappa$, there is $\mathbb{P} \in \mathcal{P}$ s.t., for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are transitive $M \subseteq \mathsf{V}[\mathbb{G}]$ and $j \subseteq \mathsf{V}[\mathbb{G}]$ s.t. $j : \mathsf{V} \overset{\prec}{\to} M$, $crit(j) = \kappa$, $j(\kappa) > \lambda$, $j''\lambda \in M$.

Proposition 3. *Suppose that κ is a supercompact cardinal and $\mathbb{P}$ is a ccc p.o. which can be seen as a finite support iteration of length κ of ccc p.o.s of size $< \kappa$. Then*

$$\Vdash_{\mathbb{P}} \text{“}\,\kappa \text{ is ccc-generically supercompact”}.$$

Theorem 4. *Suppose that κ is generically supercompact by μ-cc p.o.s for some $\mu < \kappa$. Then $(*)^{int+PKL}_{<\kappa,\lambda}$ holds for all regular $\lambda \geq \kappa$. (By Theorem 2, this is equivalent to $\mathsf{SDLS}_+^{int}(\mathcal{L}_{stat}^{PKL}, <\kappa)$).*

Consistency of $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, < 2^{\aleph_0})$

Proposition 3. *Suppose that κ is a supercompact cardinal and $\mathbb{P}$ is a ccc p.o. which can be seen as a finite support iteration of length κ of ccc p.o.s of size $< \kappa$. Then*

$$\Vdash_{\mathbb{P}} \text{“}\kappa \text{ is ccc-generically supercompact”}.$$

Theorem 4. *Suppose that κ is generically supercompact by μ-cc p.o.s for some $\mu < \kappa$. Then $(*)^{int+PKL}_{<\kappa,\lambda}$ holds for all regular $\lambda \geq \kappa$. (By Theorem 2, this is equivalent to $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, < \kappa)$).*

Corollary 5. $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, < 2^{\aleph_0})$ *is consistent, assuming the consistency of “ZFC + existence of a supercompact cardinal”.*

Laver-generically supercompact cardinal

- For a class $\mathcal{P}$ of p.o.s, a cardinal κ is a Laver-generically supercompact for $\mathcal{P}$ if, for all regular $\lambda \geq \kappa$ and $\mathbb{P} \in \mathcal{P}$ there is $\mathbb{Q} \in \mathcal{P}$ with $\mathbb{P} \lessdot \mathbb{Q}$, s.t., for any $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are a inner model $M \subseteq \mathsf{V}[\mathbb{H}]$, and an elementary embedding $j : \mathsf{V} \to M$ s.t. $crit(j) = \kappa$, $j(\kappa) > \lambda$, $\mathbb{P}, \mathbb{H} \in M$, $j''\lambda \in M$.

Trichotomy Theorem 6. *Suppose that κ is Laver-generically supercompact cardinal for a class $\mathcal{P}$ of p.o.s.*

(A) *If elements of $\mathcal{P}$ are ω_1-preserving and do not add any reals, and $\mathrm{Col}(\omega_1, \{\omega_2\}) \in \mathcal{P}$, then $\kappa = \aleph_2$ and* CH *holds.*

(B) *If elements of $\mathcal{P}$ are ω_1-preserving and contain all proper p.o.s, then $\mathsf{PFA}^{+\omega_1}$ holds and $\kappa = 2^{\aleph_0} = \aleph_2$.*

(C) *If elements of $\mathcal{P}$ are μ-cc for some $\mu < \kappa$ and $\mathbb{P}$ contains a p.o. which adds a reals then κ is fairly large and $\kappa \leq 2^{\aleph_0}$.*

Laver-generically supercompact cardinal (2/2)

Theorem 7. *Suppose that κ is Laver-generically supercompact for $\mathcal{P}$ s.t. all $\mathbb{P} \in \mathcal{P}$ are ccc and at least one of $\mathbb{P} \in \mathcal{P}$ adds a real. Then*

(1) $\kappa \leq 2^{\aleph_0}$,

(2) SCH *holds above* $2^{<\kappa}$,

(3) *For all regular* $\lambda \geq \kappa$, *there is an* $<\aleph_1$*-saturated normal filter over* $\mathcal{P}_\kappa(\lambda)$, *and*

(4) $\mathsf{MA}^{+\mu}$ *holds for all* $\mu < \kappa$.

Theorem 8. *If κ is tightly Laver-generically superhuge for ccc p.o.s, then* $\kappa = 2^{\aleph_0}$.

Consistency of Laver-generically supercompact cardinals

Theorem 9. (1) *Suppose that* ZFC + *"there exists a supercompact cardinal" is consistent. Then* ZFC + *"there exists a Laver-generically supercompact cardinal for* σ*-closed p.o.s" is consistent as well.*

(2) *Suppose that* ZFC + *"there exists a superhuge cardinal" is consistent. Then* ZFC + *"there exists a Laver-generically super almost-huge cardinal for proper p.o.s" is consistent as well.*

(3) *Suppose that* ZFC + *"there exists a supercompact cardinal" is consistent. Then* ZFC + *"there exists a strongly Laver-generically supercompact cardinal for c.c.c. p.o.s" is consistent as well.*

- **Idea of the proof:** Iteration of length κ with appropriate support along with a Laver function enumerating the corresponding p.o.s.

Some open problems

Problem 1. If κ is Laver-generically supercompact for (all) ccc p.o.s, does it follow that $\kappa = 2^{\aleph_0}$?

Let $\mathcal{C} = \{\mathrm{Fn}(\mu, 2) : \mu \in \mathsf{On}\}$.

Theorem 10.(A.Dow, F.Tall, and W.Weiss) *If κ is generic supercompact for $\mathcal{C}$, then for any non metrizable space with $\chi(X) < \kappa$, there is a subspace Y of X of cardinality $< \kappa$.*

Problem 2. Does Laver-generic supercompactness of κ imply the reflection of non-metrizability as in Theorem 9?

Thank you for your attention.

tightly Laver generically superhuge cardinals

- For a class $\mathcal{P}$ of p.o.s, a cardinal κ is a tightly Laver-generically superhuge for $\mathcal{P}$ if, for all regular $\lambda \geq \kappa$ and $\mathbb{P} \in \mathcal{P}$ there is $\mathbb{Q} \in \mathcal{P}$ with $\mathbb{P} \lessdot \mathbb{Q}$, s.t., for any $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are a inner model $M \subseteq \mathsf{V}[\mathbb{H}]$, and an elementary embedding $j : \mathsf{V} \to M$ s.t.

  (1) $crit(j) = \kappa$, $j(\kappa) > \lambda$.

  (2) $\mathbb{P}, \mathbb{H} \in M$,

  (3) $j''j(\kappa) \in M$, and

  (4) $|\mathbb{Q}| \leq j(\kappa)$.

戻る