Set-theoretic reflection principles

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The ultimate objectives

- The ultimate objectives of this research are to give better mathematical answers to the questions like:

What is $\aleph_1$?
What is (or should be) the role of $\aleph_1$ among uncountable cardinals ?

What does (or should) it mean to be of size $< 2^{\aleph_0}$?
How about "$\leq 2^{\aleph_0}$" ?

- We consider these and other questions here in terms of reflection properties around these cardinals.

- New results in this talk are obtained in a joint work with Hiroshi Sakai and André Ottenbreit-Machio-Rodrigues.

Mathematical Framework

- Suppose that we have an uncountable (possibly higher order) structure $\mathfrak{A}$ with certain bad property $\mathcal{P}$.

  One of the natural questions:

- Is there a substructure $\mathfrak{B}$ of $\mathfrak{A}$ of smaller cardinality but also with the same bad property $\mathcal{P}$?

  A similar but more general question:

- Suppose that $\mathcal{C}$ is a class of structures and κ is a cardinal. For any $\mathfrak{A} \in \mathcal{C}$, if $\mathfrak{A} \models \mathcal{P}$ for some (bad) property $\mathcal{P}$, is it true that there is always substructures $\mathfrak{B}$ of $\mathfrak{A}$ in $\mathcal{C}$ of cardinality $< \kappa$ with $\mathfrak{B} \models \mathcal{P}$?

- What is the minimal such κ ?

  — We shall call the minimal cardinal κ (or ∞ if there is no such a cardinal κ at all) *the reflection cardinal* of the property $\mathcal{P}$ in the class of structures $\mathcal{C}$.

Example I: Non-metrizability of topological spaces

Fact 1. (A. Hajnal and I. Juhász, 1976) *For any uncountable cardinal κ there is a non-metrizable space X of size κ s.t. all subspaces Y of X of cardinality $< \kappa$ are metrizable.*

Proof

- Thus, the reflection cardinal of the non-metrizability in all topological spaces is ∞.

Theorem 2. (A. Dow, 1988) *For any compact Hausdorff space X if all subspaces of X of cardinality $\leq \aleph_1$ are metrizable then X is also metrizable.*

- This means that the reflection cardinal of the non-metrizability in compact Hausdorff spaces is $\leq \aleph_2$.
- The compact space $\omega_1 + 1$ with the order topology witnesses that the reflection cardinal is $\geq \aleph_2$.

Example I: Non-metrizability of topological spaces (2/3)

- The reflection cardinal of non-metrizability in topological spaces $= \infty$
- The reflection cardinal of non-metrizability in compact Hausdorff spaces $= \aleph_2$

Fact 3. (Folklore ?) *It is consistent that the reflection cardinal of non-metrizability in locally compact Hausdorff spaces is ∞.*

Proof

Theorem 4. ([F., Juhász et al.,2010],
[F., Sakai, Soukup and Usuba])
The statement

"the reflection cardinal of non-metrizability in locally compact Hausdorff spaces $= \aleph_2$"

*is consistent modulo a large large cardinal and is equivalent to the Fodor-type Reflection Principle (*FRP*) over* ZFC.

Example I: Non-metrizability of topological spaces (3/3)

- The reflection cardinal of non-metrizability in topological spaces $= \infty$
- The reflection cardinal of non-metrizability in compact Hausdorff spaces $= \aleph_2$
- The reflection cardinal of non-metrizability in locally compact Hausdorff spaces can be $\aleph_2$ or ∞, actually can also be many other regular cardinals between them.

▷ The consistency of the statement "The reflection cardinal of non-metrizability in first countable topological spaces is $\aleph_1$" is still open (Hamburger's problem).

Theorem 5. ([Dow, Tall and Weiss, 1990]) *(Assuming the consistency of a supercompact cardinal) the statement*

> *"The reflection cardinal of non-metrizability in first countable topological spaces is $\leq 2^{\aleph_0}$"*

is consistent.

Sketch of a proof

Example II: Reflection cardinals of graph coloring

Theorem 6. ([F., Juhász et al.,2010],
[F., Sakai, Soukup and Usuba])
The statement

"the reflection cardinal of the property [of coloring number $> \aleph_0$] in the class of all graphs $= \aleph_2$"

is also equivalent to FRP *over* ZFC.

Example II: Reflection cardinals of graph coloring (2/3)

- A graph G is called an interval graph if there is a linear ordering $\langle L, <_L \rangle$ s.t. G consists of intervals in L and I, $I' \in G$ are adjacent iff $I \neq I'$ and $I \cap I' \neq \emptyset$.

Theorem 7. ([Todorcevic]) *Let κ be a regular cardinal.*

The reflection cardinal of the property [of chromatic number $> \kappa$] in the class of interval graphs

$=$ *the reflection cardinal of the property [not κ-special] in the class of trees*

- We denote the reflection cardinal in Theorem 7 by $\mathfrak{Refl}^{\kappa}_{RC}$.
- Rado's Conjecture (RC) is the assertion $\mathfrak{Refl}^{\aleph_0}_{RC} = \aleph_2$.

Example II: Reflection cardinals of graph coloring (3/3)

Theorem 8. ([F., Sakai, Torres and Usuba])
The reflection cardinal of the property [of coloring number $> \aleph_0$] in the class of all graphs $\leq \mathfrak{Refl}_{RC}^{\aleph_0}$

Corollary 9.
The reflection cardinal of the property [of coloring number $> \aleph_0$] in the class of all graphs
$\leq$ *the reflection cardinal of the property [of chromatic number $> \aleph_0$] in the class of all graphs*

Proof. By Theorem 8 and Theorem 7. □

Corollary 10. RC *implies* FRP.

Proof. By Theorem 8 and Theorem 6. □

Stationary subsets of $[X]^{\aleph_0}$

- For a cardinal κ and a set X,

$$[X]^{\kappa} = \{x \subseteq X \ : \ x \text{ is of cardinality } \kappa\}.$$

- $C \subseteq [X]^{\aleph_0}$ is club in $[X]^{\aleph_0}$ if (1) for every $u \in [X]^{\aleph_0}$, there is $v \in C$ with $u \subseteq v$; and (2) for any countable increasing chain $\mathcal{F}$ in C we have $\bigcup \mathcal{F} \in C$.

- $S \subseteq [X]^{\aleph_0}$ is stationary in $[X]^{\aleph_0}$ if $S \cap C \neq \emptyset$ for all club $C \subseteq [X]^{\aleph_0}$.

- $M \in \mathcal{P}(\mathcal{H}(\lambda))$ is internally unbounded if $M \cap [M]^{\aleph_0}$ is cofinal in $[M]^{\aleph_0}$ (w.r.t. $\subseteq$)

- $M \in \mathcal{P}(\mathcal{H}(\lambda))$ is internally club if $M \cap [M]^{\aleph_0}$ contains a club in $[M]^{\aleph_0}$.

Stationary subsets of $[X]^{\aleph_0}$ (2/2)

- The following are variations of the "Reflection Principle" in [Jech, Millennium Book].

RP$_{\mathrm{IC}}$ For any uncountable cardinal λ, stationary $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ and any countable expansion $\mathfrak{A}$ of the structure $\langle \mathcal{H}(\lambda), \in \rangle$, there is an internally club $M \in [\mathcal{H}(\lambda)]^{\aleph_1}$ s.t. (1) $\mathfrak{A} \restriction M \prec \mathfrak{A}$; and (2) $S \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$.

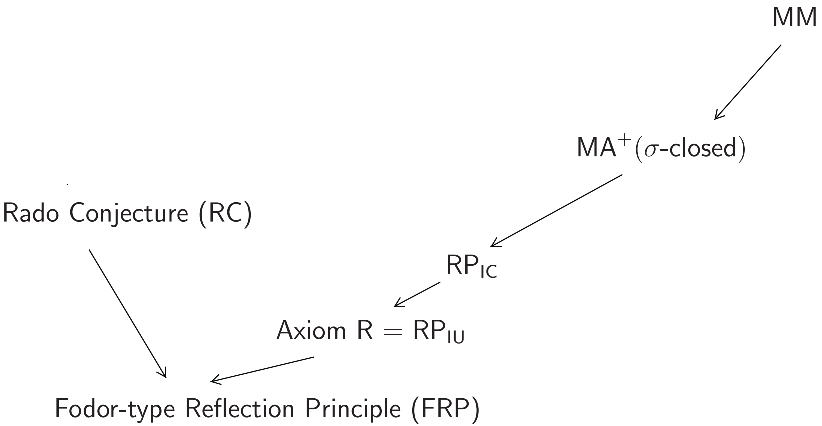
RP$_{\mathrm{IU}}$ For any uncountable cardinal λ, stationary $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ and any countable expansion $\mathfrak{A}$ of the structure $\langle \mathcal{H}(\lambda), \in \rangle$, there is an internally unbounded $M \in [\mathcal{H}(\lambda)]^{\aleph_1}$ s.t. (1) $\mathfrak{A} \restriction M \prec \mathfrak{A}$; and (2) $S \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$.

Since every internally club M is internally unbounded, we have:

Lemma 11. RP$_{\mathrm{IC}}$ *implies* RP$_{\mathrm{IU}}$.

RP$_{\mathrm{IU}}$ is also called Axiom R in the literature.

Theorem 12. ([F., Juhász et al.,2010]) RP$_{\mathrm{IU}}$ *implies* FRP.


MM
$MA^+(\sigma\text{-closed})$
Rado Conjecture (RC)
RP_{IC}
Axiom R = RP_{IU}
Fodor-type Reflection Principle (FRP)

Löwenheim-Skolem Theorems on stationary logics

- The logics:

$\mathcal{L}^{\aleph_0,II}$ denotes second order logic extending the usual first order logic with the interpretation of the second order variables such that they run over countable subsets of the underlining set of the considered structure. The logic permits quantification $\exists X$, $\forall X$ over second order variables and the logical predicate $x \,\varepsilon\, X$ where x is a first order variable and X a second order variable.

$\mathcal{L}^{\aleph_0}$ is the logic as above but without the quantification over second order variables.

$\mathcal{L}^{\aleph_0,II}_{stat}$ is the logic $\mathcal{L}^{\aleph_0,II}$ with the new quantifier $stat\, X$ where $\mathfrak{A} \models stat\, X\, \varphi(X, ...)$ is defined to be "$\{U \in [A]^{\aleph_0} : \mathfrak{A} \models \varphi(U, ...)\}$ is stationary in $[A]^{\aleph_0}$".

$\mathcal{L}^{\aleph_0}_{stat}$ is the logic $\mathcal{L}^{\aleph_0,II}_{stat}$ without second order quantifiers $\exists X$, $\forall X$.

Löwenheim-Skolem Theorems on stationary logics (2/4)

- Let $\mathcal{L}$ be one of the logics defined in the previous slide.
  - For a structure $\mathfrak{A}$ and its substructure $\mathfrak{B}$, we write $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$ if, for any $\mathcal{L}$-formula $\varphi = \varphi(x_0, \ldots, x_{m-1}, X_0, \ldots, X_{n-1})$, $a_0, \ldots, a_{m-1} \in B$ and $U_0, \ldots, U_{n-1} \in [B]^{\aleph_0}$ we have $\mathfrak{A} \models \varphi(a_0, \ldots, a_{m-1}, U_0, \ldots, U_{n-1})$ $\Leftrightarrow \mathfrak{B} \models \varphi(a_0, \ldots, a_{m-1}, U_0, \ldots, U_{n-1})$.
  - $\mathfrak{B} \prec_{\mathcal{L}^-} \mathfrak{A}$ is defined similarly except we only consider $\mathcal{L}$-formulas without any free second order variables.
- We define the following strong Downward Löwenheim-Skolem property for $\mathcal{L}$:

$\mathrm{SDLS}^-(\mathcal{L}, < \kappa)$: For any structure $\mathfrak{A}$ of countable signature, there is a substructure $\mathfrak{B}$ of of $\mathfrak{A}$ of cardinality $< \kappa$ s.t. $\mathfrak{B} \prec_{\mathcal{L}^-} \mathfrak{A}$.

$\mathrm{SDLS}(\mathcal{L}, < \kappa)$: For any structure $\mathfrak{A}$ of countable signature, there is a substructure $\mathfrak{B}$ of of $\mathfrak{A}$ of cardinality $< \kappa$ s.t. $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$.

Löwenheim-Skolem Theorems on stationary logics (3/4)

- In connection with "the reflection down to $< \aleph_2$" we obtain the following principles:

  $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}, < \aleph_2)$, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0,II}, < \aleph_2)$, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0,II}_{stat}, < \aleph_2)$, $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}, < \aleph_2)$, $\mathsf{SDLS}(\mathcal{L}^{\aleph_0,II}, < \aleph_2)$, $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$, $\mathsf{SDLS}(\mathcal{L}^{\aleph_0,II}_{stat}, < \aleph_2)$.

Lemma 13. $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}, < \aleph_2)$ *follows from the usual Downward Löwenheim Skolem Theorem and hence it holds in* ZFC.

Observation 14. ([Magidor, 2016]) $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ *implies the Fodor-type Reflection Principle. Actually it implies* $\mathsf{RP}_{\mathsf{IC}}$.

Löwenheim-Skolem Theorems on stationary logics (4/4)

- The situation is not so chaotic as it looks:

Theorem 15. *The following are equivalent:* (a) CH; (b) $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}, <\aleph_2)$; (c) $\mathsf{SDLS}^{-}(\mathcal{L}^{\aleph_0, II}, <\aleph_2)$; (d) $\mathsf{SDLS}(\mathcal{L}^{\aleph_0, II}, <\aleph_2)$.

Proof

Theorem 16. *The following are equivalent:* (a) *Diagonal Reflection Principle for internally clubness (in the sense of [Cox, 2012]),* (b) $\mathsf{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$.

Theorem 17. *The following are equivalent:* (a) *Diagonal Reflection Principle for internally clubness (in the sense of [Cox, 2012])* + CH,
(b) CH *and* $\mathsf{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$;
(c) $\mathsf{SDLS}^{-}(\mathcal{L}^{\aleph_0, II}_{stat}, <\aleph_2)$;
(d) $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$;
(e) $\mathsf{SDLS}(\mathcal{L}^{\aleph_0, II}_{stat}, <\aleph_2)$.

Game Reflection Principle

- The Game Reflection Principle (GRP) of Bernhard König (Strong Game Reflection Principle in his terminology in [König, 2004]) is defined using the following notion of infinite games:

  For any uncountable set A and $\mathcal{A} \subseteq {}^{\omega_1 >}A$, $\mathcal{G}^{{}^{\omega_1 >}A}(\mathcal{A})$ is the game of length ω_1 for Players I and II. A match in $\mathcal{G}^{{}^{\omega_1 >}A}(\mathcal{A})$ looks like the following:

  $$\begin{array}{c|cccccccccc} \text{I} & a_0 & & a_1 & & a_2 & & \cdots & a_\xi & & \cdots \\ \hline \text{II} & & b_0 & & b_1 & & b_2 & \cdots & & b_\xi & \cdots \end{array} \qquad (\xi < \omega_1)$$

  where $a_\xi,\ b_\xi \in A$ for $\xi < \omega_1$.
  II wins this match if $\langle a_\xi, b_\xi \ :\ \xi < \omega_1\rangle \in [\mathcal{A}]$ where $\langle a_\xi, b_\xi \ :\ \xi < \omega_1\rangle$ is the sequence $f \in {}^{\omega_1}A$ s.t. $f(2\xi) = a_\xi$ and $f(2\xi+1) = b_\xi$ for all $\xi < \omega_1$ and
  $[\mathcal{A}] = \{f \in {}^{\omega_1}A \ :\ f \restriction \alpha \in \mathcal{A} \text{ for all } \alpha < \omega_1\}$.

Game Reflection Principle (2/2)

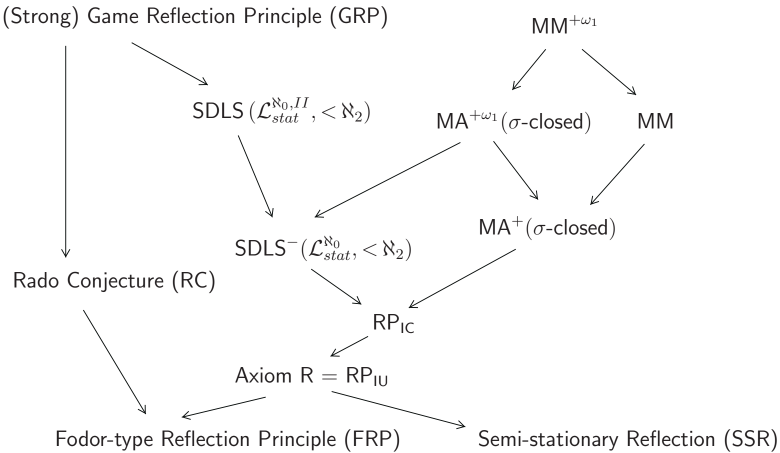
GRP: For all uncountable set A and ω_1-club $\mathcal{C} \subseteq [A]^{\aleph_1}$, if the player II has no winning strategy in $\mathcal{G}^{\omega_1 > A}(\mathcal{A})$, there is $B \in \mathcal{C}$ s.t. II has no winning strategy in $\mathcal{G}^{\omega_1 > B}(\mathcal{A} \cap {}^{\omega_1 >}B)$.

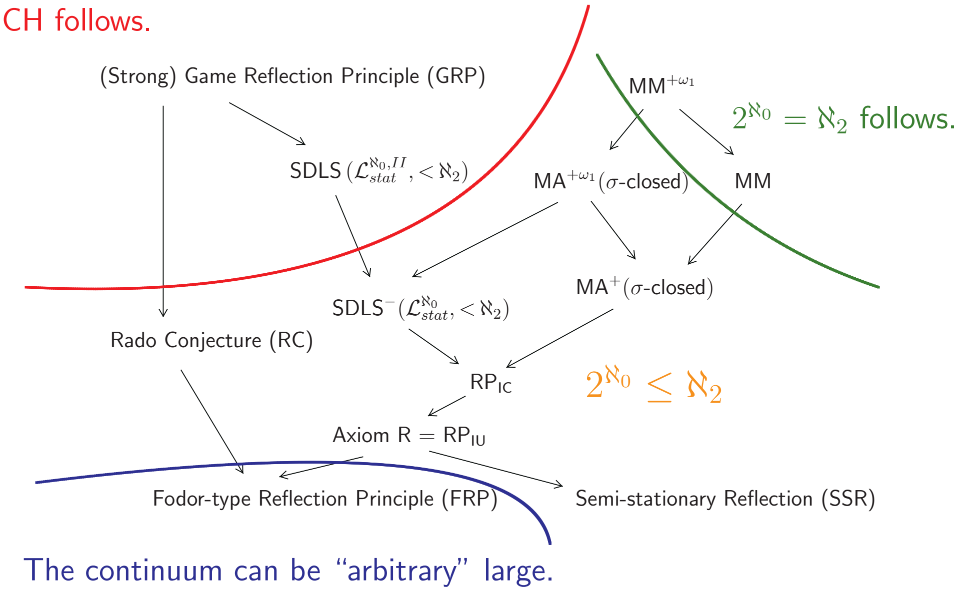
Theorem 18. ([König, 2004]) (a) GRP *implies* CH.

(b) GRP *implies Rado's Conjecture.*

(c) *GRP is forced by starting from a supercompact κ and collapsing it to $\aleph_2$ by the standard σ-closed collapsing poset.*

Theorem 19. GRP implies the Diagonal Reflection Principle for internally closedness.


(Strong) Game Reflection Principle (GRP)
$\mathsf{MM}^{+\omega_1}$
$\mathsf{SDLS}\,(\mathcal{L}^{\aleph_0,II}_{stat}, <\aleph_2)$
$\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$
MM
$\mathsf{MA}^{+}(\sigma\text{-closed})$
$\mathsf{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$
Rado Conjecture (RC)
$\mathsf{RP}_{\mathsf{IC}}$
Axiom R $= \mathsf{RP}_{\mathsf{IU}}$
Fodor-type Reflection Principle (FRP)
Semi-stationary Reflection (SSR)


CH follows.
(Strong) Game Reflection Principle (GRP)
$\mathsf{MM}^{+\omega_1}$
$2^{\aleph_0} = \aleph_2$ follows.
SDLS $(\mathcal{L}_{stat}^{\aleph_0, II}, <\aleph_2)$
$\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$
MM
$\mathsf{MA}^{+}(\sigma\text{-closed})$
$\mathsf{SDLS}^{-}(\mathcal{L}_{stat}^{\aleph_0}, <\aleph_2)$
Rado Conjecture (RC)
$\mathsf{RP}_{\mathsf{IC}}$
$2^{\aleph_0} \leq \aleph_2$
Axiom R $= \mathsf{RP}_{\mathsf{IU}}$
Fodor-type Reflection Principle (FRP)
Semi-stationary Reflection (SSR)
The continuum can be "arbitrary" large.

Further Results and open problems

- If we replace the reflection down to $< \aleph_2$ by reflection down to $< 2^{\aleph_0}$ and/or down to $\leq 2^{\aleph_0}$, most of the principles are consistent under very large (e.g. weakly inaccessible and much more) continuum.
  - Strong reflection properties seem to support CH and large continuum but not $2^{\aleph_0} = \aleph_2$.
- Our reflection priniples are connected to stationarity of subsets of $[\lambda]^{\aleph_0}$. Some of the reflection principles can be generalized to the corresponding principles connected to stationarity of subsets of $[\lambda]^{\mu}$ with certain cardinal arithmetical assumptions.
- The results in connection with what is mentioned above are still not in the final form and there seems to be many open questions.
- Hamburger's Problem and Galvin Conjecture are still open!

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Jag tackar för er uppmörksamhet.

御清聴ありがとうございました。

Fodor-type Reflection Principle (FRP)

(FRP) For any regular $\kappa > \omega_1$, any stationary $E \subseteq E^\kappa_\omega$ and any mapping $g : E \to [\kappa]^{\aleph_0}$ with $g(\alpha) \subseteq \alpha$ for all $\alpha \in E$, there is $\gamma \in E^\kappa_{\omega_1}$ s.t.

(*) for any $I \in [\gamma]^{\aleph_1}$ closed w.r.t. g and club in γ, if $\langle I_\alpha : \alpha < \omega_1 \rangle$ is a filtration of I then $\sup(I_\alpha) \in E$ and $g(\sup(I_\alpha)) \subseteq I_\alpha$ hold for stationarily many $\alpha < \omega_1$.

▷ $\mathcal{F} = \langle I_\alpha : \alpha < \lambda \rangle$ is a filtration of I if $\mathcal{F}$ is a continuously increasing $\subseteq$-sequence of subsets of I of cardinality $< |\, I\,|$ s.t. $I = \bigcup_{\alpha<\lambda} I_\alpha$.

- FRP follows from Martin's Maximum or Rado's Conjecture. $\mathrm{MA}^+(\sigma\text{-closed})$ already implies FRP but PFA does not imply FRP since PFA does not imply stationary reflection of subsets of $E^{\omega_2}_\omega$ (Magidor, Beaudoin) which is a consequence of FRP.
- FRP is a large cardinal property: By Fact 3. and Theorem 4., FRP implies the total failure of the square principle.

戻る

Proof of Fact 1

Fact 1. (A. Hajnal and I. Juhász, 1976) *For any uncountable cardinal* κ *there is a non-metrizable space* X *of size* κ *s.t. all subspaces* Y *of* X *of cardinality* $< \kappa$ *are metrizable.*

Proof.

- Let $\kappa' \geq \kappa$ be of cofinality $\geq \kappa,\ \omega_1$.
- The topological space $(\kappa' + 1, \mathcal{O})$ with

$$\mathcal{O} = \mathcal{P}(\kappa') \cup \{(\kappa' \setminus x) \cup \{\kappa'\} : x \subseteq \kappa',\ x \text{ is bounded in } \kappa'\}$$

  is non-metrizable since the point κ' has character $= \mathrm{cf}(\kappa') > \aleph_0$.
- Any subspace of $\kappa' + 1$ of size $< \kappa$ is discrete and hence metrizable.

□

戻る

Proof of Fact 3

- It is enough to prove the following:

> **Lemma.** (Folklore ?, see [F., Juhász et al.,2010]) *For a regular cardinal $\kappa \geq \aleph_2$ if, there is a non-reflectingly stationary $S \subseteq E^{\kappa}_{\omega}$, then there is a non meta-lindelöf (and hence non metrizable) locally compact and locally countable topological space X of cardinality κ s.t. all subspace Y of X of cardinality $< \kappa$ are metrizable.*

Proof.

- Let $I = \{\alpha + 1 : \alpha < \kappa\}$ and $X = S \cup I$.
- Let $\langle a_\alpha : \alpha \in S \rangle$ be s.t. $a_\alpha \in [I \cap \alpha]^{\aleph_0}$, a_α is of order-type ω and cofinal in α. Let $\mathcal{O}$ be the topology on X introduced by letting

  (1) elements of I are isolated; and

  (2) $\{a_\alpha \cup \{\alpha\} \setminus \beta : \beta < \alpha\}$ a neighborhood base of each $\alpha \in S$.

- Then $(X, \mathcal{O})$ is not meta-lindelöf (by Fodor's Lemma) but each $\alpha < \kappa$ as subspace of X is metrizable (by induction on α). □ 戻る

Sketch of a Proof of Theorem 5

Theorem 5. ([Dow, Tall and Weiss, 1990]) *(Assuming the consistency of a supercompact cardinal) the statement*

> *"The reflection cardinal of non-metrizability in first countable topological spaces is $\leq 2^{\aleph_0}$"*

is consistent.

Proof.

- The standard models of real-valued measurability, real-valued Cohenness etc. (i.e. starting from a model with a supercompact cardinal and add that many random (or Cohen) reals etc. (side-by-side)). establish the inequality. □

- The consistency of "The reflection cardinal $= 2^{\aleph_0}$" can be also obtained if we start from a model which satisfies the square principles at cofinally many cardinals below the supercompact κ.

Coloring number and chromatic number of a graph

- For a cardinal $\kappa \in \mathrm{Card}$, a graph $G = \langle G, K \rangle$ has coloring number $\leq \kappa$ if there is a well-ordering $\sqsubseteq$ on G s.t. for all $p \in G$ the set

$$\{q \in G \;:\; q \sqsubseteq p \text{ and } q \, K \, p\}$$

  has cardinality $< \kappa$. 戻る

- The coloring number $col(G)$ of a graph G is the minimal cardinal among such κ as above.

- The chromatic number $chr(G)$ of a graph $G = \langle G, K \rangle$ is the minimal cardinal κ s.t. G can be partitioned into κ pieces $G = \bigcup_{\alpha<\kappa} G_\alpha$ s.t. each G_α is pairwise non adjacent (independent).

- For all graph G we have $chr(G) \leq col(G)$.

戻る

κ-special trees

- For a cardinal κ, a tree T is said to be κ-special if T can be represented as a union of κ subsets T_α, $\alpha < \kappa$ s.t. each T_α is an antichain (i.e. pairwise incomparable set).

戻る

Stationary subset of E^κ_ω

- For a cardinal κ,

$$E^\kappa_\omega = \{\gamma < \kappa : \mathrm{cf}(\gamma) = \omega\}.$$

- A subset $C \subseteq \xi$ of an ordinal ξ of uncountable cofinality, C is closed unbounded (club) in ξ if (1): C is cofinal in ξ (w.r.t. the canonical ordering of ordinals) and (2): for all $\eta < \xi$, if $C \cap \eta$ is cofinal in η then $\eta \in C$.

- $S \subseteq \xi$ is stationary if $S \cap C \neq \emptyset$ for all club $C \subseteq \xi$.

- A stationary $S \subseteq \xi$ if reflectingly stationary if there is some $\eta < \xi$ of uncountable cofinality s.t.$S \cap \eta$ is stationary in η. Thus:

- A stationary $S \subseteq \xi$ if non reflectingly stationary if $S \cap \eta$ is non stationary for all $\eta < \xi$ of uncountable cofinality.

Meta-Lindelöf spaces

- A topological space X is meta-lindelöf if every open cover $\mathcal{U}$ of X has a point countable open refinemet, ie. such an open cover $\mathcal{U}_0$ that (0) If $u \in \mathcal{U}_0$ then $u \subseteq v$ for some $v \in \mathcal{U}$; (1) for any $x \in X$, the set $\{u \in \mathcal{U}_0 : x \in u\}$ is countable.

Theorem (A.H. Stone). Every metrizable space is meta-lindelöf.

戻る

Proof of Theorem 15.

CH $\Rightarrow$ SDLS$(\mathcal{L}^{\aleph_0,II}, <\aleph_2)$: For a structure $\mathfrak{A}$ with a countable signature L and underlying set A, let θ be large enough and $\tilde{\mathfrak{A}} = \langle \mathcal{H}(\theta), \mathfrak{A}, \in \rangle$. where $A = \mathtt{A}^{\tilde{\mathfrak{A}}}$. Let $\tilde{\mathfrak{B}} \prec \tilde{\mathfrak{A}}$ be s.t. $|\, B \,| = \aleph_1$ for the underlying set B of $\mathfrak{B}$ and $[B]^{\aleph_0} \subseteq B$. $\mathfrak{B} = \mathfrak{A} \restriction \mathtt{A}^{\tilde{\mathfrak{B}}}$ is then as desired.

SDLS$(\mathcal{L}^{\aleph_0}, <\aleph_2) \Rightarrow$ CH: Suppose $\mathfrak{A} = \{\omega_2 \cup [\omega_2]^{\aleph_0}, \in\}$. Consider the $\mathcal{L}^{\aleph_0}$-formula $\exists x \forall y\, (y \in x \leftrightarrow y \,\varepsilon\, X)$.

The rest is easy.

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