Monte Carlo Strategies for Guessing Games and Takeuti's Reflection Axiom

Sakaé Fuchino **(渕野 昌)**

Kobe University, Japan

https://fuchino.ddo.jp/index.html

(2021 年 05 月 26 日 (18:26 JST) version)

2021 年 03 月 09 日 (JST, 於 Set-theory zoom conference in Kobe)

This presentation is typeset by upLATEX with beamer class, and
given on UP2 Version 2.0.0 by Ayumu Inoue in ipad pro (10.5inch).

The most up-to-date version of these slides is downloadable as
https://fuchino.ddo.jp/slides/kobe-workshop-2021-pf.pdf

This research is supported by
Kakenhi Grant-in-Aid for Scientific Research (C) 20K03717

The two “Axioms” in Takeuti’s 1999 article

In the 1999 article [1999], Takeuti discusses Riis’ Axiom [Riis] and his own Reflection Axiom [Takeuti].

In the following, we examine these axioms and try to put them in a large continuum context.

[Riis] Søren Riis, FOM: A proof of not-CH, Sun Sep 13 12:24:49 EDT (1998).

[Takeuti] Gaishi Takeuti, Hypotheses on power set, Proceedings of Symposia in Pure Mathematics, Vol.13, Part I, American Mathematical Society, Providence, R.I., (1971), 439–446.

[1999] 竹内外史 (Takeuti, Gaishi), ランダム実数と連続体仮説，数学セミナー，1999 年 5 月号，(1999), 34–37.

Gaishi Takeuti's article in 数学セミナー (Sugaku Seminar) in 1999.05

ランダム実数と連続体仮説

竹内外史

いま A は100個の元 $a_1, a_2, \cdots, a_{100}$ の集合であるとする．このときIとIIの2人の間の次の2人ゲームを考えることにする．

Iが A のなかから一つの元 a を選ぶ．同時にIIはIの選んだものを知らないで，A から3個の元を選ぶ．ここでもしIが選んだ a がIIが選んだ3個の元に入っていれば，IIの勝ち，そうでなければ，Iの勝ちとゲームを定義する．

明らかにIIが勝つ確率は $\frac{3}{100}$ であってIが勝つ確率は $\frac{97}{100}$ である．ここで A の元の個数が多いほど，そしてIIが選んでよい元の個数が少ないほどIが勝つ確率は多くなる．

■ 確率論的ゲーム

これからわれわれが考える2人ゲームはもう少し複雑な場合，すなわち実数についてのゲームである．

以下に実数 a とは，$a \in [0,1]$ とする．また $[0,1]$ の上のルベーグ測度を考え，それによる確率空間で考える．詳しくいえば次のようになる．

ルベーグ測度 μ は実数空間についてもっとも自然な測度である．すなわち閉区間 $[a,b]$ について $\mu([a,b])$ はその長さ $b-a$ と定義する．それから完全加法性によって定義された，もっとも直観的な自然な測度がルベーグ測度である．$\mu([0,1])=1$ であるから μ は $[0,1]$ の上の確率を表す．すなわち $[0,1]$ の元を表す変数を X で表すと(以下に確率を考えるという意味で確率変数と呼ぶ)，X が性質 $A(X)$ を満たす確率 $P(A(X))$ を

$$P(A(X)) = \mu(\{X \in [0,1] | A(X)\})$$

で定義して $[0,1]$ の上の確率空間が得られる．

Iが一つの実数 $a \in [0,1]$ を選ぶ．同時にIIは $b_1, b_2, \cdots \in [0,1]$ と高々可付番個の実数 $b_1, b_2, \cdots$ を選ぶ．

もし a が $b_1, b_2, \cdots$ の一つであれば，IIの勝ちとし，そうでなければIの勝ちとする．

このゲームに金を賭けるとする．IとIIのどちらに賭けるだろうか？　読者ならばIにかけるのではないだろうか？

このゲームについてIに賭けるということが充分根拠のあることとして，これを公理として提出する．

■ 公理R

いま $[0,1]$ から $[0,1]$ への関数の列 $f_1, f_2, f_3, \cdots$ を考える．すなわち $X \in [0,1]$ ならば $f_i(X) \in [0,1]$ である．

このとき，この $f_1, f_2, f_3, \cdots$ をIIのstrategy(策戦)と呼ぶ．

IIのstrategy $f_1, f_2, \cdots$ がIIの確率1の意味でのwinning strategy(必勝法)ということを，Iがどんな実数 $a \in [0,1]$ を選んでも

$$P(a = f_1(X) \vee a = f_2(X) \vee a = f_3(X) \vee \cdots)$$
$$= 1$$

が成立することと定義する．ここで $P(A(X)) = 1$ ということは，前に説明したように確率変数 X が A を満たす確率が1であることを意味する．

すなわち上の式は a が $f_1(X), f_2(X), f_3(X), \cdots$ という可付番個の元の一つになっている確率が1であることを意味する．

次に新しい公理として次の公理を提出する．

公理R． 上のゲームにおいて，IIに確率1の意味におけるwinning strategyは存在しない．

この公理の意味は次のとおりである．もしIIに確率1の意味におけるwinning strategyが存在するとすれば，すべての人がIIに賭けるであろう．したがって我々がIに賭けると思うことは，少なくともIIに確率

Riis' Axiom — The guessing game

- $\mathbb{I} := \{r \in \mathbb{R} : 0 \leq r \leq 1\}$,
  $\mathcal{N}$:= the ideal of null sets $\subseteq \mathbb{I}$.
- We consider the following guessing game between Player I and Player II: Player I guesses a real $a \in \mathbb{I}$; simultaneously, Player II guesses a countable set $A \in [\mathbb{I}]^{\aleph_0}$.
- Player II wins, if $a \in A$.
- A sequence $\langle A_r : r \in \mathbb{I} \rangle$ of countable sets is called a Monte Carlo strategy of Player II if, for any $a \in \mathbb{I}$,

$$\{r \in \mathbb{I} : a \notin A_r\} \in \mathcal{N}.$$

- Player II wins the game as above with the probability 1, if it chooses a real $r \in \mathbb{I}$ randomly and take A_r as its move.

Player II

Riis' Axiom

- Søren Riis thought that it is impossible that Player II has such a strategy in the game and formulated:

(Riis' Axiom [Riis]) There is no Monte Carlo st. for Player II in the game as in the previous slide.

- Riis' Axiom has several interesting consequences like:

Theorem 1. (Riis' Axiom) CH does not hold.

Proof. Suppose CH holds. Let $\{I_\alpha : \alpha \in \omega_1\}$ be a filtration of $\mathbb{I}$. Let $\iota : \mathbb{I} \to \omega_1$ a bijection.

▷ For $r \in \mathbb{I}$, let $A_r = I_{\iota(r)}$. Then $\langle A_r : r \in \mathbb{I}\rangle$ is a Monte Carlo st. for Player II in our game. □

[Riis] Søren Riis, FOM: A proof of not-CH, Sun Sep 13 12:24:49 EDT (1998).


11:07 Fri Feb 12
90%
Google
TRINITY SQUARE
Monte Carlo St
11 Monte Carlo St

Riis' Axiom — A more general setting

- For ideals I, $J \subseteq \mathcal{P}(\mathbb{I})$,

(R_I^J): There is a sequence $\langle A_r \,:\, r \in \mathbb{I}\rangle$ of elements of J s.t., for any $a \in \mathbb{I}$, we have $\{r \in \mathbb{I} \,:\, a \notin A_r\} \in I$.

  - $\langle A_r \,:\, r \in \mathbb{I}\rangle$ in the statement of R_I^J is called a Monte Carlo st. for (I, J).

- We write "$<\kappa$" to denote the ideal $[\mathbb{I}]^{<\kappa}$; $\mathcal{N}$:= the ideal of null sets $\subseteq \mathbb{I}$. With this notation

$$\text{Riis' Axiom} \;\Leftrightarrow\; \neg\, \mathsf{R}_{\mathcal{N}}^{<\aleph_1}.$$

- The following monotonicity is trivial:

Lemma 2. For ideals I, I', J, $J' \subseteq \mathcal{P}(\mathbb{I})$, if $I \subseteq I'$ and $J \subseteq J'$, then

$$\mathsf{R}_I^J \;\Rightarrow\; R_{I'}^{J'}.$$

□

A characterization of CH

Theorem 3. (Lajos Soukup) $\mathrm{R}^{<\aleph_1}_{<\aleph_1} \Leftrightarrow \mathrm{CH}$.

Proof. ▶ "$\Leftarrow$" follows from Theorem 1 (and Lemma 2).

▶ "$\Rightarrow$": Assume $2^{\aleph_0} > \aleph_1$. Toward a contradiction, suppose that $\mathrm{R}^{<\aleph_1}_{<\aleph_1}$ holds and let $\langle A_r : r \in \mathbb{I}\rangle$ be a Monte Carlo st. for $([\mathbb{I}]^{<\aleph_1}, [\mathbb{I}]^{<\aleph_1})$.

▷ Let $\langle a_\xi : \xi < \omega_1\rangle$ be a 1-1 sequence of elements of $\mathbb{I}$. For each $\xi < \omega_1$, $S_\xi = \{r \in \mathbb{I} : a_\xi \notin A_r\}$ is countable. Let

$$S = \bigcup_{\xi<\omega_1} S_\xi.$$

▷ Since $|\,S\,| \leq \aleph_1 < 2^{\aleph_0}$, there is $r \in \mathbb{I} \setminus S$. But $\{a_\xi : \xi < \omega_1\} \subseteq A_r$. ↯ ☐ (Theorem 3.)

▶ The same proof shows that

Theorem 4. $\mathrm{R}^{<\kappa}_{<\kappa} \Leftrightarrow 2^{\aleph_0} \leq \kappa$.

A characterization of R_I^J

Theorem 5. (Yasuo Yoshinobu) For ideals I, $J \subseteq \mathcal{P}(\mathbb{I})$, the principle R_I^J is equivalent to the following statement:

$\bar{\mathsf{R}}_I^J$: There is a sequence $\langle E_a : a \in \mathbb{I} \rangle$ in I s.t., for any $S \in \mathcal{P}(\mathbb{I}) \setminus J$, we have $\bigcup_{a \in S} E_a = \mathbb{I}$.

Proof.

* [Suggestion to the speaker]: Skip the proof

A characterization of R^J_I

Theorem 5. (Yasuo Yoshinobu) For ideals I, $J \subseteq \mathcal{P}(\mathbb{I})$, the principle R^J_I is equivalent to the following statement:

$\bar{\mathrm{R}}^J_I$: There is a sequence $\langle E_a : a \in \mathbb{I} \rangle$ in I s.t., for any $S \in \mathcal{P}(\mathbb{I}) \setminus J$, we have $\bigcup_{a \in S} E_a = \mathbb{I}$.

Corollary 6. $R^J_I \ \Rightarrow \ cov(I) \leq non(J)$.

Proof. Clear by $\bar{\mathrm{R}}^J_I$. ☐ (Corollary 6.)

Corollary 7. $R^J_I \ \Rightarrow \ cov(J) \leq non(I)$

Proof. ► Assume R^J_I and let $\langle E_a : a \in \mathbb{I} \rangle$ be a witness for $\bar{\mathrm{R}}^J_I$ (i.e. $E_a \in I$ for all $a \in \mathbb{I}$ and (*) $\bigcup_{a \in S} E_a = \mathbb{I}$ for all $S \in \mathcal{P}(\mathbb{I}) \setminus J$).

► Suppose, for a contradiction, that there is $U \in \mathcal{P}(\mathbb{I}) \setminus I$ s.t. (**) $|\,U\,| < cov(J)$. ▷ Fix $\mathbb{I} \ni a \mapsto r_a \in U$ with $r_a \in U \setminus E_a$. For $r \in U$, let $S_r = \{a \in \mathbb{I} : r_a = r\}$. Since $\bigcup_{r \in U} S_r = \mathbb{I}$, there is $r^* \in U$ s.t. $S_{r^*} \notin J$ by (**). ► $\bigcup_{a \in S_{r^*}} E_a = \mathbb{I}$ by (*). But

► $r^* \notin \bigcup_{a \in S_{r^*}} E_a$ by the definition of S_{r^*}. ↯ ☐ (Corollary 7.)

R_I^J under MA $+\neg$ CH

Corollary 8. For any $\kappa < 2^{\aleph_0}$,

$\mathsf{R}_I^{<\kappa} \;\Rightarrow\; cov(I) \leq \kappa < 2^{\aleph_0}$ and $non(I) = 2^{\aleph_0}$.

Proof. We have $non([\mathbb{I}]^{<\kappa}) = \kappa$ and $cov([\mathbb{I}]^{<\kappa}) = 2^{\aleph_0}$. Thus the inequalities follow from Corollary 6 and 7. □ (Corollary 8.)

Proposition 9. If $non(J) = 2^{\aleph_0}$ and $non(I) = 2^{\aleph_0}$ then R_I^J holds.

Proof. Let $\langle I_\alpha : \alpha < 2^\omega \rangle$ be a filtration of $\mathbb{I}$. For a bijection $\iota : \mathbb{I} \to 2^\omega$ and $A_r = I_{\iota(r)}$, for $r \in \mathbb{I}$, the sequence $\langle A_r : r \in \mathbb{I} \rangle$ is a Monte Carlo st. for (I, J). □ (Proposition 9.)

- $\mathcal{N}$:= null ideal; $\mathcal{M}$:= meager ideal.

Theorem 10. Assume MA $+\neg$ CH. Then (1) $\overbrace{\neg\mathsf{R}_{\mathcal{N}}^{<\aleph_1}}^{\text{Riis' Axiom}}$, $\underbrace{\neg\mathsf{R}_{\mathcal{M}}^{<\aleph_1}}_{\text{top. dual of Riis' Axiom}}$

Moreover, (2) $\neg\mathsf{R}_{\mathcal{N}}^{<\kappa}$, $\neg\mathsf{R}_{\mathcal{M}}^{<\kappa}$ for all $\kappa < 2^{\aleph_0}$.

(3) for all I, $J \in \{\mathcal{M}, \mathcal{N}\}$, we have R_I^J.

Proof. Under MA $+\neg$CH we have, $non(\mathcal{N}) = \mathrm{cov}(\mathcal{N}) = non(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0} > \aleph_1$. Thus Corollary 8 and Proposition 9 imply (1)+(2), and (3), respectively. □ (Theorem 10.)

Epistemological(?) discussions

- Compare the statement of the negation of Riis' Axiom with that of Banach-Tarski theorem:

Theorem 11. (Banach - Tarski 1924; Wilson 2005) Unit ball B in $\mathbb{R}^3$ can be partitioned into finitely many pieces, s.t. these pieces can be moved continuously and isometrically without collision to each other to be rearranged into two copies of B.

- If $\mathsf{R}_{\mathcal{N}}^{<\aleph_1}$ (the negation of Riis' Axiom) is considered to be "unnatural", then Banach-Tarski Theorem must be considered to be even more unnatural! ▷Thus, the standpoint of the interpretation that Riis' Axiom is "true" should first negate AC!

- ►► The feeling that $\neg\mathsf{R}_{\mathcal{N}}^{<\aleph_1}$ and $\neg\mathsf{R}_{\mathcal{M}}^{<\aleph_1}$ (the Riis' Axiom and its top. dual) is "natural", can be seen perhaps as one of the arguments supporting $\mathsf{MA} + \neg\mathsf{CH}$?

Problem. What do we obtain if we restrict ourselves to definable (e.g. projective) Monte Carlo st.s?

Takeuti's Reflection Axioms

Reflection Axiom ([1999]) For any ordinal $\alpha_0 > \omega_1$ and $A \subseteq \mathcal{P}(\omega)$, there is a transitive set M^* s.t.

(1) $\alpha_0 \in M^*$, (2) $\mathcal{P}(\omega) \notin M^*$, and
(3) $\langle M^*, A \cap M^*, \alpha_0, \in, \alpha \rangle_{\alpha \in \omega_1} \equiv \langle \mathsf{V}, A, \alpha_0, \in, \alpha \rangle_{\alpha \in \omega_1}$.

Axiom in [Cohen] claimed to be one of Takeuti's Axioms For any $A \subseteq \mathcal{P}(\omega)$, there is a transitive set M^* s.t.

(1) $\omega_1 \in M^*$, (2) $\mathcal{P}(\omega) \notin M^*$, and
(3) $\langle M^*, A \cap M^*, \in, \alpha \rangle_{\alpha \in \omega_1} \equiv \langle \mathsf{V}, A, \in, \alpha \rangle_{\alpha \in \omega_1}$.

[Cohen] Paul E. Cohen, A Large Power Set Axiom, The Journal of Symbolic Logic, Vol.40, No.1, (1975), 48–54.

[Takeuti] Gaishi Takeuti, Hypotheses on power set, Proceedings of Symposia in Pure Mathematics, Vol.13, Part I, American Mathematical Society, Providence, R.I., (1971), 439–446.

[1999] 竹内外史 (Takeuti, Gaishi)，ランダム実数と連続体仮説，数学セミナー，1999 年 5 月号，(1999), 34–37.

Significance and problems of Takeuti's Axioms

- Takeuti's Axioms can be considered as significant since they represent the intuition that the power set of ω is very rich so that it cannot be captured by all transitive set models even though the models considered should reflect the full truth of the universe.

- These axioms have a fatal flaw: They are inconsistent in their original formulation because of the Theorem of Undefinability of the Truth by Tarski! [CONSISTENCY]

- Besides this problem (which can be avoided by going to a weaker reflection statement), the condition $\mathcal{P}(\omega) \notin M^*$ (which is equivalent to $\mathcal{P}(\omega) \not\subseteq M^*$ if M^* satisfies the powerset axiom) does not say anything about what $\mathcal{P}(\omega) \setminus M^*$ should be. [REALS OUTSIDE M^*]

[CONSISTENCY] — A consistent version of [Cohen]

($T_{0,\kappa}$) **([Cohen] modified (An axiom schema))**
For any formula $\varphi = \varphi(x_0, \ldots, x_{\ell-1})$ in $\mathcal{L}_{\varepsilon,\underset{\sqcup}{A}} = \{\underset{\sqcup}{A}, \varepsilon\}$, and for any $A \subseteq \mathcal{P}(\omega)$, there is a transitive set M^* s.t. (1) $\kappa \in M^*$,
(2) $\mathcal{P}(\omega) \notin M^*$, and
(3) for all $\alpha_0, \ldots, \alpha_{\ell-1} \in \kappa$, we have

$$\langle M^*, A \cap M^*, \in\rangle \models \varphi[\alpha_0, \ldots, \alpha_{\ell-1}] \quad \Leftrightarrow \quad \langle \mathsf{V}, A, \in\rangle \models \varphi[\alpha_0, \ldots, \alpha_{\ell-1}].$$

- Since a parameter can be used as a switch, the axiom (schema) above is equivalent to the following:

($T^*_{0,\kappa}$) **(An axiom schema)** For any formulas $\varphi_0 = \varphi_0(x_0, , \ldots, x_{\ell_0-1})$, $\ldots, \varphi_{k-1} = \varphi_{k-1}(x_0, \ldots, x_{\ell_{k-1}-1})$ in $\mathcal{L}_{\varepsilon,\underset{\sqcup}{A}}$, and for any $A \subseteq \mathcal{P}(\omega)$, there is a transitive set M^* s.t. (1) $\kappa \in M^*$, (2) $\mathcal{P}(\omega) \notin M^*$, and, $(3')$ for all $i \in k$ and $\alpha_0, \ldots, \alpha_{\ell_i-1} \in \kappa$, we have

$$\langle M^*, A \cap M^*, \in\rangle \models \varphi_i[\alpha_0, \ldots, \alpha_{\ell_i-1}] \;\Leftrightarrow\; \langle \mathsf{V}, A, \in\rangle \models \varphi_i[\alpha_0, \ldots, \alpha_{\ell_i-1}].$$

[CONSISTENCY] — A consistent version of [Cohen]

$(T_{0,\kappa})$ (Takeuti's Axiom in [Cohen] modified (an axiom schema))
For any formula $\varphi = \varphi(x_0, ..., x_{\ell-1})$ in $\mathcal{L}_{\varepsilon,\underset{\sqcup}{A}} = \{\underset{\sqcup}{A}, \varepsilon\}$, and for any $A \subseteq \mathcal{P}(\omega)$, there is a transitive set M^* s.t. (1) $\kappa \in M^*$,

(2) $\mathcal{P}(\omega) \notin M^*$, and,

(3) for all $\alpha_0, ..., \alpha_{\ell-1} \in \kappa$, we have

$$\langle M^*, A \cap M^*, \in\rangle \models \varphi[\alpha_0, ..., \alpha_{\ell-1}] \quad \Leftrightarrow \quad \langle \mathsf{V}, A, \in\rangle \models \varphi[\alpha_0, ..., \alpha_{\ell-1}].$$

Theorem 12. (ZFC) $T_{0,\kappa}$ is equivalent to $\kappa < 2^{\aleph_0}$.

Proof.

Concerning [REALS OUTSIDE M^*]

($T_{1,\kappa}$) **(A sterngthening of $T_{0,\kappa}$ (an axiom schema))** Suppose that $\varphi = \varphi(x_0, \ldots, x_{\ell-1})$ is an arbitrary formula in $\mathcal{L}_{\varepsilon,\underset{\sqcup}{A}} = \{\underset{\sqcup}{A}, \varepsilon\}$. For any $A \subseteq \mathcal{P}(\omega)$ and a c.c.c. p.o. $\mathbb{P}$ of size $\leq \kappa$, there is a transitive set M^* s.t. (1) $\kappa \in M^*$,

(2) there is a p.o. $\mathbb{P}' \in M^*$ with $\mathbb{P}' \cong \mathbb{P}$ and an $(M^*, \mathbb{P}')$-generic filter $\mathbb{G}$ $(\in \mathsf{V})$, and,

(3) for all $\alpha_0, \ldots, \alpha_{\ell-1} \in \kappa$, we have

$$\langle M^*, A \cap M^*, \in\rangle \models \varphi[\alpha_0, \ldots, \alpha_{\ell-1}] \quad \Leftrightarrow \quad \langle \mathsf{V}, A, \in\rangle \models \varphi[\alpha_0, \ldots, \alpha_{\ell-1}].$$

Theorem 13. (ZFC) $T_{1,\kappa}$ is equivalent to MA_κ.

Proof. Similarly to the proof of Theorem 12. □ (Theorem 13.)

Corollary 14. (ZFC) "$T_{1,\kappa}$ for all $\omega_1 \leq \kappa < 2^{\aleph_0}$" is equivalent to MA.

A step or two toward a consistent verion of [1999]

(T_2) **(A strengthening of $T_{1,\kappa}$ (an axiom schema))** Suppose that $\varphi = \varphi(x_0, \ldots, x_{\ell-1})$ is an arbitrary formula in $\mathcal{L}_{\varepsilon, \underset{\sqcup}{A}} = \{\underset{\sqcup}{A}, \varepsilon\}$. For any $A \subseteq \mathcal{P}(\omega)$, $\kappa < 2^{\aleph_0}$, and any c.c.c. p.o. $\mathbb{P}$ of size $\leq 2^{\aleph_0}$, there is a transitive set M^* s.t. (1) $2^{\aleph_0} \in M^*$,

(2) there is a p.o. $\mathbb{P}' \in M^*$ with $\mathbb{P}' \cong \mathbb{P}$ and an $(M^*, \mathbb{P}')$-generic filter $\mathbb{G}$ $(\in \mathsf{V})$, and,

(3) for all $\alpha_0, \ldots, \alpha_{\ell-1} \in \kappa \cup \{2^{\aleph_0}\}$, we have

$$\langle M^*, A \cap M^*, \in \rangle \models \varphi[\alpha_0, \ldots, \alpha_{\ell-1}] \quad \Leftrightarrow \quad \langle \mathsf{V}, A, \in \rangle \models \varphi[\alpha_0, \ldots, \alpha_{\ell-1}].$$

Theorem 15. (ZFC + there exists a **Laver-generically superhuge cardinal for c.c.c. p.o.s**) T_2 holds.

Proof.

A step or two toward a consistent verion of [1999] (2/2)

(T_3) (A strengthening of T_2 even closer to [1999] (an axiom schema))
Suppose that $\varphi = \varphi(x_0, ..., x_{\ell-1})$ is an arbitrary formula in $\mathcal{L}_{\varepsilon,\underline{A}} = \{\underline{A}, \varepsilon\}$.
For any $A \subseteq \mathcal{P}(\omega)$, $\kappa < 2^{\aleph_0}$, $\alpha \in \mathrm{On} \setminus 2^{\aleph_0}$ and any c.c.c. p.o. $\mathbb{P}$ of size $\leq 2^{\aleph_0}$, there are $\alpha_0 \in \mathrm{On} \setminus \alpha$ and a transitive set M^* s.t.

(1) $\alpha_0 \in M^*$,

(2) there is a p.o. $\mathbb{P}' \in M^*$ with $\mathbb{P}' \cong \mathbb{P}$ and an $(M^*, \mathbb{P}')$-generic filter $\mathbb{G}$ $(\in \mathsf{V})$, and,

(3) for all $\alpha_0, ..., \alpha_{\ell-1} \in \kappa \cup \{2^{\aleph_0}, \alpha_0\}$, we have
$$\langle M^*, A \cap M^*, \in\rangle \models \varphi[\alpha_0, ..., \alpha_{\ell-1}] \quad \Leftrightarrow \quad \langle \mathsf{V}, A, \in\rangle \models \varphi[\alpha_0, ..., \alpha_{\ell-1}].$$

Theorem 16. (ZFC + there exists a **Laver-generically superI2 cardinal for c.c.c. p.o.s**) T_3 holds.

Proof. Similarly to the proof of Theorem 15. □ (Theorem 16.)

Conclusions

- Existence of a Laver-generically large cardinal unifies strong but "natural" assertions about the largeness of $\mathcal{P}(\omega)$. For the scenario of very large continuum. This can be expressed with a Laver-generically large cardinal for c.c.c. p.o.s (or some other natural class of p.o.s preserving cardinals below the large cardinal):

Theorem 17. (Proposition 2.8 in [II]) Suppose that μ is generically supercompact for c.c.c. p.o.s. Then, (1) SCH holds. (2) there is an ω_1-saturated normal filter over $\mathcal{P}_\mu(\lambda)$ for all $\lambda \geq \mu$.

Theorem 18. (Theorem 5.7 in [II]) Suppose that μ is generically supercompact for c.c.c. p.o.s. Then, $\mathsf{MA}^{++\kappa}(c.c.c.)$ holds for all $\kappa < \mu$. In particular, we have $\neg R^{<\aleph_1}_{\mathcal{N}}$ and $\neg R^{<\aleph_1}_{\mathcal{M}}$, as well as: $R^{<J}_{<I}$ for all I, $J \in \{\mathcal{N}, \mathcal{M}\}$ holds.

- If we assume the existence of a Laver-generically superI2 cardinal, then even a verion of [1999] is integrated into this picture.

[II] S.F., André Ottenbreit Maschio Rodrigues and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, II — reflection down to the continuum, to appear in Archive for Mathematical Logic (2021).

Thank you for your attention!
ご清聴ありがとうございました．

n

Laver-generic superI2

- A cardinal μ is Laver-generically superI2 for a class $\mathcal{P}$ of p.o.s, if, for any $\lambda \geq \mu$ and $\mathbb{P} \in \mathcal{P}$, there are $\alpha_0 > \lambda$ $\mathbb{Q} \in \mathcal{P}$, $\mathbb{P} \mathrel{\leq\!\!\!\circ} \mathbb{Q}$ with $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$ and j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t.

(1) $j : V \overset{\preccurlyeq}{\rightarrow} M$,

(2) $crit(j) = \mu$, $\alpha_0 = j(\alpha_0) > j(\mu) > \lambda$,

(3) $|\,\mathbb{Q}\,| \leq j(\mu)$,

(4) $\mathbb{P}$, $\mathbb{H} \in M$ and

(5) $j''\alpha_0 \in M$

- I still have to check the following:

Theorem (?) The consistency of the existence of a Laver-generic superI2 cardinal for c.c.c. p.o.s follows from I3.

Back

Size of a Laver-generic large cardinal and the continuum

Lemma A1. (Lemma 2.6 in [II]) If μ is generically measurable for some p.o. $\mathbb{P}$, then μ is regular.

Lemma A2. (Lemma 5.6 in [II]) If μ is generically supercompact by a class $\mathbb{P}$ whose elements do not add any reals, then $2^{\aleph_0} < \mu$.

Lemma A3. (Lemma 5.5 in [II]) If μ is Laver-generically supercompact for a class $\mathcal{P}$ containing at least one p.o. adding a reals then $\mu \leq 2^{\aleph_0}$.

Lemma A4. (Lemma 5.4 in [II]) If μ is Laver-generically supercompact for a class $\mathcal{P}$ s.t. all $\mathbb{P} \in \mathcal{P}$ preserve ω_1 and $\mathrm{Col}(\omega_1, \omega_1) \in \mathbb{P}$, then $\mu = \aleph_2$.

Theorem A5. (Theorem 5.8 in [II]) If μ is Laver-generically superhuge for c.c.c. p.o.s, then $\mu = 2^{\aleph_0}$.

[II] S.F., André Ottenbreit Maschio Rodrigues and Hiroshi Sakai,
Strong downward Löwenheim-Skolem theorems for stationary logics, II
— reflection down to the continuum,
to appear in Archive for Mathematical Logic (2021).

Back to the proof

Proof of Theorem 15.

Theorem 15. (ZFC + there exists a **Laver-generically superhuge cardinal for c.c.c. p.o.s**) T_2 holds.

Proof. Assume that there is a Laver-gen. superhuge caredinal μ for c.c.c. p.o.s. Then $\mu = 2^{\aleph_0}$. We may assume that φ in the assertion of T_2 expresses everything we need below.

- Suppose that $A \subseteq \mathcal{P}(\omega)$, $\kappa < 2^{\aleph_0}$ and $\mathbb{P} = \langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ is a c.c.c. p.o. of size $\leq \kappa$. W.l.o.g., the underlying set of $\mathbb{P} \subseteq \kappa$.
- $\mathbb{Q}$ be a c.c.c. p.o. with $\mathbb{P} \lessdot \mathbb{Q}$, with $\mathbb{H}$, j, M be as in the definition of Laver-generic superhugeness.
- In V, let M_0^* be a transitive set s.t. ① $V_{2^{\aleph_0}} \subseteq M_0^*$, ② A, $j(\mu) \in M_0^*$, ③ φ is absolute over M_0^* (possible by Montague-Lévy Theorem), ④ $|\, M_0^* \,| = j(\mu)$ (possible by Löwenheim-Skolem Theorem).
- By the closedness property of M, $M_1^* = \langle j''M_0^*, j''A, \in \rangle \in M$. Let $M_2^* \in M$ be the transitive collapse of M_0^*.
- Then, in M, $M_2^* \models (1), (2), (3)$ of T_2 for φ, $j(A)$, κ $(< 2^{\aleph_0})$, $j(\mathbb{P})$.
- By elementarity, there is M^* in V satisfying $(1), (2), (3)$ for φ, A, κ, $\mathbb{P}$.

Back

Laver-generically large cardinals

- A cardinal μ is Laver-generically supercompacrt (Laver-generically superhuge resp.) for a class $\mathcal{P}$ of p.o.s, if, for any $\lambda \geq \mu$ and $\mathbb{P} \in \mathcal{P}$, there are $\mathbb{Q} \in \mathcal{P}$, $\mathbb{P} \mathrel{\underline{<\!\!\circ}} \mathbb{Q}$ with $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$ and j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t.

(1) $j : V \overset{\preccurlyeq}{\to} M$,
(2) $crit(j) = \mu$, $j(\mu) > \lambda$,
(3) $|\,\mathbb{Q}\,| \leq j(\mu)$,
(4) $\mathbb{P}$, $\mathbb{H} \in M$ and
(5) $j''\lambda \in M$ $\quad$ ($j''j(\mu) \in M$ resp.)

- The notion of Laver-generically large cardinals was introduced in [II] without the condition (3). The large cardinal with the all the conditions (1)∼(5) is called there tightly Laver-generically supercompact (superhuge resp.).

[II] S.F., André Ottenbreit Maschio Rodrigues and Hiroshi Sakai,
Strong downward Löwenheim-Skolem theorems for stationary logics, II
— reflection down to the continuum,
to appear in Archive for Mathematical Logic (2021).

Back Back to the proof

Proof of Theorem 12.

Theorem 12. (ZFC) $T_{0,\kappa}$ is equivalent to $\kappa < 2^{\aleph_0}$.

Proof. "$\Rightarrow$": Suppose that $T_{0,\kappa}$ holds and assume, for contradiction, that $2^{\aleph_0} \leq \kappa$ also holds. Let $A \subseteq \mathcal{P}(\omega)$ be a set coding an enumeration $\langle a_\alpha : \alpha < \kappa \rangle$ of $\mathbb{P}(\omega)$. Let φ be an $\mathcal{L}_{\varepsilon,\underset{\sim}{A}}$-formula which capture all the properties used below. Let M^* be the transitive set as in the statement of $T_{0,\kappa}$ for this φ. By the choice of φ, we have, for each $\alpha < \kappa$

$\langle M^*, A \cap M^*, \in \rangle \models \underset{\sim}{A}$ codes a sequence of reals of length $> \alpha$.

Since the property "α th element of $\underset{\sim}{A}$ contains n" is coded in an instance of φ, we have $a_\alpha \in M^*$ for all $\alpha \in \kappa$. Thus $\mathcal{P}(\omega) \subseteq M^*$ ↯.

"$\Leftarrow$": Assume that $\kappa < 2^{\aleph_0}$. Let φ be an arbitrary $\mathcal{L}_{\varepsilon,\underset{\sim}{A}}$-formula and let $\alpha \in \mathrm{On} \setminus \kappa$ be s.t. φ reflects over V_α (Montague-Lévy Reflection Theorem). Let $M_0^* \prec V_\alpha$ be s.t. $\kappa \subseteq M_0^*$, $A \in M_0^*$ and $|\, M_0^* \,| = \kappa$. Then the transitive collapse M^* of M_0^* is as desired in the statement of $T_{0,\kappa}$ for the formula φ. □ (Theorem 12.) Back

Undefinability of the Truth

Theorem. (Undefinability of the Truth, Tarski (1933)) Suppose that T is a concretely given theory in a language $\mathcal{L}$ s.t. Diagonal Lemma can be formulated in $\mathcal{L}$ and is true in T. Then, there is no $\mathcal{L}$-formula $\chi = \chi(x)$ s.t. $T \vdash \varphi \leftrightarrow \chi(\ulcorner\varphi\urcorner)$ for all $\mathcal{L}$-sentences φ (as far as T is consistent).

- **Takeuti's Axiom in the original formulation is inconsistent:**

  Suppose that Takeuti's Axiom (either the one in [1999] or the version in [Cohen]) holds then the formula expressing:

  There exists an M^* as in Takeuti's Axiom and $\langle M^*, \in\rangle \models \ulcorner\varphi\urcorner$

  would be a truth definition. □

Back

A characterization of R_I^J

Theorem 5. (Yasuo Yoshinobu) For ideals I, $J \subseteq \mathcal{P}(\mathbb{I})$, the principle R_I^J is equivalent to the following statement:

$\bar{\mathsf{R}}_I^J$: There is a sequence $\langle E_a : a \in \mathbb{I} \rangle$ in I s.t., for any $S \in \mathcal{P}(\mathbb{I}) \setminus J$, we have $\bigcup_{a \in S} E_a = \mathbb{I}$.

Proof. ▶ "⇒": Suppose that $\langle A_r : r \in \mathbb{I} \rangle$ witnesses R_I^J.

▷ For each $a \in \mathbb{I}$, let $E_a = \{r \in \mathbb{I} : a \notin A_r\}$. Then $E_a \in I$.

▷ For $S \in \mathcal{P}(\mathbb{I}) \setminus J$, we have $\bigcup_{a \in S} E_a = \mathbb{I}$: Suppose otherwise, and let $r \in \mathbb{I} \setminus \bigcup_{a \in S} E_a$. Then for all $a \in S$, $r \notin Ea$ (i.e. $a \in A_r$). Thus $S \subseteq A_r$. A contraction to $A_r \in J$.

▶ This shows that $\langle E_a : a \in \mathbb{I} \rangle$ witnesses $\bar{\mathsf{R}}_I^J$.

A characterization of R_I^J

Theorem 5. (Yasuo Yoshinobu) For ideals I, $J \subseteq \mathcal{P}(\mathbb{I})$, the principle R_I^J is equivalent to the following statement:

$\bar{\mathsf{R}}_I^J$: There is a sequence $\langle E_a : a \in \mathbb{I} \rangle$ in I s.t., for any $S \in \mathcal{P}(\mathbb{I}) \setminus J$, we have $\bigcup_{a \in S} E_a = \mathbb{I}$.

Proof. ▶ "$\Leftarrow$": Suppose that $\langle E_a : a \in \mathbb{I} \rangle$ witnesses $\bar{\mathsf{R}}_I^J$.

▷ For each $r \in \mathbb{I}$, let $A_r = \{a \in \mathbb{I} : r \notin E_a\}$.

▷ $A_r \in J$ holds for all $r \in \mathbb{I}$: Suppose otherwise, i.e. $A_r \notin J$ for some $r \in \mathbb{I}$. By the definition of A_r, $r \notin \bigcup_{a \in A_r} E_a$. This is a contradiction to the choice of $\langle E_a : a \in \mathbb{I} \rangle$.

▷ For all $a \in \mathbb{I}$, $E'_a := \{r \in \mathbb{I} : a \notin A_r\} \in I$: This follows from $E'_a = E_a \in I$.

▷ The equality holds because, $r \in E'_a \;\Leftrightarrow\; a \notin A_r \;\; (\Leftrightarrow \;\; r \in E_a)$.

▶ This shows that $\langle A_r : r \in \mathbb{I} \rangle$ is a wittess of R_I^J. ☐ (Theorem 5)

Back