Laver-generically large cardinals, their first order definability and some applications of the method

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Existence of Laver-generically supercompact cardinals

- We consider the following "axioms".

- Suppose that $\mathcal{P}$ is an iterable class of p.o.s. I.e. $\mathcal{P}$ is a class (property) s.t. all $\mathbb{P} \in \mathcal{P}$ are p.o.s, $\mathcal{P}$ is closed w.r.t. restriction; and

$$\text{if } \mathbb{P} \in \mathcal{P} \text{ and } \underset{\sim}{\mathbb{Q}} \in \mathsf{V}^{\mathbb{P}} \text{ with } \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{\mathbb{Q}} \in \mathcal{P} \text{”, then } \mathbb{P} * \underset{\sim}{\mathbb{Q}} \in \mathcal{P}.$$

LGSC ($\mathcal{P}$): there is a cardinal κ s.t. for any $\lambda \geq \kappa$ and for any $\mathbb{P} \in \mathcal{P}$, there is $\underset{\sim}{\mathbb{Q}} \in \mathsf{V}^{\mathbb{P}}$ s.t. $\Vdash_{\mathbb{P}}$ “$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$” and

$$\Vdash_{\mathbb{P} * \underset{\sim}{\mathbb{Q}}} \text{“there is } j : \mathsf{V} \xrightarrow{\preceq} M \subseteq \mathsf{V}^{\mathbb{P} * \underset{\sim}{\mathbb{Q}}} \text{ s.t. } crit(j) = \kappa,\ j(\kappa) > \lambda,\ \mathbb{P}, \underset{\sim}{\mathbb{H}} \in M,\ j''\lambda \in M.\text{”}$$

- If LGSC ($\mathcal{P}$) holds, then the κ as in the definition above is called the Laver-generically (L-g for short) supercompact cardinal for $\mathcal{P}$.

- Actually, under LGSC ($\mathcal{P}$) for most $\mathcal{P}$, κ as above is unique.

to 8/10

Background

- Strong reflection principles decide the size of the continuum.

[α] Downward Löwenheim-Skolem Theorem for the stationary logic down to $< 2^{\aleph_0}$ for elementarity w.r.t. first- and second-order parameters (SDLS($\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2$) in the notation of [I]) implies $2^{\aleph_0} = \aleph_1$.

— S.F., A.Ottenbreit Maschio Rodrigues, and H.Sakai [I].

[β] Downward Löwenheim-Skolem Theorem for the stationary logic down to $< 2^{\aleph_0}$ for elementarity w.r.t. only first-order parameters (SDLS$^-$($\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0}$) in the notation of [I]) implies $2^{\aleph_0} = \aleph_2$.

— S.F., A.Ottenbreit Maschio Rodrigues, and H.Sakai [II].

[γ] Downward Löwenheim-Skolem Theorem for a $\mathcal{P}_\kappa(\lambda)$ version of stationary logic down to $< 2^{\aleph_0}$ (SDLS$^{int}_+$($\mathcal{L}^{PKL}_{stat}, < 2^{\aleph_0}$) in the notation of [II]) implies that the continuum is very large (e.g. wealky hyper Mahlo and more).

— S.F., A.Ottenbreit Maschio Rodrigues, and H.Sakai [II].

Background (2/3) — [δ] replaces [α]

- [α] can be seen as a consequence of the following [δ]:

[δ] Game Reflection Principle ($\Leftrightarrow$ ω_2 is generically supercompact by σ-closed forcing) implies CH (i.e. $2^{\aleph_0} = \aleph_1$).

— B.König [König].

[β] Downward Löwenheim-Skolem Theorem for the stationary logic down to $< 2^{\aleph_0}$ for elementarity w.r.t. only first-order parameters ($\mathrm{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0})$ in the notation of [I]) implies $2^{\aleph_0} = \aleph_2$.

— S.F., A.Ottenbreit Maschio Rodrigues, and H.Sakai [II].

[γ] Downward Löwenheim-Skolem Theorem for a $\mathcal{P}_\kappa(\lambda)$ version of stationary logic down to $< 2^{\aleph_0}$ ($\mathrm{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, < 2^{\aleph_0})$ in the notation of [II]) implies that the continuum is very large (e.g. wealky hyper Mahlo and more).

— S.F., A.Ottenbreit Maschio Rodrigues, and H.Sakai [II].

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The scenarios of refl. principles are covered by instances of LGSC

Theorem 1 (S.F, A. O.M.Rodrigues., and H.Sakai [II])

[δ^+] LGSC (σ-closed) implies that the Laver generic $\kappa = \aleph_2$ and CH holds.

[ε] LGSC (proper) implies that the Laver generic $\kappa = \aleph_2$ and $2^{\aleph_0} = \aleph_2$ holds.

[ζ] LGSC (ccc) implies that the Laver generic κ is extremely large and $\kappa \leq 2^{\aleph_0}$ holds.

With a slight strengthening of LGSC (ccc), we obtain $\kappa = 2^{\aleph_0}$.

Theorem 2 (S.F, A. O.M.Rodrigues., and H.Sakai [II])

[δ^+] LGSC (σ-closed) implies Game Reflection Principle and $\mathsf{MA}^{+\omega_1}$(σ-closed).

[ε] LGSC (proper) implies $\mathsf{MA}^{+\omega_1}$(proper). In particular, this implies $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$.

[ζ] LGSC (ccc) with the assertion that the Lever generic κ is $2^{\aleph_0}$ implies $\mathsf{MA}^{+\mu}$(ccc) for all $\mu < 2^{\aleph_0}$ and $\mathsf{SDLS}^{int}_+(\mathcal{L}^{PKL}_{stat}, < 2^{\aleph_0})$.

LGSC is frist-order formalizable

- If LGSC$(\mathcal{P})$ should be treated as a new axiom to be added to ZFC, it should be formalizable in the language of ZFC.

Theorem 3. (S.F., and H.Sakai [Def]) Suppose that $\mathcal{P}$ is an iterable class of p.o.s. Then the following are equivalent:

(a) κ is L-g supercompact for $\mathcal{P}$ (so LGSC$(\mathcal{P})$ holds).

(b) For any λ, and for any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \,\varepsilon\, \mathcal{P}$" s.t.

$\Vdash_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}}$ "there are a regular cardinal $\theta > \kappa, \lambda$, a transitive set N, and a mapping j_0 s.t.

(1) $j_0 : \mathcal{H}(\theta)^{\mathsf{V}} \overset{\preceq}{\to} N$, (2) $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in \mathcal{H}(\theta)^{\mathsf{V}}$,

(3) $crit(j_0) = \kappa$, $j_0(\kappa) > \lambda$,

(4) for any $b \,\varepsilon\, N$, there is $a \,\varepsilon\, \mathcal{H}(\theta)^{\mathsf{V}}$ s.t. $b \,\varepsilon\, j_0(a)$

(5) $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\underset{\sim}{\mathbb{H}} \in N$, and (6) $j_0{}''\lambda \in N$ ".

A sketch of the proof of "(b) $\Rightarrow$ (a)"

- We imitate the ultraproduct construction: Assume that (b) holds.
- Let $\mathbb{H}$ be a $(\mathsf{V}, \mathbb{P} * \mathbb{Q})$-generic filter. Working in $\mathsf{V}[\mathbb{H}]$, let
  - $\mathcal{F} := \{f \in \mathsf{V} : f : \mathrm{dom}(f) \to \mathsf{V}, \mathrm{dom}(f) \in \mathcal{H}(\theta)^{\mathsf{V}}\}$, and
  - $\Pi := \{\langle f, a\rangle : f \in \mathcal{F},\ a \in j_0(\mathrm{dom}(f))\}$.

  For $\langle f, a\rangle$, $\langle g, b\rangle \in \Pi$, let
  - $\langle f, a\rangle \sim \langle g, b\rangle \ :\Leftrightarrow\ \langle a, b\rangle \in j_0(S_{f(x)=g(y)})$, where $S_{f(x)=g(y)} := \{\langle u, v\rangle : u \in \mathrm{dom}(f),\ v \in \mathrm{dom}(g),\ f(u) = g(v)\}$;

  and
  - $\langle f, a\rangle\ E\ \langle g, b\rangle \ :\Leftrightarrow\ \langle a, b\rangle \in j_0(S_{f(x)\varepsilon g(y)})$, where $S_{f(x)\varepsilon g(y)} := \{\langle u, v\rangle : u \in \mathrm{dom}(f),\ v \in \mathrm{dom}(g),\ f(u) \in g(v)\}$.

- ① $\sim$ is a congruent relation to E; Let $\check{f}_u : \{\emptyset\} \to \{u\}$ for $u \in \mathsf{V}$, then ② $i : \mathsf{V} \to \Pi/\!\sim;\ u \mapsto \langle \check{f}_u, \emptyset\rangle/\!\sim$ is an elementary embedding; ③ $(\Pi/\!\sim, E/\!\sim)$ is well-founded and set-like; and ④ The Mostowski collapse M of $(\Pi/\!\sim, E/\!\sim)$ together with the canonical embedding j of V into M is as desired. □


Je vous remercie de votre attention.
ご清聴ありがとうございました.
Thank you for your attention!

$(\Pi/\sim, E/\sim)$ is well-founded

Suppose not and let $\langle f_n, b_n \rangle \in \Pi$, $n \in \omega$ (in $V[\mathbb{H}]$) be s.t.

$$\langle f_0, b_0 \rangle \ni \langle f_1, b_1 \rangle \ni \langle f_2, b_2 \rangle \ni \cdots .$$

Let $\underset{\sim}{f}_n$, $n \in \omega$ be $\mathbb{P}$-names of f_n, $n \in \omega$ (note that we can choose $\underset{\sim}{f}_n$, $n \in \omega$ s.t. $\langle \underset{\sim}{f}_n : n \in \omega \rangle \in \mathsf{V}$ and $\Vdash_{\mathbb{P}}$"$\underset{\sim}{f}_n \in \mathsf{V}$" for each $n \in \omega$), and let

$$\mathcal{Q} := \{ \langle \mathbb{p}, n, u \rangle : \mathbb{p} \in \mathbb{P},\ n \in \omega,\ u \in \mathcal{H}(\theta)^{\mathsf{V}},\ \mathbb{p} \text{ decides } \underset{\sim}{f}_n, \text{ and } \mathbb{p} \Vdash_{\mathbb{P}} \text{"} u \,\varepsilon\, \mathrm{dom}(\underset{\sim}{f}_n) \text{"} \}.$$

Since θ is regular, $\mathcal{Q} \in \mathcal{H}(\theta)^{\mathsf{V}}$.
For $\langle \mathbb{p}_0, n_0, u_0 \rangle$, $\langle \mathbb{p}_1, n_1, u_1 \rangle \in \mathcal{Q}$, let

$$\langle \mathbb{p}_0, n_0, u_0 \rangle \sqsubset \langle \mathbb{p}_1, n_1, u_1 \rangle \;:\Leftrightarrow\; \mathbb{p}_0 \leq_{\mathbb{P}} \mathbb{p}_1,\quad n_0 = n_1 + 1, \text{ and } \mathbb{p}_0 \Vdash_{\mathbb{P}} \text{"} \underset{\sim}{f}_{n_0}(u_0) \,\varepsilon\, \underset{\sim}{f}_{n_1}(u_1) \text{"}.$$

$(\Pi/\sim, E/\sim)$ is well-founded (2/2)

In $V[\mathbb{H}]$, let $\langle \mathbb{p}_n : n \in \omega \rangle$ be a descending sequence in $\mathbb{H}$ w.r.t. $\leq_{\mathbb{P}}$ s.t. each $\mathbb{p}_n$ decides $\underset{\sim}{f}_n$ to be f_n.
Then $\langle \langle j_0(\mathbb{p}_n), n, b_n \rangle : n \in \omega \rangle$ is a descending chain in $j_0(\langle \mathcal{Q}, \sqsubset \rangle)$ w.r.t. $j_0(\sqsubset)$.
Since $V[\mathbb{H}]$ can see the sequence $\langle \mathbb{p}_n : n \in \omega \rangle$, we have $V[\mathbb{H}] \models$" $j_0(\langle \mathcal{Q}, \sqsubset \rangle)$ is not well-founded".
Since being well-founded is Δ_1, it follows that $N \models$" $j_0(\langle \mathcal{Q}, \sqsubset \rangle)$ is not well-founded". By elementarity, it follows that $\mathcal{H}(\theta)^V \models$" $\langle \mathcal{Q}, \sqsubset \rangle$ is not well-founded". However, if $\langle \langle \mathbb{q}_n, k_n, u_n \rangle : n \in \omega \rangle$ is a descending chain in $\langle \mathcal{Q}, \sqsubset \rangle$, then we would have

$$g_{k_0}(u_0) \ni g_{k_1}(u_1) \ni g_{k_2}(u_2) \ni \cdots$$

where g_{k_n}, for each $n \in \omega$, is the element of $\mathcal{F}$ which is decided to be $\underset{\sim}{f}_{k_n}$ by $\mathbb{p}_n$. This is a contradiction.

Back

Game Reflection Principle and SDLS

- The Game Reflection Principle in [δ] implies The Downward Löwenheim-Skolem Theorem in [α] for stationary logic down to $< \aleph_2$ SDLS$(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$.

  — S.F., A.Ottenbreit Maschio Rodrigues, and H.Sakai [I].

- SDLS$^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ is equivalent to Diagonal Reflection Principle (for internally clubness) of S. Cox [Cox].

- SDLS$(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ is equivalent to CH + Diagonal Reflection Principle (for internally clubness) of S. Cox [Cox].

  — S.F., A.Ottenbreit Maschio Rodrigues, and H.Sakai [I].

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Back to 5/10