On Downward Löwenheim-Skolem Theorems of some non first-order logics

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Downward Löwenheim-Skolem Theorem for First-Order Logic

- We use the following notation: A structure $\mathfrak{A}$ is a (first-order) structure of countable signature (if not mentioned otherwise).

- For a structure $\mathfrak{A}$, we denote with $|\mathfrak{A}|$ the underlying set of $\mathfrak{A}$, and $\|\mathfrak{A}\|$ the cardinality (of the underlying set) of $\mathfrak{A}$.

  Cf.: if X is a set, we denote with $|X|$ the cardinality of X.

Theorem 1. (Downward Löwenheim-Skolem Theorem) For any uncountable cardinal κ and a structure $\mathfrak{A}$ (of countable signature) if $S \subseteq |\mathfrak{A}|$ is of cardinality $< \kappa$, then there is $\mathfrak{B} \prec \mathfrak{A}$ s.t. $S \subseteq |\mathfrak{B}|$ and $\|\mathfrak{B}\| < \kappa$.

Löwenheim-Skolem Spectrum of a Logic

- Let $\mathcal{L}$ be a logic with the notion $\prec_{\mathcal{L}}$ of elementary substructure. The Löwenheim-Skolem spectrum of the logic $\mathcal{L}$ is defined as:

$$\mathrm{LSS}(\mathcal{L}) := \{\mu \in \mathrm{Card} : \text{for any structure } \mathfrak{A} \text{ of a countable signature and } S \subseteq |\mathfrak{A}| \text{ with } |S| < \mu, \text{ there is } \mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A} \text{ s.t. } S \subseteq |\mathfrak{B}| \text{ and } \|\mathfrak{B}\| < \mu\}.$$

- Denoting the first-order logic with L, (the classical) Downward Löwenheim-Skolem Theorem can be reformulated as:

Theorem 2. $\mathrm{LSS}(L) = \{\kappa \in \mathrm{Card} : \kappa \geq \aleph_1\}$.

On the restriction to countable signatures

Lemma 2a. For a logic $\mathcal{L}$ (with natural properties expected to a "logic"), we have

$$\mathrm{LSS}(\mathcal{L}) = \{\mu \in \mathrm{Card} : \text{for any structure } \mathfrak{A} \text{ with a signature of size } < \mu, \text{ there is } \mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A} \text{ s.t. } \|\mathfrak{B}\| < \mu\}.$$

Proof. "$\subseteq$": Suppose that $\mu \in \mathrm{LSS}(\mathcal{L})$ and let $\mathfrak{A}$ be a structure with a signature of size $\nu < \mu$. W.l.o.g., we may assume that $\mathfrak{A}$ is a relational structure and $\mathfrak{A} = \langle |\mathfrak{A}|, R_{n,\alpha}\rangle_{n\in\omega,\alpha<\nu}$ where $R_{n,\alpha}$ is an n-ary relation on $|\mathfrak{A}|$ for $n \in \omega$ and $\alpha < \nu$. We may also assume, w.l.o.g., that $\|\mathfrak{A}\| \geq \mu$ and $\nu \subseteq |\mathfrak{A}|$.

▷ Let $R_n := \bigcup_{\alpha<\nu}\{\alpha\} \times R_{n,\alpha}$ for each $n \in \omega$. Let $\mathfrak{A}^- := \langle |\mathfrak{A}|, R_n\rangle_{n\in\omega}$. Applying our assumption on μ, we find $\mathfrak{B}^- \prec_{\mathcal{L}} \mathfrak{A}^-$ with $\|\mathfrak{B}^-\| < \mu$ and $\nu \subseteq |\mathfrak{B}^-|$. By the last condition, we can reconstruct a submodel $\mathfrak{B}$ of $\mathfrak{A}$ from $\mathfrak{B}^-$ with the same underlying set and $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$.

On the restriction to countable signatures (2/2)

Lemma 2a. For a logic $\mathcal{L}$ (with natural properties expected to a "logic"), we have

$$\mathsf{LSS}(\mathcal{L}) = \{\mu \in \mathsf{Card} : \text{for any structure } \mathfrak{A} \text{ with a signature of size } < \mu, \text{ there is } \mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A} \text{ s.t. } \|\mathfrak{B}\| < \mu\}.$$

Proof. "$\subseteq$": Suppose that $\mu \in \mathsf{LSS}(\mathcal{L})$ and let $\mathfrak{A}$ be a structure with a signature of size $\nu < \mu$. W.l.o.g., we may assume that $\mathfrak{A}$ is a relational structure and $\mathfrak{A} = \langle |\mathfrak{A}|, R_{n,\alpha}\rangle_{n\in\omega,\alpha<\nu}$ where $R_{n,\alpha}$ is an n-ary relation on $|\mathfrak{A}|$ for $n \in \omega$ and $\alpha < \nu$. We may also assume, w.l.o.g., that $\|\mathfrak{A}\| \geq \mu$ and $\nu \subseteq |\mathfrak{A}|$.
Let $R_n := \bigcup_{\alpha<\nu}\{\alpha\} \times R_{n,\alpha}$ for each $n \in \omega$. Let $\mathfrak{A}^- := \langle |\mathfrak{A}|, R_n\rangle_{n\in\omega}$. Applying our assumption on μ, we find $\mathfrak{B}^- \prec_{\mathcal{L}} \mathfrak{A}^-$ with $\|\mathfrak{B}^-\| < \mu$ and $\nu \subseteq |\mathfrak{B}^-|$. By the last condition, we can reconstruct an $\mathcal{L}$-elementary submodel $\mathfrak{B}$ of $\mathfrak{A}$ from $\mathfrak{B}^-$ with the same underlying set.

"$\supseteq$": Suppose now that μ is in the set on the right side of the equality. Let $\mathfrak{A}$ be a structure of size $\geq \mu$ with a countable signature, and $S \in [\,|\mathfrak{A}|\,]^{<\mu}$.
Let $\mathfrak{A}^+ = \langle \mathfrak{A}, a\rangle_{a\in S}$. Applying the assumption on μ, we obtain $\mathfrak{B}^+ \prec_{\mathcal{L}} \mathfrak{A}^+$ of size $< \mu$. Denoting by $\mathfrak{B}$ the $\mathfrak{B}^+$ reduced to the original language, we have $\|\mathfrak{B}\| < \mu$, $S \subseteq |\mathfrak{B}|$ and $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$.

□ (Lemma 2a)

Löwenheim-Skolem Spectrum of $L(Q)$

- Let $L(Q)$ be the logic obtained from the first-order logic by adding a new unary (first-order) quantifier Q which is interpreted by

$$\mathfrak{A} \models Qx\,\varphi(x, \ldots) \;\Leftrightarrow\; \text{there are } \underline{\text{uncountably many}}\ a \in |\mathfrak{A}| \text{ s.t. } \mathfrak{A} \models \varphi(a, \ldots).$$

- $\prec_{L(Q)}$ is defined just as in the first-order logic for formulas of $L(Q)$.

Theorem 3. $\mathrm{LSS}(L(Q)) = \{\kappa \in \mathrm{Card} : \kappa \geq \aleph_2\}$.

Proof. Suppose that $\kappa \geq \aleph_2$ and $\mathfrak{A}$ is a structure with a countable signature with $\|\mathfrak{A}\| \geq \kappa$.
Let θ be a sufficiently large regular cardinal $> \omega_1$ with $\mathfrak{A} \in \mathcal{H}(\theta)$.
For $S \in [\,|\mathfrak{A}|\,]^{<\kappa}$, let $M \prec \mathcal{H}(\theta)$ be s.t.

(1) $\mathfrak{A} \in M$,

(2) $\omega_1, S \subseteq M$, and

(3) $|M| < \kappa$.

Let $B := |\mathfrak{A}| \cap M$ and $\mathfrak{B} := \mathfrak{A} \restriction B$.

Löwenheim-Skolem Spectrum of $L(Q)$ (2/2)

Theorem 3. $\mathrm{LSS}(L(Q)) = \{\kappa \in \mathrm{Card} : \kappa \geq \aleph_2\}$.

Proof. Suppose that $\kappa \geq \aleph_2$ and $\mathfrak{A}$ is a structure with a countable signature with $\|\mathfrak{A}\| \geq \kappa$.
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For $S \in [\,|\mathfrak{A}|\,]^{<\kappa}$, let $M \prec \mathcal{H}(\theta)$ be s.t.

(1) $\mathfrak{A} \in M$,

(2) $\omega_1, S \subseteq M$, and

(3) $|M| < \kappa$.

Let $B := |\mathfrak{A}| \cap M$ and $\mathfrak{B} := \mathfrak{A} \restriction B$.

$S \subseteq B = |\mathfrak{B}|$, $\|\mathfrak{B}\| < \kappa$.

Thus we are done by:

Claim. $\mathfrak{B} \prec_{L(Q)} \mathfrak{A}$.

$\vdash$ It is enough to show:

- $M \models$"$\mathfrak{A} \models \varphi(b_0, \ldots, b_{n-1})$" $\Leftrightarrow$ $\mathfrak{B} \models \varphi(b_0, \ldots, b_{n-1})$
  for any $L(Q)$-formula $\varphi = \varphi(x_0, \ldots, x_{n-1})$ and $b_0, \ldots, b_{n-1} \in B$.

▷ The crucial step of the induction proof:

$M \models$"$\mathfrak{A} \models Qx\psi(x, b_0, \ldots, b_{n-1})$" $\Leftrightarrow$ $\mathcal{H}(\theta) \models$"$\mathfrak{A} \models Qx\psi(x, b_0, \ldots, b_{n-1})$"

$\Leftrightarrow$ $\mathcal{H}(\theta) \models$"there is 1-1 $f : \omega_1 \to \{a \in |\mathfrak{A}| : \mathfrak{A} \models \psi(a, b_0, \ldots, b_{n-1})\}$"

$\Leftrightarrow$ $M \models$"there is 1-1 $f : \omega_1 \to \{a \in |\mathfrak{A}| : \mathfrak{A} \models \psi(a, b_0, \ldots, b_{n-1})\}$"

$\Rightarrow$ $\{b \in |\mathfrak{A}| \cap M : M \models$"$\mathfrak{A} \models \psi(b, b_0, \ldots, b_{n-1})$"$\}$ is uncountable

$\underbrace{\Leftrightarrow}$ $\{b \in |\mathfrak{B}| : \mathfrak{B} \models \psi(b, b_0, \ldots, b_{n-1})\}$ is uncountable

by induction hypothesis and by the def. of $\mathfrak{B}$

$\Leftrightarrow$ $\mathfrak{B} \models Qx\psi(x, b_0, \ldots, b_{n-1})$, and

Löwenheim-Skolem Spectrum of $L(Q)$ (2/2)

Theorem 3. $\mathrm{LSS}(L(Q)) = \{\kappa \in \mathrm{Card} : \kappa \geq \aleph_2\}$.

Proof. Suppose that $\kappa \geq \aleph_2$ and $\mathfrak{A}$ is a structure with a countable signature with $\|\mathfrak{A}\| \geq \kappa$.

Let θ be a sufficiently large regular cardinal $> \omega_1$ with $\mathfrak{A} \in \mathcal{H}(\theta)$.

For $S \in [\,|\mathfrak{A}|\,]^{<\kappa}$, let $M \prec \mathcal{H}(\theta)$ be s.t.

(1) $\mathfrak{A} \in M$,

(2) $\omega_1, S \subseteq M$, and

(3) $|M| < \kappa$.

Let $B := |\mathfrak{A}| \cap M$ and $\mathfrak{B} := \mathfrak{A} \restriction B$.

$S \subseteq B = |\mathfrak{B}|$, $\|\mathfrak{B}\| < \kappa$.

Thus we are done by:

Claim. $\mathfrak{B} \prec_{L(Q)} \mathfrak{A}$.

$\vdash$ It is enough to show:

- $M \models$"$\mathfrak{A} \models \varphi(b_0, \ldots, b_{n-1})$" $\Leftrightarrow$ $\mathfrak{B} \models \varphi(b_0, \ldots, b_{n-1})$ for any $L(Q)$-formula $\varphi = \varphi(x_0, \ldots, x_{n-1})$ and $b_0, \ldots, b_{n-1} \in B$.

▷ The crucial step of the induction proof (reverse direction):

$M \not\models$"$\mathfrak{A} \models Qx\psi(x, b_0, \ldots, b_{n-1})$" $\Leftrightarrow$ $\mathcal{H}(\theta) \not\models$"$\mathfrak{A} \models Qx\psi(x, b_0, \ldots, b_{n-1})$"

$\Leftrightarrow$ $\mathcal{H}(\theta) \models$"there is a 1-1 $f : \{a \in |\mathfrak{A}| : \mathfrak{A} \models \psi(a, b_0, \ldots, b_{n-1})\} \to \omega$"

$\Leftrightarrow$ $M \models$"there is 1-1 $f : \{a \in |\mathfrak{A}| : \mathfrak{A} \models \psi(a, b_0, \ldots, b_{n-1})\} \to \omega$"

$\Rightarrow$ $\{b \in |\mathfrak{A}| \cap M : M \models$"$\mathfrak{A} \models \psi(b, b_0, \ldots, b_{n-1})$"$\}$ is countable

$\Leftrightarrow$ $\{b \in |\mathfrak{B}| : \mathfrak{B} \models \psi(b, b_0, \ldots, b_{n-1})\}$ is countable

by induction hypothesis and by the def. of $\mathfrak{B}$

$\Leftrightarrow$ $\mathfrak{B} \not\models Qx\psi(x, b_0, \ldots, b_{n-1})$.

Theorem 5 is going to be proved similarly.

$\dashv$ $\square$ ((Theorem 3.))

Full second order logic

- $\mathcal{L}^{\mathrm{II}}$ denotes the (monadic, full) second-order logic with second-order variables X, Y, Z etc. running over all subsets of the underlying set of a structure. In addition to the constructs of the first-order logic, we have the symbol ε as a logical binary predicate and allow the expression "$x \mathrel{\varepsilon} X$" for a first order variable x and a second-order variable X as an atomic formula. We also allow the quantification of the form "$\exists X$" (and its dual "$\forall X$") over the second-order variables X.
- The relation symbol ε is interpreted as the (real) element relation and the interpretation of the quantifier $\exists X$ in $\mathcal{L}^{\mathrm{II}}$ is defined by:

$\mathfrak{A} \models \exists X \varphi(a_0, \dots, a_{m-1}, B_0, \dots, B_{n-1}, X)$ $:\Leftrightarrow$

there exists a $B \in \mathcal{P}(\,|\mathfrak{A}|\,)$ s.t. $\mathfrak{A} \models \varphi(a_0, \dots, a_{m-1}, B_0, \dots, B_{n-1}, B)$

for a first-order structure $\mathfrak{A}$, an $\mathcal{L}^{\mathrm{II}}$-formula φ in the signature of the structure $\mathfrak{A}$ with $\varphi = \varphi(x_0, \dots, x_{m-1}, X_0, \dots, X_{n-1}, X)$ where $x_0, \dots, x_{m-1}$ and $X_0, \dots, X_{n-1}$, X are first- and second-order variables, $a_0, \dots, a_{m-1} \in |\mathfrak{A}|$, and $B_0, \dots, B_{n-1} \in \mathcal{P}(\,|\mathfrak{A}|\,)$.

Full second order logic (2/4)

$\mathfrak{B} \prec_{\mathcal{L}^{\mathrm{II}}} \mathfrak{A} \;:\Leftrightarrow\; \mathfrak{B} \models \varphi(b_0, \ldots, b_{n-1})$ holds if and only if $\mathfrak{A} \models \varphi(b_0, \ldots, b_{n-1})$ holds for all formulas $\varphi = \varphi(x_0, \ldots)$ in $\mathcal{L}^{\mathrm{II}}$ without free second-order variables, and for all $b_0, \ldots, b_{n-1} \in |\mathfrak{B}|$.

- ▷ Exclusion of second-order free variables and parameters in this context is natural because of the following trivial example:

Example 4. Let $\mathfrak{B} \subsetneq \mathfrak{A}$. Let $B = |\mathfrak{B}|$. Then

$\mathfrak{A} \models \exists x\, (x \not\in B)$ but $\mathfrak{B} \models \neg\exists x\, (x \not\in B)$.

Theorem 5. (M. Magidor [1971])
$\mathrm{LSS}(\mathcal{L}^{\mathrm{II}}) = \{\kappa : \kappa$ is supercompact or a limit of supercompact cardinals$\}$.

- ▶ A cardinal κ is supercompact if, for any $\lambda \geq \kappa$, there are transitive class M and elementary embedding $j : \mathsf{V} \to M$ s.t. κ is the smallest ordinal moved by j (critical point of j: we denote these conditions as $j : \mathsf{V} \overset{\prec}{\to}_{\kappa} M$), $j(\kappa) > \lambda$ and $[M]^{\lambda} \subseteq M$.

Full second order logic (3/4)

Theorem 5. (M. Magidor [1971])

$\mathrm{LSS}(\mathcal{L}^{\mathrm{II}}) = \{\kappa : \kappa$ is supercompact or a limit of supercompact cardinals$\}$.

- A cardinal κ is supercompact if, for any $\lambda \geq \kappa$, there are transitive class M and elementary embedding $j : \mathsf{V} \to M$ s.t. κ is the smallest ordinal moved by j (critical point of j: we denote these conditions as $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$), $j(\kappa) > \lambda$ and $[M]^\lambda \subseteq M$.

Proof. "$\supseteq$": Suppose that κ is supercompact and $\mathfrak{A}$ a structure in a countable signature. W.l.o.g., $|\mathfrak{A}|$ is a cardinal λ and let $S \subseteq [\lambda]^{<\kappa}$.

- Let $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$ be s.t. $j(\kappa) > \lambda$ and $[M]^\lambda \subseteq M$.
- Then $\mathfrak{A}$, $j(\mathfrak{A}) \restriction j''\lambda$, $j \restriction \lambda \in M$, $M \models j \restriction \lambda : \mathfrak{A} \overset{\cong}{\to} j(\mathfrak{A}) \restriction j''\lambda$ and $\mathcal{P}(\,|\mathfrak{A}|\,)^{\mathsf{V}} = \mathcal{P}(\,|\mathfrak{A}|\,)^M$. For any $\mathcal{L}^{\mathrm{II}}$-formula $\varphi = \varphi(x_0, \ldots)$ without free second order variables and any $a_0, \ldots \in |\mathfrak{A}|$,

$$M \models j(\mathfrak{A}) \models \varphi(j(a_0), \ldots) \quad \Leftrightarrow \quad \mathsf{V} \models \mathfrak{A} \models \varphi(a_0, \ldots)$$
$$\Leftrightarrow \quad M \models \mathfrak{A} \models \varphi(a_0, \ldots) \quad \Leftrightarrow \quad M \models j(\mathfrak{A}) \restriction j''\lambda \models \varphi(j(a_0), \ldots,).$$

The idea of this proof is similar to the proof of Theorem 3.

- Thus $M \models j(\mathfrak{A}) \restriction j''\lambda \prec_{\mathcal{L}^{\mathrm{II}}} j(\mathfrak{A})$, $\|j(\mathfrak{A}) \restriction j''\lambda\| < j(\kappa)$,

$$j(S) = j''S \subseteq |j(\mathfrak{A}) \restriction j''\lambda|\,.$$

- By elementarity, it follows that $\mathsf{V} \models$ there is $\mathfrak{B} \prec_{\mathcal{L}^{\mathrm{II}}} \mathfrak{A}$ s.t. $S \subseteq |\mathfrak{B}|$ and $\|\mathfrak{B}\| < \kappa$.

Theorem 11. is going to be proved analogously.

Full second order logic (4/4)

Theorem 5. (M. Magidor [1971])
$\mathrm{LSS}(\mathcal{L}^{\mathrm{II}}) = \{\kappa : \kappa$ is supercompact or a limit of supercompact cardinals$\}$.

- A cardinal κ is supercompact if, for any $\lambda \geq \kappa$, there are transitive class M and elementary embedding $j : \mathsf{V} \to M$ s.t. κ is the smallest ordinal moved by j (critical point of j: we denote these conditions as $j : \mathsf{V} \overset{\preceq}{\to}_{\kappa} M$), $j(\kappa) > \lambda$ and $[M]^{\lambda} \subseteq M$.

- Since $\mathrm{LSS}(\mathcal{L})$ is closed for any logic $\mathcal{L}$, the inclusion "$\supseteq$" follows from this.

"$\subseteq$": The proof of this direction requires a heavier tool of set theory. I will discuss about this proof in my next talk at:

- Kobe Set Theory Seminar

  May 25, 2022 (We) | 16:00 – (zoom)

Sakaé Fuchino: On Magidor's characterization of supercompact cardinals as Löwenheim-Skolem numbers of the second order logic

(Theorem 5)

Weak second-order logics

- $\mathcal{L}^{\aleph_0,\mathrm{II}}$ denotes the weak (monadic) second-order logic with second-order variables X, Y, Z etc. whose intended interpretation is that they run over countable subsets of the underlying set of the structure.
- Similarly to the full second-order logic, we introduce, also in $\mathcal{L}^{\aleph_0,\mathrm{II}}$, the element relation symbol ε as a logical predicate and allow the expression "$x \mathrel{\varepsilon} X$" for a first order variable x and a weak second-order variable X as an atomic formula. We also allow the quantification of the form "$\exists X$" (or its dual "$\forall X$") over the weak second-order variables X.
- The relation symbol ε here is also interpreted as the element relation and the interpretation of the quantifier $\exists X$ in $\mathcal{L}^{\aleph_0,\mathrm{II}}$ is defined by

$$\mathfrak{A} \models \exists X \varphi(a_0, \dots, a_{m-1}, B_0, \dots, B_{n-1}, X) \;\; :\Leftrightarrow$$

$$\text{there exists a } B \in [\,|\mathfrak{A}|\,]^{\aleph_0} \text{ s.t. } \mathfrak{A} \models \varphi(a_0, \dots, a_{m-1}, B_0, \dots, B_{n-1}, B)$$

for a first-order structure $\mathfrak{A}$, an $\mathcal{L}^{\aleph_0,\mathrm{II}}$-formula φ in the signature of the structure $\mathfrak{A}$ with $\varphi = \varphi(x_0, \dots, x_{m-1}, X_0, \dots, X_{n-1}, X)$ where $x_0, \dots, x_{m-1}$, and $X_0, \dots, X_{n-1}$, X are first- and second-order variables, $a_0, \dots, a_{m-1} \in |\mathfrak{A}|$, and $B_0, \dots, B_{n-1} \in [\,|\mathfrak{A}|\,]^{\aleph_0}$.

Weak second-order logics (2/4)

- If we allow the weak second-order variables in $\aleph_0$-interpretation and the logical relation symbol ε but no quantification over the weak second-order variables, the resulting logic is called $\mathcal{L}^{\aleph_0}$.
- $\mathcal{L}^{\aleph_0}_{stat}$ is the logic obtained from $\mathcal{L}^{\aleph_0}$ by adding the stationarity quantifier "*stat X*" (and its dual "*aa X*" (there are club many) but neither the existential nor universal quantification over second-order variables). The semantics of the logic is defined by

$$\mathfrak{A} \models stat\, X \varphi(a_0, \ldots, a_{m-1}, B_0, \ldots, B_{n-1}, X) \;:\Leftrightarrow$$
$$\{B \in [\,|\mathfrak{A}|\,]^{\aleph_0} : \mathfrak{A} \models \varphi(a_0, \ldots, a_{m-1}, B_0, \ldots, B_{n-1}, B)\}$$
is stationary

for a first-order structure $\mathfrak{A}$, an $\mathcal{L}^{\aleph_0}_{stat}$-formula φ in the signature of $\mathfrak{A}$ with $\varphi = \varphi(x_0, \ldots, x_{m-1}, X_0, \ldots, X_{n-1}, X)$, $a_0, \ldots, a_{m-1} \in |\mathfrak{A}|$ and $B_0, \ldots, B_{n-1} \in [A]^{\aleph_0}$.

- $\mathcal{L}^{\aleph_0, \mathrm{II}}_{stat}$ is the logic $\mathcal{L}^{\aleph_0}_{stat}$ with weak second-order quantifiers $\exists X$, $\forall X$.

Weak second-order logics (3/4)

- Let $\mathcal{L}$ be one of the logics introduced above. In contrast to the full second-order logic, the notion of elementary submodels in terms of first and second order parameters makes sense for $\mathcal{L}$.
  - For a logic $\mathcal{L}$ with weak second-order variables, and structures $\mathfrak{A}$, $\mathfrak{B}$ with $\mathfrak{B} \subseteq \mathfrak{A}$:

$\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$ $:\Leftrightarrow$ $\mathfrak{B} \models \varphi(b_0, \ldots, b_{m-1}, A_0, \ldots, A_{n-1})$ holds if and only if $\mathfrak{A} \models \varphi(b_0, \ldots, b_{m-1}, A_0, \ldots, A_{n-1})$ holds for all $\mathcal{L}$-formulas $\varphi = \varphi(x_0, \ldots, X_0, \ldots)$, for all $b_0, \ldots, b_{m-1} \in |\mathfrak{B}|$, and for all $A_0, \ldots, A_{n-1} \in [\,|\mathfrak{B}|\,]^{\aleph_0}$.

- We obtain a weaker notion of elementarity by dropping the second-order parameters.

$\mathfrak{B} \prec^{-}_{\mathcal{L}} \mathfrak{A}$ $:\Leftrightarrow$ $\mathfrak{B} \models \varphi(b_0, \ldots, b_{m-1})$ holds if and only if $\mathfrak{A} \models \varphi(b_0, \ldots, b_{m-1})$ holds for all $\mathcal{L}$-formulas $\varphi = \varphi(x_0, \ldots, x_{m-1})$ without free second-order variables, and for all $b_0, \ldots, b_{m-1} \in |\mathfrak{B}|$.

Weak second-order logics (4/4)

- We we consider $\mathcal{L}^{\aleph_0,\mathrm{II}}$, $\mathcal{L}^{\aleph_0}_{stat}$ etc. with $\prec_{\mathcal{L}^{\aleph_0,\mathrm{II}}}$, $\prec_{\mathcal{L}^{\aleph_0}_{stat}}$ etc. by default. When we consider $\mathcal{L}^{\aleph_0,\mathrm{II}}$, etc. together with $\prec^{-}_{\mathcal{L}^{\aleph_0,\mathrm{II}}}$ etc. we shall write $\mathcal{L}^{\aleph_0,\mathrm{II}\,-}$, $\mathcal{L}^{\aleph_0\,-}_{stat}$ etc.
- We call a cardinal κ $\cdot^{\aleph_0}$-closed if $\mu^{\aleph_0} < \kappa$ holds for all $\mu < \kappa$.

Proposition 6. (Fuchino, Ottenbreit Maschio Rodrigues, and Sakai [2021])
$\mathsf{LSS}(\mathcal{L}^{\aleph_0}) = \mathsf{LSS}(\mathcal{L}^{\aleph_0,\mathrm{II}\,-}) = \mathsf{LSS}(\mathcal{L}^{\aleph_0,\mathrm{II}}) = \{\kappa \in \mathsf{Card} : \kappa \text{ is } \cdot^{\aleph_0}\text{-closed}\}$.

Proof. The non-trivial direction (of inclusion) is proved similarly to Theorem 3 or Theorem 5, using $M \prec \mathcal{H}(\theta)$ with $[M]^{\aleph_0} \subseteq M$.
- Note that, if $\mu < \theta$ is $\cdot^{\aleph_0}$-closed then there is $M \prec \mathcal{H}(\theta)$ as above with $|M| = \mu$. ⧉ (Proposition 6.)

Corollary 7. (Fuchino, Ottenbreit Maschio Rodrigues, and Sakai [2021])
$\aleph_2 \in \mathsf{LSS}(\mathcal{L}^{\aleph_0}) \Leftrightarrow \mathsf{CH}$. ⧉

Some independence results around $\mathrm{LSS}(\mathcal{L}^{\aleph_0,\mathrm{II}-}_{stat})$

Theorem 8. (see Fuchino, Ottenbreit Maschio Rodrigues, and Sakai [2021])

For any $n \in \mathbb{N}$, $n \geq 2$, the statements "$\aleph_n \in \mathrm{LSS}(\mathcal{L}^{\aleph_0,\mathrm{II}-}_{stat})$" and "$\aleph_n \in \mathrm{LSS}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat})$" are independent from ZFC (modulo consistency strength of the caliber "supercompact". Known lower bound: class many Woodin cardinals).

Theorem 9. (Fuchino, Ottenbreit Maschio Rodrigues, and Sakai [2022])

"$2^{\aleph_0} \in \mathrm{LSS}(\mathcal{L}^{\aleph_0-}_{stat})$" is consistent with ZFC (modulo consistency strength similar to above) and it implies $2^{\aleph_0} = \aleph_2$.

- ▷ The consistency in Theorem 8 and Theorem 9 will be shown in next slides.
- ▷ The independence of Theorem 8 can be shown e.g. by $V = L$. But we can further localize the reason of $\aleph_n \notin \mathrm{LSS}(\mathcal{L}^{\aleph_0,\mathrm{II}-}_{stat})$.

$\mathsf{LSS}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat})$ can contain "small" cardinals

- A cardinal κ is said to be generically supercompact by σ-closed p.o.s (or σ-closed gen. supercompact, for short) if, for any $\lambda \geq \kappa$, there are σ-closed p.o. $\mathbb{P}$ $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, j, $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. $\mathsf{V}[\mathbb{G}] \models j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M$ $j(\kappa) > \lambda$ and $j''\lambda \in M$.

Lemma 9a. (Easy) If κ is σ-closed gen. supercompact then κ is regular and $> 2^{\aleph_0}$.

Lemma 10. (Folklore ?) If κ is supercompact and $\mathbb{P} = \mathrm{Col}(\mu, \kappa)$ for a regular $\mu < \kappa$, Then $\mathbb{P}$ forces "$\kappa = \mu^+$ is σ-closed gen. supercompact (actually $< \mu$-closed gen. supercompact)".

Theorem 11. If κ is σ-closed gen. supercompact, then $\kappa \in \mathsf{LSS}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat})$.

Corollary 12. Suppose that (ZFC +) "there is a supercompact cardinal" is consistent, then for each $n \geq 2$, $\aleph_n \in \mathsf{LSS}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat})$ $(\subseteq \mathsf{LSS}(\mathcal{L}^{\aleph_0}_{stat}))$ is consistent.

A Proof of Theorem 11

Theorem 11. If κ is σ-closed gen. supercompact, then $\kappa \in \mathsf{LSS}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat})$.

Proof. This can be shown similarly to the proof of Theorem 5., "$\subseteq$" .

- Assume that κ is σ-closed gen. supercompact. Suppose $\mathfrak{A}$ is a structure with $\|\mathfrak{A}\| \geq \kappa$ and $S \in [\,|\mathfrak{A}|\,]^{<\kappa}$. W.l.o.g., assume $|\mathfrak{A}| = \|\mathfrak{A}\|$.
- Let $\mathbb{P}$ be a σ-closed p.o. s.t. for a $(\mathsf{V},\mathbb{P})$-generic $\mathbb{G}$, there are j, $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$, $j(\kappa) > \|\mathfrak{A}\|$ and $j''\,\|\mathfrak{A}\| \in M$.
- Then $\mathfrak{B} := j(\mathfrak{A}) \restriction j''\,\|\mathfrak{A}\| \in M$. Since $j \restriction |\mathfrak{A}| \in M$ we also have $\mathfrak{A} \in M$ and $M \models j \restriction |\mathfrak{A}| : \mathfrak{A} \overset{\cong}{\to} \mathfrak{B}$.
- By σ-closedness of $\mathbb{P}$ we have $([\,|\mathfrak{A}|\,]^{\aleph_0})^{\mathsf{V}} = ([\,|\mathfrak{A}|\,]^{\aleph_0})^M$. Also, all stationary subsets (club subsets resp.) of $([\,|\mathfrak{A}|\,]^{\aleph_0})^{\mathsf{V}}$ remain stationary (club resp.) in M.
- Thus, $M \models \mathfrak{B} \prec_{\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}} j(\mathfrak{A})$, $\|\mathfrak{B}\| < j(\kappa)$, $j(S) \subseteq |\mathfrak{B}|$.
  By elementarity, in V, there is $\mathfrak{C} \prec_{\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}} \mathfrak{A}$ s.t. $\|\mathfrak{C}\| < \kappa$, $S \subseteq |\mathfrak{C}|$.

□ (Theorem 11.)

Uncountable Coloring number of graphs

- I learned the following theorem in a tutorial lecture of Menachem Magidor:

Theorem 13. Suppose $\kappa = \min \mathsf{LSS}(\mathcal{L}^{\aleph_0-}_{stat})$. Then for any graph $G = \langle G, E \rangle$ with $col(G) > \aleph_0$, there is $G_0 \in [G]^{<\kappa}$ s.t. $col(G_0) > \aleph_0$. Or, equivalently, for any graph $G = \langle G, E \rangle$, if $col(G_0) \leq \aleph_0$ for all $G_0 \in [G]^{<\kappa}$, then $col(G) \leq \aleph_0$.

- I will discuss about this and some other applications of Löwenheim-Skolem Theorems of non first-order logics in:

- Kobe Set Theory Seminar

  June 1 2022 (We) | 16:00 – (zoom)

Sakaé Fuchino: On Löwenheim-Skolem number and compactness number of some non first-order logics

관심을 가져 주셔서 감사합니다
Thank you for your attention!
ご清聴ありがとうございました.

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