On Löwenheim-Skolem number and compactness number of some non first-order logics

slides of the last talk

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Löwenheim-Skolem Spectrum of a Logic

- Notation: A structure $\mathfrak{A}$ is a (first-order) structure of countable signature (if not mentioned otherwise).

▷ For a structure $\mathfrak{A}$, we denote with $|\mathfrak{A}|$ the underlying set of $\mathfrak{A}$, and $\|\mathfrak{A}\|$ the cardinality (of the underlying set) of $\mathfrak{A}$.

Cf.: if X is a set, we denote with $|X|$ the cardinality of X.

- Let $\mathcal{L}$ be a logic with a notion $\prec_{\mathcal{L}}$ of elementary substructure. The Löwenheim-Skolem spectrum of the logic $\mathcal{L}$ is defined as:

$$\begin{aligned} \mathrm{LSS}(\mathcal{L}) := \{\mu \in \mathrm{Card} \ : \ & \text{for any structure } \mathfrak{A} \text{ of a countable signature} \\ & \text{and } S \subseteq |\mathfrak{A}| \text{ with } |S| < \mu, \\ & \text{there is } \mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A} \text{ s.t. } S \subseteq |\mathfrak{B}| \text{ and } \|\mathfrak{B}\| < \mu\}. \end{aligned}$$

Löwenheim-Skolem Spectrum of a Logic (2/2)

$\mathrm{LSS}(\mathcal{L}) := \{\mu \in \mathrm{Card} : $ for any structure $\mathfrak{A}$ of a countable signature and $S \subseteq |\mathfrak{A}|$ with $|S| < \mu$, there is $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$ s.t. $S \subseteq |\mathfrak{B}|$ and $\|\mathfrak{B}\| < \mu\}$.

Lemma 1. For a logic $\mathcal{L}$ (with natural properties expected to a "logic"), we have

$\mathrm{LSS}(\mathcal{L}) = \{\mu \in \mathrm{Card} : $ for any structure $\mathfrak{A}$ with a signature of size $< \mu$, there is $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$ s.t. $\|\mathfrak{B}\| < \mu\}$.

Proof

Lemma 2. For any logic $\mathcal{L}$, $\mathrm{LSS}(\mathcal{L})$ is a closed class of cardinals.

Proof

Stationary logic with two notions of elementary substructure

- $\mathcal{L}^{\aleph_0}_{stat}$ is the monadic second order logic whose second-order variables run over countable subsets of the underlying set of the structure, with new quantifier with the quantification $stat\ \underbrace{X}_{\text{second-order variable}}$ whose interpretation is "there are stationarily many X s.t. …"

$\mathfrak{B} \prec_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A} :\Leftrightarrow \quad \mathfrak{B} \models \varphi(b_0, \ldots, b_{m-1}, A_0, \ldots, A_{n-1})$ holds if and only if $\mathfrak{A} \models \varphi(b_0, \ldots, b_{m-1}, A_0, \ldots, A_{n-1})$ holds for all $\mathcal{L}^{\aleph_0}_{stat}$-formulas $\varphi = \varphi(x_0, \ldots, X_0, \ldots)$, for all $b_0, \ldots, b_{m-1} \in |\mathfrak{B}|$, and for all $A_0, \ldots, A_{n-1} \in [\,|\mathfrak{B}|\,]^{\aleph_0}$.

$\mathfrak{B} \prec^{-}_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A} :\Leftrightarrow \quad \mathfrak{B} \models \varphi(b_0, \ldots, b_{m-1})$ holds if and only if $\mathfrak{A} \models \varphi(b_0, \ldots, b_{m-1})$ holds for all $\mathcal{L}^{\aleph_0}_{stat}$-formulas $\varphi = \varphi(x_0, \ldots, x_{m-1})$ <u>without</u> <u>free second-order variables</u>, and for all $b_0, \ldots, b_{m-1} \in |\mathfrak{B}|$.

- In the following, we consider the logic $\mathcal{L}^{\aleph_0}_{stat}$ as equipped with $\prec_{\mathcal{L}^{\aleph_0}_{stat}}$.
  The logic $\mathcal{L}^{\aleph_0}_{stat}$ equipped with $\prec^{-}_{\mathcal{L}^{\aleph_0}_{stat}}$ is denoted with $\mathcal{L}^{\aleph_0\,-}_{stat}$.

Stationary logic with two notions of elementary substructure (2/2)

- $\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}$ is like $\mathcal{L}^{\aleph_0}_{stat}$ but the quantification $\exists X$ and $\forall X$ of the second order variables X is allowed in addition. These quantifiers are then interpreted as: "there exists a countable subset X of the underlying set of the structure s.t. ..." and "for all countable subsets X of the underlying set of the structure ..." respectively.
- $\prec_{\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}}$ and $\prec^{-}_{\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}}$ are defined as before.
- Also, we regard the logic $\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}$ as equipped with $\prec_{\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}}$.
- The logic $\mathcal{L}^{\aleph_0}_{stat}$ equipped with $\prec^{-}_{\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}}$ is denoted with $\mathcal{L}^{\aleph_0,\mathrm{II}\,-}_{stat}$.

Full second order logic

- $\mathcal{L}^{\mathrm{II}}$ denotes the (monadic, full) second-order logic with second-order variables X, Y, Z etc. running over all subsets of the underlying set of a structure. In addition to the constructs of the first-order logic, we have the symbol ε as a logical binary predicate and allow the expression "$x \mathrel{\varepsilon} X$" for a first order variable x and a second-order variable X as an atomic formula. We also allow the quantification of the form "$\exists X$" (and its dual "$\forall X$") over the second-order variables X.
- The relation symbol ε is interpreted as the (real) element relation and the interpretation of the quantifier $\exists X$ in $\mathcal{L}^{\mathrm{II}}$ is defined by:

$\mathfrak{A} \models \exists X \varphi(a_0, \ldots, a_{m-1}, B_0, \ldots, B_{n-1}, X) \ :\Leftrightarrow$

there exists a $B \in \mathcal{P}(\,|\mathfrak{A}|\,)$ s.t. $\mathfrak{A} \models \varphi(a_0, \ldots, a_{m-1}, B_0, \ldots, B_{n-1}, B)$

for a first-order structure $\mathfrak{A}$, an $\mathcal{L}^{\mathrm{II}}$-formula φ in the signature of the structure $\mathfrak{A}$ with $\varphi = \varphi(x_0, \ldots, x_{m-1}, X_0, \ldots, X_{n-1}, X)$ where $x_0, \ldots, x_{m-1}$ and $X_0, \ldots, X_{n-1}$, X are first- and second-order variables, $a_0, \ldots, a_{m-1} \in |\mathfrak{A}|$, and $B_0, \ldots, B_{n-1} \in \mathcal{P}(\,|\mathfrak{A}|\,)$.

Full second order logic (2/2)

$\mathfrak{B} \prec_{\mathcal{L}^{\mathrm{II}}} \mathfrak{A} \;:\Leftrightarrow\; \mathfrak{B} \models \varphi(b_0, \ldots, b_{n-1})$ holds if and only if $\mathfrak{A} \models \varphi(b_0, \ldots, b_{n-1})$ holds for all formulas $\varphi = \varphi(x_0, \ldots)$ in $\mathcal{L}^{\mathrm{II}}$ without free second-order variables, and for all $b_0, \ldots, b_{n-1} \in |\mathfrak{B}|$.

- ▷ Exclusion of second-order free variables and parameters in this context is natural because of the following trivial example:

Example 3. Let $\mathfrak{B} \subsetneq \mathfrak{A}$. Let $B = |\mathfrak{B}|$. Then

$\mathfrak{A} \models \exists x\,(x \not\in B)$ but $\mathfrak{B} \models \neg\exists x\,(x \not\in B)$.

Theorem 4. (M. Magidor [1971])
$\mathrm{LSS}(\mathcal{L}^{\mathrm{II}}) = \{\kappa \,:\, \kappa$ is supercompact, or a limit of supercompact cardinals$\}$.

- ▶ A cardinal κ is supercompact if, for any $\lambda \geq \kappa$, there are transitive class M and elementary embedding $j : \mathrm{V} \to M$ s.t. κ is the smallest ordinal moved by j (critical point of j: we denote these conditions as $j : \mathrm{V} \overset{\prec}{\to}_{\kappa} M$), $j(\kappa) > \lambda$ and $[M]^{\lambda} \subseteq M$.

Back to the proof of Proposition 12.

Back to the Example.

A slight modification of Magidor's theorem

Theorem 4. (M. Magidor [1971])
$\mathsf{LSS}(\mathcal{L}^{\mathrm{II}}) = \{\kappa : \kappa$ is supercompact, or a limit of supercompact cardinals$\}$.

- ▷ Let $\mathcal{L}^{\mathrm{HO}}$ denote the higher order logic that is the union of nth order logics for all $n \in \omega$.
- ▷ Note that, if $\mathcal{L}'$ has more expressive power than $\mathcal{L}$ then we have $\mathsf{LSS}(\mathcal{L}') \subseteq \mathsf{LSS}(\mathcal{L})$.

Corollary 5. (M. Magidor [1971])
$\mathsf{LSS}(\mathcal{L}^{\mathrm{II}}) = \mathsf{LSS}(\mathcal{L}^{\mathrm{HO}})$
$= \{\kappa : \kappa$ is supercompact, or a limit of supercompact cardinals$\}$.

Proof. $\mathsf{LSS}(\mathcal{L}^{\mathrm{II}}) \underbrace{\supseteq}_{\text{by the remark above}} \mathsf{LSS}(\mathcal{L}^{\mathrm{HO}}) \underbrace{\supseteq}_{\text{by a modification of the proof of Theorem 4. “}\supseteq\text{”}} \{\kappa : \cdots\} \underbrace{\supseteq}_{\text{Theorem 4. “}\subseteq\text{”}} \mathsf{LSS}(\mathcal{L}^{\mathrm{II}})$

(Corollary 8)

The spectrum of compactness numbers of a logic

- For a logic $\mathcal{L}$, the compactness spectrum of $\mathcal{L}$ is defined as:

$\mathsf{CS}(\mathcal{L}) := \{\kappa \in \mathsf{Card}$: for any $\mathcal{L}$-theory T (possibly of an uncountable signature), of size κ, T is satisfiable if and only if all $S \in [T]^{<\kappa}$ are satisfiable$\}$.

- The strong compactness number of a logic $\mathcal{L}$ is defined as:

$\mathsf{scn}(\mathcal{L}) := \min(\{\kappa \in \mathsf{Card}$: for any $\mathcal{L}$-theory T (possibly of an uncountable signature) of any size, T is satisfiable if and only if all $S \in [T]^{<\kappa}$ are satisfiable$)\}$.

Lemma 6. For any logic $\mathcal{L}$,

(1) $\{\kappa \in \mathsf{Card} : \mathsf{scn}(\mathcal{L}) \leq \kappa\} \subseteq \mathsf{CS}(\mathcal{L})$.

(2) $\min(\mathsf{CS}(\mathcal{L})) \leq \mathsf{scn}(\mathcal{L})$.

Theorem 7. (M. Magidor [1971], see also Theorem 23.4 in [Kanamori])
For a cardinal $\kappa > \aleph_0$, the following are equivalent:

(a) κ is extendible.
(b) $\kappa = \mathsf{scn}(\mathcal{L}^{\mathrm{II}}_{\kappa,\omega})$.

- A cardinal κ is extendible if, for any $\eta > 0$, there is $j : V_{\kappa+\eta} \overset{\prec}{\rightarrow}_{\kappa} V_{\zeta}$ for some ζ with $\eta < j(\kappa)$.

- In the definition above, the condition "$\eta < j(\kappa)$" can be dropped: (Proposition 23.15 in [Kanamori]). Also, if $\eta > \kappa$, we may replace $\kappa + \eta$ by η. If the condition above holds for an η we say that κ is η-extendible.

- For a cardinal κ, $\mathcal{L}^{\mathrm{II}}_{\kappa,\omega}$ is the logic defined as $\mathcal{L}^{\mathrm{II}}$ but also allowing conjunction and disjunction of $<\kappa$ many formulas (with the restriction that the number of free variables in such conjunction or disjunction is always kept finite).

Back to Theorem 8.

Proof of Theorem 7: "(a) $\Rightarrow$ (b)": Suppose that κ is extendible.

- We first show $\kappa \geq \mathrm{scn}(\mathcal{L}^{\mathrm{II}}_{\kappa,\omega})$.
- Suppose that T is a set of sentences in $\mathcal{L}^{\mathrm{II}}_{\kappa,\omega}$ of size $\lambda \geq \kappa$ s.t. T is $<\kappa$-satisfiable. We want to show that T is satisfiable.
- We may assume that all the symbols used in T are elements of λ and the coding of formulas is done in an appropriate way.
- Let $j : V_\lambda \overset{\prec}{\to}_\kappa V_\zeta$ for some ζ and $\lambda < j(\kappa)$. By Theorem 4 and since κ is supercompact (Proposition 23.6 in [Kanamori]), all $T_0 \in [T]^{<\kappa}$ has a model of size $<\kappa$. In particular $V_\lambda \models$"T is $<\kappa$-satisfiable".
- $V_\zeta \models$"$j''T \in [j(T)]^{<j(\kappa)}$". By elementarity of j,

  $$V_\zeta \models \text{“} j''T \text{ has a model (of size } < j(\kappa)\text{)”}.$$

  Let $\mathfrak{A} \in V_\zeta$ be a structure s.t. $V_\zeta \models$"$\mathfrak{A} \models j''T$ and $\|\mathfrak{A}\| < j(\kappa)$". Then $\mathfrak{A} \models j''T$ (V_ζ interprets model relation of $\mathcal{L}^{\mathrm{II}}_{\kappa,\omega}$ correctly at the structure $\mathfrak{A}$ because of $\mathcal{P}(\lambda)^{V_\zeta} = \mathcal{P}(\lambda)^{\mathsf{V}}$ etc.). By renaming the interpretations of the non-logical symbols $\subseteq j''\lambda$ by corresponding symbols $\subseteq \lambda$ we obtain a model $\mathfrak{A}^*$ of T from $\mathfrak{A}$.
- Thus, T is satisfiable.

- To show that $\kappa \leq \mathsf{scn}(\mathcal{L}^{\mathrm{II}}_{\kappa,\omega})$, note that $\mathsf{scn}(\mathcal{L}^{\mathrm{II}}_{\kappa,\omega})$ exists by the proof above.
- So it is enough to show that any $\mu < \kappa$ is not $\mathsf{scn}(\mathcal{L}^{\mathrm{II}}_{\kappa,\omega})$. Suppose $\mu < \kappa$ and consider the theory

$$T := \{\forall_x (\bigvee\!\!\!\!\bigvee_{\alpha<\mu} x \equiv c_\alpha)\} \cup \{d \not\equiv c_\alpha \,:\, \alpha < \mu\}.$$

Then T is not satisfiable but any subset of T of size $< \mu$ is satisfiable. In particular, μ cannot be $\mathsf{scn}(\mathcal{L}^{\mathrm{II}}_{\kappa,\omega})$.

"(b) $\Rightarrow$ (a)": Suppose that $\kappa = \mathsf{scn}(\mathcal{L}^{\mathrm{II}}_{\kappa,\omega})$.

- Let φ^* be the $\mathcal{L}^{\mathrm{II}}$-sentence in the language $\{\underline{\in}\}$ of set theory s.t.

$$\langle\, |\mathfrak{A}|\,, \underline{\in}^{\mathfrak{A}} \rangle \models \varphi^* \;\Leftrightarrow\; \underline{\in}^{\mathfrak{A}} \text{ is well-founded and extensional binary relation and } mc(\langle\, |\mathfrak{A}|\,, \underline{\in}^{\mathfrak{A}} \rangle) = \langle V_\gamma, \in \rangle \text{ for some } \gamma$$

where mc denotes the Mostowski collapse function.

Strong compactness number of the infinitary second-order logics (4/4)

For regular $\lambda > \kappa$, let

$$
\begin{aligned}
T := \{\varphi^*\} \cup\ & \{\varphi(\underset{\sqcup}{j}(\underset{\sqcup}{a}_0), \ldots) \ : \ \varphi \text{ is a first-order formula in the language of set theory, } a_0, \ldots \in V_\lambda \text{ and } V_\lambda \models \varphi(a_0, \ldots)\} \\
& \cup \{\text{``}\underset{\sqcup}{\alpha} \text{ is an ordinal''} \ : \ \alpha < \kappa + 1\} \\
& \cup \{\forall_x (x \mathrel{\underset{\sqcup}{\in}} \underset{\sqcup}{\alpha} \ \leftrightarrow \ \textstyle\bigvee\!\!\!\bigvee_{\beta < \alpha} x \equiv \underset{\sqcup}{\beta}) \ : \ \alpha < \kappa\} \\
& \cup \{\underset{\sqcup}{j}(\underset{\sqcup}{\kappa}) > \underset{\sqcup}{\kappa}\}.
\end{aligned}
$$

- It is easy to see that T is $< \kappa$-satisfiable.

▷ By the choice of κ, it follows that T is satisfiable. A model of T witnesses that κ is λ-extendible. Since this holds for arbitrary λ, it follows that κ is extendible. □ (Theorem 7)

Theorem 8. (M. Magidor [1971]) Suppose that κ is the least extendible cardinal. Then we have $\mathsf{scn}(\mathcal{L}^{\mathrm{II}}) = \mathsf{scn}(\mathcal{L}^{\mathrm{II}}_{\kappa,\omega}) = \kappa$.

Proof. (1) $\mathsf{scn}(\mathcal{L}^{\mathrm{II}}_{\kappa,\omega}) = \kappa$ follows from Theorem 7 .

- (2) $\mathsf{scn}(\mathcal{L}^{\mathrm{II}}) \leq \mathsf{scn}(\mathcal{L}^{\mathrm{II}}_{\kappa,\omega})$ since $\mathcal{L}^{\mathrm{II}}_{\kappa,\omega}$ extends $\mathcal{L}^{\mathrm{II}}$.
- Thus it is enough to show that $\kappa^* := \mathsf{scn}(\mathcal{L}^{\mathrm{II}})$ is extendible.
- By (1) and (2), there is no extendible cardinal below κ^*. Thus, for each $\mu < \kappa^*$, we can choose a cardinal λ_μ s.t. μ is not λ_μ-extendible. Note that it follows that μ is not ξ-extendible for any $\xi \geq \lambda_\mu$. Let $\lambda > \kappa^*$ be s.t.

$$\lambda \geq \sup(\{\lambda_\mu \,:\, \mu < \kappa^*\}).$$

Strong compactness number of the full second-order logic (2/2)

- Let

$$
\begin{aligned}
T := \{\varphi^*\} \cup \{\varphi(\underset{\sqcup}{j}(\underset{\sqcup}{a}_0), \ldots) \;:\; & \varphi \text{ is a first-order formula in the} \\
& \text{language of set theory,} \\
& a_0, \ldots \in V_\lambda \text{ and } V_\lambda \models \varphi(a_0, \ldots)\} \\
\cup\ \{\text{“}\underset{\sqcup}{\alpha} \text{ is an ordinal”} \;:\; & \alpha < \kappa^* + 1\} \\
\cup\ \{\underset{\sqcup}{\alpha} < \underset{\sqcup}{\beta} \;:\; & \alpha < \beta < \kappa^* + 1\} \\
\cup\ \{\underset{\sqcup}{j}(\underset{\sqcup}{\kappa}^*) > \underset{\sqcup}{\kappa}^*\}. &
\end{aligned}
$$

- Again T is $<\kappa^*$-satisfiable. By the definition of κ^*, it follows that T has a model.
- ▷ A model $\mathfrak{A}$ of T witnesses the λ-extendibility of some $\mu \leq \kappa^*$. By the choice of λ however it is only possible that κ^* is λ-extendible.

(Theorem 8)

Examples showing non-trivial situations

- Suppose that κ is extendible and λ ($\kappa < \lambda$) is the first supercompact cardinal above κ.

This is stronger than just one extendible cardinal. See [Kanamori] Exercise 23.9.

- Let μ ($\lambda < \mu$) be the first inaccessible above λ. We have

$$V_\mu \models \mathsf{ZFC}.$$

- In V_μ, there is no extendible cardinal since an extendible cardinal would imply cofinally many inaccessible cardinals. By Theorem 8 , we have

$$V_\mu \models \text{“}\mathsf{scn}(\mathcal{L}^{\mathrm{II}}) \text{ does not exist”}.$$

- By Theorem 4 , sufficiently many witnesses for $\mathsf{CS}(\mathcal{L}^{\mathrm{II}}) \cap \lambda$ survive in $V_\lambda \subseteq V_\mu$. In particular

$$V_\mu \models \kappa \in \mathsf{CS}(\mathcal{L}^{\mathrm{II}}).$$

Fun Questions: Is it consistent that $\mathsf{CS}(\mathcal{L}^{\mathrm{II}}) = \emptyset$? (Yes. Cf.: by Theorem 4 , it is very easy to obtain a model of $\mathsf{LSS}(\mathcal{L}^{\mathrm{II}}) = \emptyset$)

- What is (or can be) $\mathsf{CS}(\mathcal{L}^{\mathrm{II}})$ in $V = L$?

Examples showing non-trivial situations (2/2)

- Let $\kappa_0 < \kappa_1$ be two consecutive extendible cardinals. Let λ, be a supercompact cardinal strictly between κ_0 and κ_1.
- ▷ Let $\mathbb{G}$ be $(\mathsf{V}, \mathbb{P})$-generic for a sufficiently distributive p.o. $\mathbb{P}$ of size $< \kappa_1$ forcing a square sequence for a κ strictly between λ and κ_1.
- ▷ In $V[\mathbb{G}]$, $\mathsf{CS}(\mathcal{L}^{\mathrm{II}}) \cap \lambda$ survives (see Theorem 4). In particular

$$\mathsf{V}[\mathbb{G}] \models \text{“} \min(\mathsf{CS}(\mathcal{L}^{\mathrm{II}})) \leq \kappa_0 \text{”}.$$

- ▷ κ_1 is now the smallest extendible cardinal Thus, by Theorem 8,

$$\mathsf{V}[\mathbb{G}] \models \text{“} \mathsf{scn}(\mathcal{L}^{\mathrm{II}}) = \kappa_1 \text{”}.$$

Note that $\square_\kappa$ does not hold above a supercompact.

Generic large cardinals

- A cardinal κ is said to be generically supercompact by σ-closed p.o.s (or σ-closed gen. supercompact, for short) if, for any $\lambda \geq \kappa$, there are σ-closed p.o. $\mathbb{P}$ $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, j, $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. $\mathsf{V}[\mathbb{G}] \models j : \mathsf{V} \overset{\prec}{\to}_\kappa M$ $j(\kappa) > \lambda$ and $j''\lambda \in M$.

Lemma 9. (Easy) If κ is σ-closed gen. supercompact then κ is regular and $> 2^{\aleph_0}$. □

Lemma 10. (Folklore ?) If κ is supercompact and $\mathbb{P} = \mathrm{Col}(\mu, \kappa)$ for a regular $\mu < \kappa$, Then $\mathbb{P}$ forces "$\kappa = \mu^+$ is σ-closed gen. supercompact (actually $<\mu$-closed gen. supercompact)". □

Theorem 11. If κ is σ-closed gen. supercompact, then $\kappa \in \mathsf{LSS}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat})$.

Corollary 12. Suppose that (ZFC +) "there is a supercompact cardinal" is consistent, then for each $n \geq 2$, $\aleph_n \in \mathsf{LSS}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat})$ $(\subseteq \mathsf{LSS}(\mathcal{L}^{\aleph_0}_{stat}))$ is consistent.

Proof. By Lemma 10 and Theorem 11. □ (Corollary 12)

Theorem 11. If κ is σ-closed gen. supercompact, then $\kappa \in \mathsf{LSS}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat})$.

Proof.

- Assume that κ is σ-closed gen. supercompact. Suppose $\mathfrak{A}$ is a structure with $\|\mathfrak{A}\| \geq \kappa$ and $S \in [\,|\mathfrak{A}|\,]^{<\kappa}$. W.l.o.g., assume $|\mathfrak{A}| = \|\mathfrak{A}\| = \lambda$.
  - Let $\mathbb{P}$ be a σ-closed p.o. s.t. for a $(\mathsf{V},\mathbb{P})$-generic $\mathbb{G}$, there are j, $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. $j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$, $j(\kappa) > \lambda$ and $j''(\lambda^{\aleph_0}) \in M$.
  - Then $\mathfrak{B} := j(\mathfrak{A}) \restriction j''\,|\mathfrak{A}| \in M$. Since $j \restriction |\mathfrak{A}| \in M$ we also have $\mathfrak{A} \in M$ and $\quad M \models j \restriction |\mathfrak{A}| : \mathfrak{A} \overset{\cong}{\rightarrow} \mathfrak{B}$.
  - By σ-closedness of $\mathbb{P}$ we have $([\,|\mathfrak{A}|\,]^{\aleph_0})^{\mathsf{V}} = ([\,|\mathfrak{A}|\,]^{\aleph_0})^{M}$. Also, all stationary subsets (club subsets resp.) of $([\,|\mathfrak{A}|\,]^{\aleph_0})^{\mathsf{V}}$ remain stationary (club resp.) in M.
- Thus, $M \models \mathfrak{B} \prec_{\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}} j(\mathfrak{A})$, $\|\mathfrak{B}\| < j(\kappa)$, $j(S) = j''S \subseteq |\mathfrak{B}|$.
  By elementarity, in V, there is $\mathfrak{C} \prec_{\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}} \mathfrak{A}$ s.t. $\|\mathfrak{C}\| < \kappa$, $S \subseteq |\mathfrak{C}|$.
  Thus $\kappa \in \mathsf{LSS}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat})$. □ (Theorem 11.)

Back to Generic view of ...

Strong compactness number of stationary logic

Lemma 13. $\mathcal{L}^{\aleph_0}_{stat}$ extends $L(Q)$ (Q_x: there are uncountably many x). It follows that $\mathrm{CS}(\mathcal{L}^{\aleph_0}_{stat}) \subseteq \mathrm{CS}(L(Q))$ and $\mathrm{scn}(L(Q)) \le \mathrm{scn}(\mathcal{L}^{\aleph_0}_{stat})$.

Proof. $stat\, X\, \exists_x (x \not\in X \wedge \ \ldots)$ replaces $Q_x\,(\ldots)$. □ (Lemma 9)

Lemma 14. For all $n \in \omega$ none of $\aleph_n$, $(\beth_n)^+$ belongs to $\mathrm{CS}(L(Q))$. In particular, $\mathrm{scn}(\mathcal{L}^{\aleph_0}_{stat}) \ge \mathrm{scn}(L(Q)) \ge \beth_\omega$.

Proof. We show this for $\aleph_2$ and $(\beth_1)^+$. The general case can be proved by modifications of the theories below. $\aleph_1 \notin \mathrm{CS}(L(Q))$ is trivial.

- The following (non-satisfiable) theory witnesses $\aleph_2 \notin \mathrm{CS}(L(Q))$.

$$T_0 := \{\text{“}E \text{ is a linear ordering”}, \forall x \neg Q_y\,(y\,E\,x)\} \cup \{c_\alpha \not\equiv c_\beta \ : \ \alpha < \beta < \omega_2\}$$

- The following (non-satisfiable) theory witnesses $(2^{\aleph_0})^+ \notin \mathrm{CS}(L(Q))$.

$$T_1 := \{\neg Q_x\, N(x), \forall x \forall y\, (\forall z (N(z) \ \rightarrow \ (z \mathrel{\underset{\sqcup}{\in}} x \leftrightarrow z \mathrel{\underset{\sqcup}{\in}} y)) \ \rightarrow x \equiv y)\} \cup \{c_\alpha \not\equiv c_\beta \ : \ \alpha < \beta < (2^{\aleph_0})^+\}$$

- Lemma 14 shows that a Compactness Spectrum analogue of Theorem 11 is impossible.
- For a class $\mathcal{P}$ of p.o.s, let

$$\mathrm{scn}^{\mathcal{P}}(\mathcal{L}) := \min\{\kappa \in \mathrm{Card} : \text{for any } \mathcal{L}\text{-theory } T, \text{ if } T \text{ is } <\kappa\text{-satisfiable, then there is } \mathbb{P} \in \mathcal{P} \text{ s.t. } \Vdash_{\mathbb{P}} \text{“} \check{T} \text{ is satisfiable in an inner model contianing } \check{T}\text{”}\}.$$

- For a class $\mathcal{P}$ of p.o.s, a cardinal κ is said to be generically supercompact by $\mathcal{P}$ (or $\mathcal{P}$-gen. supercompact, for short) if, for any $\lambda \geq \kappa$, there are $\mathbb{P} \in \mathcal{P}$, $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, j, $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. $\mathsf{V}[\mathbb{G}] \models j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M$ $j(\kappa) > \lambda$ and $j''\lambda \in M$.

Theorem 15. If κ is $\mathcal{P}$-gen. supercompact, then $\kappa \geq \mathrm{scn}^{\mathcal{P}}(\mathcal{L}^{\mathrm{HO}})$.

Theorem 15. If κ is $\mathcal{P}$-gen. supercompact, then $\kappa \geq \mathsf{scn}^{\mathcal{P}}(\mathcal{L}^{\mathrm{HO}})$.

Proof. Suppose that T is an $\mathcal{L}^{\mathrm{HO}}$-theory with $|\,T\,| = \lambda$ and T is $<\kappa$-satisfiable. W.l.o.g., we assume that all non-logical symbols appearing in T are elements of λ and the coding of the formulas is done in an appropriate way.

- Let $\mathbb{P} \in \mathcal{P}$ be s.t., for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are j, $M \subseteq \mathsf{V}[\mathbb{G}]$ with $j : \mathsf{V} \xrightarrow{\preceq}_{\kappa} M$, $j(\kappa) > \lambda$ and (*) $j''\lambda \in M$.
- By elementarity, we have $M \models$"all $T_0 \in [j(T)]^{<j(\kappa)}$ has a model". By the closure property (*) of M, $j''T \in M$. Since $M \models$"$|\,j''T\,| = \lambda < j(\kappa)$" and T is obtained from $j''T$ by renaming of non-logical symbols, we have $T \in M$ and $M \models$" T has a model".

□ (Theorem 15)

Thank you for your attention!
ご清聴ありがとうございました.

관심을 가져 주셔서 감사합니다
Gracias por su atención.
Dziękuję za uwagę.
Grazie per l'attenzione.
Dank u voor uw aandacht.
Ich danke Ihnen für Ihre Aufmerksamkeit.

LSS($\mathcal{L}$) is closed

Lemma 2. For any logic $\mathcal{L}$, LSS($\mathcal{L}$) is a closed class of cardinals.

Proof. Suppose that $\langle \kappa_\alpha : \alpha < \delta \rangle$ is a strictly increasing sequence in LSS($\mathcal{L}$) and $\kappa = \sup_{\alpha<\delta} \kappa_\alpha$. We want to show that $\kappa \in$ LSS($\mathcal{L}$).

- Suppose that $\mathfrak{A}$ is a structure and $S \subseteq [\, |\mathfrak{A}|\,]^{<\kappa}$. Let $\alpha < \delta$ be s.t. $|\, S \,| < \kappa_\alpha$. Since $\kappa_\alpha \in$ LSS($\mathcal{L}$), there is a $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$ s.t. $S \subseteq |\mathfrak{B}|$ and $\|\mathfrak{B}\| < \kappa_\alpha < \kappa$. This shows that $\kappa \in$ LSS($\mathcal{L}$). ◻ (Lemma 2)

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On the restriction to countable signatures

Lemma 1. For a logic $\mathcal{L}$ (with natural properties expected to a "logic"), we have

$$\mathrm{LSS}(\mathcal{L}) = \{\mu \in \mathrm{Card} \ : \ \text{for any structure } \mathfrak{A} \text{ with a signature of size } < \mu, \text{ there is } \mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A} \text{ s.t. } \|\mathfrak{B}\| < \mu\}.$$

Proof. "$\subseteq$": Suppose that $\mu \in \mathrm{LSS}(\mathcal{L})$ and let $\mathfrak{A}$ be a structure with a signature of size $\nu < \mu$. W.l.o.g., we may assume that $\mathfrak{A}$ is a relational structure and $\mathfrak{A} = \langle |\mathfrak{A}|, R_{n,\alpha}\rangle_{n\in\omega,\alpha<\nu}$ where $R_{n,\alpha}$ is an n-ary relation on $|\mathfrak{A}|$ for $n \in \omega$ and $\alpha < \nu$. We may also assume, w.l.o.g., that $\|\mathfrak{A}\| \geq \mu$ and $\nu \subseteq |\mathfrak{A}|$.

▷ Let $R_n := \bigcup_{\alpha<\nu}\{\alpha\} \times R_{n,\alpha}$ for each $n \in \omega$. Let $\mathfrak{A}^- := \langle |\mathfrak{A}|, R_n\rangle_{n\in\omega}$. Applying our assumption on μ, we find $\mathfrak{B}^- \prec_{\mathcal{L}} \mathfrak{A}^-$ with $\|\mathfrak{B}^-\| < \mu$ and $\nu \subseteq |\mathfrak{B}^-|$. By the last condition, we can reconstruct a submodel $\mathfrak{B}$ of $\mathfrak{A}$ from $\mathfrak{B}^-$ with the same underlying set and $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$.

On the restriction to countable signatures (2/2)

Lemma 1. For a logic $\mathcal{L}$ (with natural properties expected to a "logic"), we have

$$\mathsf{LSS}(\mathcal{L}) = \{\mu \in \mathsf{Card} : \text{for any structure } \mathfrak{A} \text{ with a signature of size } < \mu, \text{ there is } \mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A} \text{ s.t. } \|\mathfrak{B}\| < \mu\}.$$

Proof. "$\subseteq$": Suppose that $\mu \in \mathsf{LSS}(\mathcal{L})$ and let $\mathfrak{A}$ be a structure with a signature of size $\nu < \mu$. W.l.o.g., we may assume that $\mathfrak{A}$ is a relational structure and $\mathfrak{A} = \langle |\mathfrak{A}|, R_{n,\alpha} \rangle_{n \in \omega, \alpha < \nu}$ where $R_{n,\alpha}$ is an n-ary relation on $|\mathfrak{A}|$ for $n \in \omega$ and $\alpha < \nu$. We may also assume, w.l.o.g., that $\|\mathfrak{A}\| \geq \mu$ and $\nu \subseteq |\mathfrak{A}|$.
Let $R_n := \bigcup_{\alpha < \nu} \{\alpha\} \times R_{n,\alpha}$ for each $n \in \omega$. Let $\mathfrak{A}^- := \langle |\mathfrak{A}|, R_n \rangle_{n \in \omega}$.
Applying our assumption on μ, we find $\mathfrak{B}^- \prec_{\mathcal{L}} \mathfrak{A}^-$ with $\|\mathfrak{B}^-\| < \mu$ and $\nu \subseteq |\mathfrak{B}^-|$. By the last condition, we can reconstruct an $\mathcal{L}$-elementary submodel $\mathfrak{B}$ of $\mathfrak{A}$ from $\mathfrak{B}^-$ with the same underlying set.

"$\supseteq$": Suppose now that μ is in the set on the right side of the equality. Let $\mathfrak{A}$ be a structure of size $\geq \mu$ with a countable signature, and $S \in [\,|\mathfrak{A}|\,]^{<\mu}$.
Let $\mathfrak{A}^+ = \langle \mathfrak{A}, a \rangle_{a \in S}$. Applying the assumption on μ, we obtain $\mathfrak{B}^+ \prec_{\mathcal{L}} \mathfrak{A}^+$ of size $< \mu$. Denoting by $\mathfrak{B}$ the $\mathfrak{B}^+$ reduced to the original language, we have $\|\mathfrak{B}\| < \mu$, $S \subseteq |\mathfrak{B}|$ and $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$.

□ (Lemma 1)

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