On reflection, hereditarity, and absoluteness of topological properties

Sakaé Fuchino (渕野 昌)

Kobe University, Japan

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2022 年 6 月 7 日 (16:00 〜 JST), 至 Set-Theoretic and Geometric Topology, and their applications to related fields

The following slides are typeset by upLaTeX with beamer class, and presented on UP2 Version 2.0.0 by Ayumu Inoue running on an ipad pro (10.5inch).

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Outline

- Reflection of topological properties
- Reflection number of a topological property
- Reflection number as degree of hereditarity and compactness
- Reflection number of non-metrizability
- Fodor-type Reflection Principle
- Generic large cardinals
- Reflection and non-reflection down to the continuum
- Potential metrizability

* We assume in the following that topological spaces are Hausdorff (to remain on the safe side).

** Some of the results mentioned at the end of the talk appear in a future joint paper with Hiroshi Sakai.

Reflection of topological properties

- For a topological space X satisfying some property P, it is very often the case that there is a subspace/ many subspaces Y of X of smaller size (in terms of cardinality) which also satisfy the property P. This is what I shall call here:

  reflection of the topological property P down to Y (or down to $|Y|$).

- The following facts are such examples:

Proposition 1. Suppose that $X = \langle X, \tau \rangle$ is non-separable. Then there is a subspace Y of X of cardinality $\aleph_1$ which is also non-separable.

Proof.

Proposition 2. ([Hajnal-Juhász 1976]) For a topological space $X = \langle X, \tau \rangle$ and $p \in X$, if $\chi(p, X) = \kappa$ for a regular uncountable κ, then there is a subspace Y of X of cardinality $\leq \kappa$ s.t. $p \in Y$ and $\chi(p, Y) = \kappa$. The condition "of cardinality $\leq \kappa$" above is optimal.

Proof. back to Examples

Reflection number of a topological property

- Suppose that P and Q are properties of topological spaces. In the following, we mainly treat cases where P is a "bad property" for topological spaces satisfying Q in the sense that if X is a topological space with the property Q and $Y \subseteq X$ is a subspace with the property P, then all intermediate spaces Z with $Y \subseteq Z \subseteq X$ satisfy the property P.
- For P and Q as above, we define the reflection number of the property P under the spaces with the property Q as

$$\mathfrak{rn}(P, Q) := \min\{\kappa \in \mathrm{Card} : \text{for any topological space } X \text{ with the property } Q, \text{ if } X \text{ satisfies the property } P, \text{ then there is a subspace } Y \text{ of } X \text{ with the property } P \text{ s.t.} |Y| < \kappa\}$$

- if the class $\{\kappa \in \mathrm{Card} : \ldots\}$ in the definition of $\mathfrak{rn}(P, Q)$ is empty, we define $\mathfrak{rn}(P, Q)$ to be ∞.

back to "Reflection number as ..."

Reflection number of a topological property (2/3)

$\mathfrak{rn}(P, Q) := \min\{\kappa \in \mathrm{Card} :$ for any topological space X with the property Q, if X satisfies the property P, then there is a subspace Y of X with the property P s.t. $|Y| < \kappa\}$

- if the class $\{\kappa \in \mathrm{Card} : ...\}$ in the definition of $\mathfrak{rn}(P, Q)$ is empty, we define $\mathfrak{rn}(P, Q)$ to be ∞.

- We shall write $\mathfrak{rn}(P, \emptyset)$ if the property Q imposes no restrictions.

Example 3. For $P =$"non-separable", Proposition 1 can be reformulated as: $\mathfrak{rn}(P, \emptyset) = \aleph_2$.

Example 4. For a regular uncountable cardinal κ and $P =$"there is a point with χ-character κ", Proposition 2 can be reformulated as: $\mathfrak{rn}(P, \emptyset) = \kappa^+$.

Reflection number of a topological property (3/3)

$\mathfrak{rn}\,(P, Q) := \min\{\kappa \in \mathrm{Card}$: for any topological space X with the property Q, if X satisfies the property P, then there is a subspace Y of X with the property P s.t. $|Y| < \kappa\}$

- if the class $\{\kappa \in \mathrm{Card} : ...\}$ in the definition of $\mathfrak{rn}\,(P, Q)$ is empty, we define $\mathfrak{rn}\,(P, Q)$ to be ∞.

- We shall write $\mathfrak{rn}\,(P, \emptyset)$ if the property Q imposes no restrictions.

Example 5. For P ="there is a point with χ-character $\geq \kappa$", we have $\mathfrak{rn}\,(P, \emptyset) = \infty$.

Proof. Let λ be an uncountable regular cardinal and $X := \lambda + 1$ be with the topology generated from

$$\tau := \{\{\alpha\} : \alpha < \lambda\} \cup \{A \cup \{\lambda\} : A \subseteq \lambda, |\lambda \setminus A| < \lambda\}.$$

- $\chi(\alpha, X) = 1$ for all $\alpha \in \lambda$ and $\chi(\lambda, X) = \lambda$.
- $\chi(\alpha, Y) = 1$ for all $\alpha \in Y$ for any $Y \in [X]^{<\lambda}$. □ (Example 5)

Reflection number as degree of hereditarity and compactness

- The reflection number $\mathfrak{rn}(P, Q)$ for a "bad" property P (in the sense of the previous slide) may be regarded as a degree of hereditarity of P: we can find always a small subspace $Y \subseteq X$ (of cardinality $< \mathfrak{rn}(P, Q)$) for the space X with $X \models P \wedge Q$ s.t. all subspace of X above Y satisfy P.

- $\mathfrak{rn}(P, Q)$ can also be seen as the compactness (in model-theoretic sense) of the property $\neg P$: the contraposition of the property in the definition of $\mathfrak{rn}(P, Q)$ implies the following.

▷ For any topological space $X \models Q$, if $Y \models \neg P$ for all subspace Y of X of cardinality $< \mathfrak{rn}(P, Q)$, then $X \models \neg P$.

Reflection number of non-metrizability

- ([Hajnal-Juhász 1976]) Example 5 actually shows that $\mathfrak{rn}(P,\emptyset)=\infty$ for $P=$ "non-metrizable".

- Note that every metrizable space is first countable. The equality $\mathfrak{rn}(P,Q)=\aleph_1$? for $P=$ "non-metrizable" and $Q=$ "first countable" is known as Hamburger's Problem and (its consistency) is still unsolved.

Proposition 6. ([Hajnal-Juhász 1976]) $\mathfrak{rn}(P,Q)=\infty$ for $P=$ "non-metrizable" and $Q=$ "first countable" is consistent.

Proof. Let κ be a regular cardinal $\geq\aleph_2$ and $S\subseteq E^\kappa_\omega$ $(=\{\alpha<\kappa : \mathrm{cf}(\alpha)=\omega\})$ be a non-reflecting stationary set. Then S as a subspace of κ with the order topology is first countable.

- S is not perfectly normal (nor meta-Lindelöf, and hence not metrizable).
- ▷ We can show by induction on $\alpha<\kappa$ that $S\cap\alpha$ is metrizable.
- Note that S as above exists for $\kappa=\lambda^+$ for λ s.t. $\square_\lambda$ holds.
- Thus, e.g. $V=L$ implies $\mathfrak{rn}(P,Q)=\infty$. □ (Proposition 6.)

Reflection number of non-metrizability (2/4)

- The proof of Theorem 6 shows that we need the consistency strength of quite large large cardinals to get $\mathfrak{rn}(P, Q) < \infty$ for P, Q as above: $\neg\Box_\lambda$ for an end-segment of Card implies the consistency of a Woodin Cardinal.

Theorem 7. ([Bagaria-Magidor 2014]) Let P = "non-metrizable" and Q = "first countable". If κ is ω_1-strongly compact, then $\mathfrak{rn}(P, Q) \leq \kappa$.

Corollary 8. For P = "non-metrizable" and Q = "first countable", $\mathfrak{rn}(P, Q) < \infty$ is independent (modulo a certain large cardinal).

- We shall discuss later more results related to Theorem 7.

Theorem 9. ([Dow 1988]) $\mathfrak{rn}(P, Q) = \aleph_2$ holds for P = "non-metrizable" and Q = "countably compact".

Reflection number of non-metrizability (3/4)

Corollary 10. Suppose that X is locally countably compact and all subspaces Y of X of cardinality $\leq \aleph_1$ are metrizable, then X is first countable.

Proof. Let $p \in X$. Then there is $p \in O \overset{\text{open}}{\subseteq} X$ s.t. $\overline{O}$ is countably compact. By Dow's theorem (Theorem 7) $\overline{O}$ is metrizable. Thus $\chi(p, \overline{O}) \leq \aleph_0$. But then $\aleph_0 \geq \chi(p, O) = \chi(p, X)$. ◻ (Corollary 8)

Proposition 11. It is consistent that $\mathfrak{rn}\,(P, Q) = \infty$ for $P =$ "non-metrizable" and $Q =$ "locally countably compact".

(Fan?) Question For a countably compact space X, if all subspaces of cardinality $\leq \aleph_1$ are first countable, does it follow that X is first countable?

Reflection number of non-metrizability (4/4)

Proposition 11. It is consistent that $\mathfrak{rn}(P, Q) = \infty$ for $P =$ "non-metrizable" and $Q =$ "locally countably compact".

Proof. Suppose that κ is an regular cardinal $\geq \aleph_2$ and $S \subseteq E^\kappa_\omega$ is non-reflecting stationary subset of E^κ_ω.

- ▶ For each $\xi \in S$ let $\ell_\xi \subseteq \xi$ be a set of successor ordinals of order-type ω cofinal in ξ.
- ▷ Let $X := \bigcup\{\ell_\xi : \xi \in S\} \cup S$ with the topology generated from $\tau := \{\{\alpha\} : \alpha \in \bigcup\{\ell_\xi : \xi \in S\}\} \cup \{\ell_\xi \setminus \beta \cup \{\xi\} : \xi \in S, \beta < \xi\}$.
- ▶ X is first countable and locally compact.
- ▷ $X \cap \beta$ is metrizable fore all $\beta < \kappa$ (by induction).
- ▷ X is not meta-Lindelöf (use Fodor's Lemma).
- ▶ Thus, similarly to the proof of Proposition 6, $V = L$ implies $\mathfrak{rn}(P, Q) = \infty$. □ (Proposition 11)

Fodor-type Reflection Principle

- Fodor-type Reflection Principle (FRP for short) is the following assertion:

FRP : For all regular $\kappa \geq \aleph_1$, any stationary $S \subseteq E^\kappa_\omega$ and mapping $g : S \to [\kappa]^{\leq \aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that

(1) $\mathrm{cf}(I) = \omega_1$;

(2) $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;

(3) for any regressive $f : S \cap I \to \kappa$ s.t. $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ s.t. $f^{-1}{}''\{\xi^*\}$ is stationary in $sup(I)$.

- FRP follows from Martin's Maximum but in contrast to it, FRP is preserved by ccc generic extension. Hence FRP is compatible with any size of the continuum (it can be also forced under CH).

Theorem 12. ([Fuchino-Juhász-Soukup-Szentmiclóssy-Usuba 2010], [Fuchino-Sakai-Soukup-Usuba ∞]) $\mathfrak{rn}\,(P, Q) = \aleph_2$ for $P =$ "non-metrizable" and $Q =$ "locally countably compact" is equivalent to FRP.

Generic large cardinals

- For a class $\mathcal{P}$ of p.o.s, a cardinal κ is said to be generically supercompact by $\mathcal{P}$ ($\mathcal{P}$-gen. supercompact for short) if, for any $\lambda \geq \kappa$, there are σ-closed p.o. $\mathbb{P}$, $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, j, $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. $\mathsf{V}[\mathbb{G}] \models j : \mathsf{V} \overset{\prec}{\to}_\kappa M$ (M is transitive, κ is the critical point of j), $j(\kappa) > \lambda$ and $j''\lambda \in M$.

Theorem 13. ([Dow-Tall-Weiss 1990]) Suppose that X is a non-metrizable space, $\delta \in \mathsf{Card}$ and $\mathbb{P} = \mathrm{Fn}(\delta, 2)$, the p.o. with finite conditions adding δ many Cohen reals. Then we have

$$\Vdash_{\mathbb{P}} \text{“}\check{X} \text{ is non-metrizable”}.$$

Proposition 14. (see [Fuchino-O.M.Rodrigues-Sakai 202?]) If κ is Cohen-gen. supercompact, then $\mathfrak{rn}(P, Q) \leq \kappa$ for $P =$ “non-metrizable” and $Q =$ “first countable”.

Proof

Corollary 15. ([Dow-Tall-Weiss 1990]) $\mathfrak{rn}(P, Q) \leq 2^{\aleph_0}$ is consistent modulo large cardinals for P, Q as above.

Proof

Reflection and non-reflection down to the continuum

- Let P := "non-metrizable", Q := "first countable" and Q_0 := locally countably compact.
- The statement of Corollary 15 can be still improved by starting from two supercompact cardinals.

Theorem 16. (see [Fuchino-O.M.Rodrigues-Sakai 202?]) $\mathfrak{rn}\,(P, Q_0) = \aleph_2 + \mathfrak{rn}\,(P, Q) \leq 2^{\aleph_0}$ is consistent modulo certain large cardinals.

Proof. Start from two supercompact cardinals κ_0, κ_1 with $\kappa_0 < \kappa_1$.

- Use κ_0 to force FRP. The forcing can be chosen to be small enough so that the supercompactness of κ_1 survives the extension.
- In the generic extension, force with $\mathbb{P} = \mathrm{Fn}(\kappa_1, 2)$. Since $\mathbb{P}$ is ccc, FRP survives in the second generic extension. Thus, by Theorem 12, $\mathfrak{rn}\,(P, Q_0) = \aleph_2$ holds in the second generic extension.
- In the second generic extension, we have $\kappa_1 = 2^{\aleph_0}$ and it is Cohen-gen. supercompact. Thus by Proposition 14 we have $\mathfrak{rn}\,(P, Q) \leq 2^{\aleph_0}$. □ (Theorem 16)

Reflection and non-reflection down to the continuum (2/3)

- Let $P :=$ "non-metrizable", $Q :=$ "first countable" and $Q_0 :=$ locally countably compact.

Theorem 17. (van Douwen, see [Fuchino-O.M.Rodrigues-Sakai 202?])
$\mathfrak{b} < \mathfrak{rn}(P, Q)$.

Theorem 18. (see [Fuchino-O.M.Rodrigues-Sakai 202?])
$\mathrm{MA} + \mathfrak{rn}(P, Q_0) = \aleph_2 + \mathfrak{rn}(P, Q) \not\leq 2^{\aleph_0}$ is consistent modulo certain large cardinals.

Proof. The proof is similar to that of Theorem 16. Start again from two supercompact cardinals $\kappa_0 < \kappa_1$.

- Use κ_0 to force FRP and then force $\mathrm{MA} + 2^{\aleph_0} = \kappa_1$ by a ccc p.o..
- ▷ FRP is preserved by ccc of the second extension.
- In the second generic extension, we forced $\mathfrak{b} = 2^{\aleph_0}$. Hence by Theorem 17, we have $\mathfrak{rn}(P, Q) \not\leq 2^{\aleph_0}$. □ (Theorem 18)

back to "Potential ... (3/3)"

Reflection and non-reflection down to the continuum (3/3)

- Let $P :=$ "non-metrizable", $Q :=$ "first countable" and $Q_0 :=$ locally countably compact.
- In the proof of Theorem 18, κ_1 is ccc-gen. supercompact.
- The construction in the proofs of Theorem 16 and Theorem 18 can be further refined by using certain new type of mixed support iteration to make $2^{\aleph_0}$ a strongform of gen. large carinal (what we called Laver-generic large cardinal) to obtain strong stationary reflection type properties or Rado Conjecture type reflection with reflection number around the continuum together with either $\mathfrak{rn}\,(P, Q) \leq 2^{\aleph_0}$ or $\mathfrak{rn}\,(P, Q) \not\leq 2^{\aleph_0}$.

  (see [Fuchino-O.M.Rodrigues-Sakai 202?])

▷ These results suggest that the reflection of non-metrizability should be regarded as a reflection of the type quite different from the other more standard reflection properties.

Potential metrizability

- The consistency of $\mathfrak{rn}(P, Q) = \aleph_2$ for P = "non-metrizable", and Q = "first countable". (Hamburger's Problem) is very difficult to establish since non-metrizability of a topological space in general is neither preserved by σ-closed p.o. nor by arbitrary ccc p.o. (cf. the proof of Proposition 14) — non-absoluteness of non-metrizability for σ-closed or ccc generic extensions.

- We can often solve a problem by changing the question itself:
  - For a class $\mathcal{P}$ of p.o.s, let

$$\mathfrak{rn}^{\mathcal{P}}(P, Q) := \min\{\kappa : \text{for any topological space } X \text{ with } X \models Q, \text{ if all subspace of } X \text{ of size} < \kappa \text{ satisfy } P, \text{ then there is } \mathbb{P} \in \mathcal{P} \text{ s.t. } \Vdash_{\mathbb{P}} \text{“} X \text{ satisfies } P \text{”}\}$$

  - Cf.

$$\mathfrak{rn}(P, Q) = \min\{\kappa : \text{for any topological space } X \text{ with } X \models Q, \text{ if all subspace of } X \text{ of size} < \kappa \text{ satisfy } P, \text{ then } X \text{ satisfies } P\}$$

Potential metrizability (2/3)

- For a class $\mathcal{P}$ of p.o.s, let

$$\mathfrak{rn}^{\mathcal{P}}(P,Q) := \min\{\kappa : \text{for any topological space } X \text{ with } X \models Q, \text{ if all subspace of } X \text{ of size } < \kappa \text{ satisfy } P, \text{ then there is } \mathbb{P} \in \mathcal{P} \text{ s.t. } \Vdash_{\mathbb{P}} \text{“} X \text{ satisfies } P\text{”}\}$$

- Cf.

$$\mathfrak{rn}(P,Q) = \min\{\kappa : \text{for any topological space } X \text{ with } X \models Q, \text{ if all subspace of } X \text{ of size } < \kappa \text{ satisfy } P, \text{ then } X \text{ satisfies } P\}$$

$$\mathfrak{rn}^{\mathcal{P}-}(P,Q) := \min\{\kappa : \text{for any topological space } X \text{ with } X \models Q, \text{ if all subspace of } X \text{ of size } < \kappa \text{ satisfy } P, \text{ then there is } \mathbb{P} \in \mathcal{P} \text{ s.t. } \Vdash_{\mathbb{P}} \text{“in an inner model, } X \text{ satisfies } P\text{”}\}$$

- If $\mathcal{P}$ contains the trivial p.o., $\mathfrak{rn}^{\mathcal{P}-}(P,Q) \leq \mathfrak{rn}^{\mathcal{P}}(P,Q) \leq \mathfrak{rn}(P,Q)$.

Potential metrizability (2/3)

Theorem 19. For regular cardinals κ, μ with $\mu < \kappa$ the following are equivalent:

(a) κ is μ^+-cc gen. supercompact.

(b) For any $\lambda > \kappa$, there are μ-cc $\mathbb{P}$, $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ and j, $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. $\mathsf{V}[\mathbb{G}] \models j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$, $j(\kappa) > \lambda$ and $([M]^\lambda)^{\mathsf{V}[G]} \subseteq M$.

- Let, again, $P :=$ "non-metrizable", and $Q :=$ "first countable".

- An argument similar to that of the proof of Proposition 14 combined with Theorem 19 proves the following:

Theorem 20. If κ is ccc gen. supercompact, then $\mathfrak{rn}^{\mathsf{ccc}}(P, Q) \leq \kappa$. □

- Theorem 19 is not available for σ-closed p.o.s. Though we still have

Theorem 21. If κ is σ-closed gen. supercompact, then $\mathfrak{rn}^{\sigma\text{-closed}-}(P, Q) \leq \kappa$. □

Potential metrizability (3/3)

- Let, again, $P :=$ "non-metrizable", $Q :=$ "first countable" and $Q_0 :=$ "locally compact".

- Theorem 20 on the last slide combined with the idea of Theorem 18 implies the following:

Corollary 22. $\mathsf{MA} + \mathfrak{rn}\,(P_0, Q) = \aleph_2$, $\mathfrak{rn}^{\,\mathsf{ccc}}(P, Q) \leq 2^{\aleph_0}$ + $\mathfrak{rn}\,(P, Q) \not\leq 2^{\aleph_0}$ is consistent modulo large cardinals.

- For a supercompact κ, if all cardinals below κ are collapsed to cardinality $\aleph_1$ by σ-closed forcing, $\kappa = \aleph_2$ in the generic extension is σ-closed gen. supercompact. This together with Theorem 21 implies:

Corollary 23. $\mathfrak{rn}^{\,\sigma\text{-closed}-}(P, Q) = \aleph_2$ is consistent modulo large cardinals.

Thank you for your attention!
ご清聴ありがとうございました.

관심을 가져 주셔서 감사합니다
Gracias por su atención.
Dziękuję za uwagę.
Grazie per l'attenzione.
Dank u voor uw aandacht.
Ich danke Ihnen für Ihre Aufmerksamkeit.

Corollary 15. (see [Fuchino-O.M.Rodrigues-Sakai 202?]) $\mathfrak{rn}(P, Q) \le 2^{\aleph_0}$ is consistent modulo large cardinals for P, Q as above.

Proof. Suppose that κ is a supercompact cardinal. Let $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$ and let $\mathbb{G}$ be a $(\mathsf{V}, \mathbb{P})$-generic filter.

- ▶ Then $\mathsf{V}[\mathbb{G}] \models$ "κ" $= 2^{\aleph_0}$ and κ is a Cohen-gen. supercompact in $\mathsf{V}[\mathbb{G}]$.
- ▷ Thus, by Proposition 14, we have $\mathsf{V}[\mathbb{G}] \models \mathfrak{rn}(P, Q) \le 2^{\aleph_0}$.

□ (Corollary 15)

back

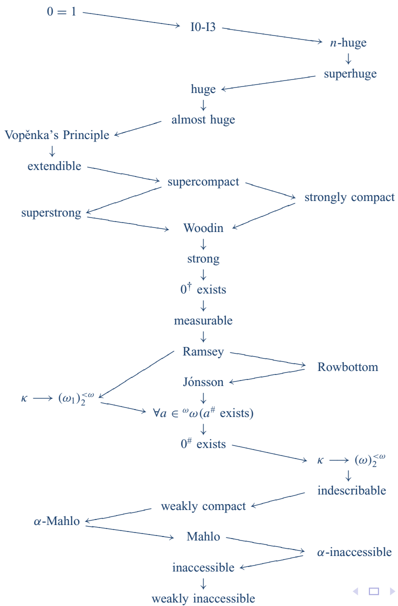
Proposition 14. (see [Fuchino-O.M.Rodrigues-Sakai 202?])
If κ is Cohen-gen. supercompact, then $\mathfrak{rn}(P, Q) \leq \kappa$ for $P =$ "non-metrizable" and $Q =$ "first countable".

Proof. Suppose that X is a non-metrizable space with
(0) $\chi(p, X) \leq \aleph_0$ for all $p \in X$.

- W.l.o.g., $X = \langle \theta, \tau \rangle$ for some ordinal θ and an open base τ on θ. Let $\lambda \geq \theta$ be sufficiently large and let $\mathbb{P} = \mathrm{Fn}(\mu, 2)$ for some cardinal μ s.t., for a $(\mathsf{V}, \mathbb{P})$-generic filer $\mathbb{G}$, there are classes j, $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. (1) $\mathsf{V}[\mathbb{G}] \models j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$, (2) $j(\kappa) > \lambda$ and (3) $j''\lambda \in M$.
- Let $\tau'' = \{j(O) \cap j''\theta : O \in \tau\}$. Then we have $\langle j''\theta, \tau'' \rangle$, $\langle \theta, \tau \rangle \in M$, and $M \models \langle \theta, \tau \rangle \cong \langle j''\theta, \tau'' \rangle$ by (3).
- By Dow-Tall-Weiss theorem (Theorem 13), $\mathsf{V}[\mathbb{G}] \models$" $\langle j''\theta, \tau'' \rangle$ is non-metrizable".
- ▷ By (0), $M \models$" $\langle j''\theta, \tau'' \rangle$ is a sub-space of $\langle j(\theta), j(\tau) \rangle$".
- Thus, $M \models$" there is a non-metrizable subspace Y of $j(X)$ of cardinality $< j(\kappa)$". By elementarity, it follows that $\mathsf{V} \models$" there is a non-metrizable subspace Y of X of cardinality $< \kappa$". □

Chart of Large cardinals in [Kanamori 2003]

The arrows indicates direct implications or relative consistency implications, often both.



Proposition 2. (Hajnal and Juhász) For a topological space $X = \langle X, \tau \rangle$ and $p \in X$, if $\chi(p, X) = \kappa$ for a regular uncountable κ, then there is a subspace Y of X of cardinality $\leq \kappa$ s.t. $p \in Y$ and $\chi(p, Y) = \kappa$. The condition "of cardinality $\leq \kappa$" above is optimal.

Proof. Let θ be sufficiently large regular cardinals and let $M \prec \mathcal{H}(\theta)$ be s.t. p, X, τ, $\kappa \in M$, $\kappa \subseteq M$ and $|M| = \kappa$.

- Let $Y := X \cap M$.
- Let $\{B_\alpha : \alpha < \kappa\}$ be a neighborhood basis of p in X of size κ s.t. $\langle B_\alpha : \alpha < \kappa \rangle \in M$ (there is such a sequence by elementarity). ▷ The following Claims says implies Y is as desired.

Claim. $\{B_\alpha \cap Y : \alpha < \kappa\}$ is a neighborhood basis of p in Y and no $\{B_\alpha \cap Y : \alpha \in I\}$ for $I \in [\kappa]^{<\kappa}$ is a neighborhood basis of p in Y.

$\vdash$ For the second half of the claim, suppose $I \in [\kappa]^{<\kappa}$. By regularity of κ, there is $\beta < \kappa$ s.t. $I \subseteq \beta$. $\mathcal{H}(\theta)$ knows that $\{B_\alpha : \alpha < \beta\}$ is not a neighborhood basis of p. Thus, M also knows it. It follows that $\{B_\alpha \cap Y : \alpha < \beta\}$ is not a neighborhood basis of p in Y. Hence neither $\{B_\alpha : \alpha \in I\}$. $\dashv$

(2/2)

Proposition 2. (Hajnal and Juhász) For a topological space $P = \langle P, \tau \rangle$ and $x \in P$, if $\chi(p, X) = \kappa$ for a regular uncountable κ, then there is a subspace Y of X of cardinality $\leq \kappa$ s.t. $p \in Y$ and $\chi(p, Y) = \kappa$. The condition "of cardinality $\leq \kappa$" above is optimal.

Proof.

- To see that the condition "of cardinality $\leq \kappa$" above is optimal (in particular, that it cannot replaced by "of cardinality $< \kappa$"), consider e.g. $X = \kappa + 1$ with order topology of the canonical ordering of $\kappa + 1$:

▷ $\chi(\kappa, X) = \kappa$.

▷ For any $Y \in [\kappa + 1]^{<\kappa}$ with $\kappa \in Y$, $\chi(\kappa, Y) \leq |\, Y \,| < \kappa$.

□ (Proposition 2)

back

Proposition 1. Suppose that $X = \langle X, \tau \rangle$ is non-separable. Then there is a subspace Y of X of cardinality $\aleph_1$ which is also non-separable.

Proof.

- Note that $|X| \geq \aleph_1$.
- Let θ be a sufficiently large regular cardinal with $\langle X, \tau \rangle \in \mathcal{H}(\theta)$. Let $M \prec \mathcal{H}(\theta)$ be internally cofinal (w.r.t. countable sets: M is said to be internally cofinal if $[M]^{\aleph_0} \cap M$ is cofinal in $[M]^{\aleph_0}$ w.r.t. $\subseteq$) s.t. $\langle X, \tau \rangle \in M$ and $|M| = \aleph_1$. Let $Y := X \cap M$.
- Then $|Y| = \aleph_1$.
- We show that Y (as a subspace of X) is non-separable: Suppose that $c \subseteq Y$ is an arbitrary countable set. Since $c \subseteq M$, there is a countable set $a \in M$ with $c \subseteq a \subseteq Y$. By elementarity, $M \models$"a is not a dense subset of X". Thus, there is $O \in \tau \cap M$ and $y \in X \cap M$ $(= Y)$ s.t. $y \in O$ and $M \models a \cap O = \emptyset$. It follows that $a \cap O = \emptyset$ by elementarity. This shows that a is not a dense subset of Y and hence $c \subseteq a$ is neither a dense subset of Y. □ (Proposition 1)

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