A reflection principle as a reverse-mathematical fixed point over the base theory ZFC

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dedicated to Professor Dr. Kazuyuki Tanaka
on the occasion of his 60th birthday

1 Reverse-mathematical fixed points

In reverse mathematics, five subtheories of second order arithmetic (the so called Big 5) are put under the spotlight. These theories are considered as being central, for one thing, because each of them is equivalent to many classical theorems over the base

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theory RCA_0. For example the axiom system ACA_0 which is often considered to be the modern representation of the system Hermann Weyl introduced in [20] (see also [11]) is known to be equivalent to the system RCA_0 + Bolzano-Weierstrass theorem or to the system RCA_0 + Ascoli's theorem (see e.g. [18] for more such examples).

The same criterion for determining the significance (or at least the relevance) of axioms (or principles) can be also applied to assertions over other base theories.

The Axiom of Choice (AC) for example can be considered to be significant because of a long list of mathematical theorems which are known to be equivalent to it over the base theory ZF.

Some of such theorems are:

- *Zorn's lemma (Max Zorn 1935, Kazimierz Kuratowski 1922);*
- *"Any product of compact spaces is compact" (John L. Kelley 1950);*
- *"Every commutative unital ring has a maximal ideal" (Wilfrid Hodges 1974);*
- *"Every linear algebra over some field has a linear basis" (Andreas Blass 1984);*
- *etc.*

Similar phenomena are also observed about the Continuum Hypothesis (CH) over ZFC. Among many others, each of the following mathematical theorems are known to be equivalent to CH over the base theory ZFC:

- *"There is a decomposition of the plane* $\mathbb{R}^2 = S_1 \dot\cup S_2$ *such that* $S_1 \cap \{\langle a, y\rangle \in \mathbb{R}^2 : y \in \mathbb{R}\}$ *is countable for all* $a \in \mathbb{R}$ *and* $S_2 \cap \{\langle x, b\rangle \in \mathbb{R}^2 : x \in \mathbb{R}\}$ *is countable for all* $b \in \mathbb{R}$*" (Wacław Sierpiński 1919).*
- *"There is an uncountable collection* $\mathcal{F}$ *of analytic (complex) functions such that the set* $\{f(z) : f \in \mathcal{F}\}$ *is countable for every* $z \in \mathbb{C}$*" (Paul Erdős, 1964).*
- *"*$\mathbb{R}$ *can be decomposed into countably many sets* X_n $n \in \omega$ *such that each* X_n *is linearly independent over* $\mathbb{Q}$*" (Paul Erdős and Shizuo Kakutani, 1943).*
- *"*$\mathbb{R}^2$ *can be covered by 3 clouds" (Péter Komjáth, 2001).*
- *etc.*

Here a subset C of the plane $\mathbb{R}^2$ is said to be a *cloud* if there is $\mathbf{a} \in \mathbb{R}^2$ such that the intersection of C with each line in $\mathbb{R}^2$ containing $\mathbf{a}$ is at most countable.

We have to emphasize here that a multitude of mathematical assertions equivalent to a statement over ZF, ZFC or some other set-theoretic axiom system does not imply the truth of the statement. This is true in particular with CH.

While the majority of mathematicians seems to believe naïvely the truth of CH possibly because of the multitude of the mathematical "theorems" which are either equivalent to CH or become true under it, many set-theorists believe (or at least prefer) the negation of it because of many other reasons.

In the following we shall call the criterion for the importance of an axiom measured by the number of mathematical assertions equivalent to it over a base theory *the reverse-mathematical criterion*.

The set-theoretic reflection principle which Lajos Soukup and I called *the Fodor-type Reflection Principle* (FRP) is also one of such statements as CH satisfying this reverse-mathematical criterion over ZFC.

FRP was introduced around 2008. The name of the principle was coined in this way because of the reminiscence of the Fodor's Lemma in the original formulation of the principle (see the definition below).

This principle proved to be a very useful infinite-combinatorial tool for proving many statements originally known to be consequences of Fleissner's Axiom R*. Then, by studying the set-theoretic characterizations of FRP, we found out that the principle is even equivalent to many of these statements and the list of the mathematical statements equivalent to FRP over ZFC is growing ever since.

The equivalence over ZFC between these mathematical statements seems to be not so obvious but it is established at once when we prove the equivalence of each of them to FRP. Since FRP can be proved to be independent of CH and consistent with arbitrary large continuum, we can conclude that all of these mathematical statements are also independent of CH and consistent with arbitrary large continuum.

In the following, we will give a survey of the results around the statements equivalent to FRP and some other related results in light of the reverse-mathematical criterion.

*It is easy to see that FRP follows from Axiom R. The reverse implication does not hold: Axiom R implies $2^{\aleph_0} \leq \aleph_2$ while FRP is consistent with $2^{\aleph_0} > \aleph_2$.

2 The Fodor-type Reflection Principle

The Fodor-type Reflection principle (FRP) was originally formulated as the following set-theoretic reflection principle:

(FRP): For any regular cardinal κ and stationary $S \subseteq E^{\omega}_{\kappa}$ and mapping $g : S \to [\kappa]^{\leq\aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that

(2.1) $cf(I) = \omega_1$;

(2.2) $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;

(2.3) for any regressive $f : S \cap I \to \kappa$ such that $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ such that $f^{-1}{}''\{\xi^*\}$ is stationary in $\sup(I)$.

Here we define $E^{\kappa}_{\lambda} = \{\alpha \in \lambda : cf(\alpha) = \kappa\}$ for cardinal λ and a regular cardinal $\kappa < \lambda$. A mapping $f : S \to \mathrm{On}$ for a set $S \subseteq \mathrm{On}$ is said to be *regressive* if $f(\alpha) < \alpha$ holds for all $\alpha \in S$. A subset S of an ordinal α is *stationary* if it intersects with all closed unbounded (club) subset of α.

There are may equivalent set-theoretical variations of the statement of FRP. For example the following formulation of FRP is recently used to obtain the equivalence of the statements (G) below to FRP over ZFC in [4]:

(FRP') For any regular $\kappa > \omega_1$, any stationary $E \subseteq E^{\omega}_{\kappa}$ and any mapping $g : E \to [\kappa]^{\aleph_0}$, there is $\alpha^* \in E^{\omega_1}_{\kappa}$ such that

(2.4) α^* is closed with respect to g (that is, $g(\alpha) \subseteq \alpha^*$ for all $\alpha \in E \cap \alpha^*$) and, for any $I \in [\alpha^*]^{\aleph_1}$ closed with respect to g, closed in α^* with respect to the order topology and with $\sup(I) = \alpha^*$, if $\langle I_\alpha : \alpha < \omega_1 \rangle$ is a filtration of I then $\sup(I_\alpha) \in E$ and $g(\sup(I_\alpha)) \cap \sup(I_\alpha) \subseteq I_\alpha$ hold for stationarily many $\alpha < \omega_1$.

The simplest way to force FRP is to start from a strongly compact cardinal κ and Lévy collapse it to ω_2 by countable conditions.

Conversely it can be seen that a quite high consistency strength is involved in FRP as follows: it is relatively easy to prove that FRP implies the following (A) and square principle at any cardinal produces a topological space which is a counter example to

(A) and hence to FRP (see [6]). Thus we obtain the theorem that FRP implies the total failure of the square principle*.

It is known that the total filure of the square principle has a very high consistency strength (at least that of infinitely many Woodin cardinals).

Actually with a modification of the total failure of the square principle we obtain another set-theoretic characterization of FRP which is used to show that the following mathematical statements (A) – (G) are equivalent to FRP.

For a regular cardinal κ, we define $\mathrm{ADS}^-(\kappa)$ to be the assertion that there is a stationary set $S \subseteq E^\omega_\kappa$ and a sequence $\langle A_\alpha : \alpha \in S\rangle$ such that

(2.5) $A_\alpha \subseteq \alpha$ and $\mathrm{otp}(A_\alpha) = \omega$ for all $\alpha \in S$;

(2.6) for any $\beta < \kappa$, there is a mapping $f : S \cap \beta \to \beta$ such that $f(\alpha) < \sup(A_\alpha)$ for all $\alpha \in S \cap \beta$ and $A_\alpha \setminus f(\alpha)$, $\alpha \in S \cap \beta$ are pairwise disjoint.

Theorem 2.1 ([7]) FRP *is equivalent over* ZFC *to the assertion that* $\mathrm{ADS}^-(\kappa)$ *does not hold for all regular* $\kappa > \omega_1$*.* ❏

The following assertions (A) – (G) are known to be equivalent with FRP over ZFC.

Theorem 2.1 is used to prove that each of the following mathematical reflection statements (A) – (G) implies FRP.

A topological space X is *countably tight* if, for every $U \subseteq X$ and $a \in \overline{U}$, there is $U_0 \in [U]^{\aleph_0}$ such that $a \in \overline{U_0}$. X is *meta-Lindelöf* if every open cover of $\mathcal{B}$ of X has a point countable open refinement where a family of sets $\mathcal{B}$ is said to be *point countable* if $\{B \in \mathcal{B} : x \in B\}$ is countable for all $x \in \bigcup \mathcal{B}$.

(A) For every locally separable countably tight topological space X, if all subspaces of X of cardinality $\leq \aleph_1$ are meta-Lindelöf, then X itself is also meta-Lindelöf ([6] and [7]).

(B) For every locally countably compact topological space X with, if all subspaces of X of cardinality $\leq \aleph_1$ are metrizable, then X itself is also metrizable ([6] and [7]).

*A more natural explanation of the implication of the total failure of the square from FRP for set-theorists would be the following: FRP implies the reflection of stationarity of subsets of E^ω_κ for all regular κ. It is known that the square principle $\square_\kappa$ produces a non-refleting stationary subset of $E^\omega_{\kappa^+}$.

A topological X space is said to be *left-separated* if there is a well-ordering $\sqsubset$ on X such that all initial segments of X with respect to $\sqsubset$ are closed.

(C) For every metrizable space X, if all subspaces of X of cardinality $\leq \aleph_1$ are left-separated then X itself is also left-separated ([2] and [7]).

The proof of the assertions (A) – (C) from FRP is relatively easy (see [6]). (B) follows from (A). The proofs of (A) and (C) from FRP are done by induction on the Lindelöf number of the spaces where regular uncountable cardinal case is done by FRP while singular case can be treated straightforwardly. The proof of the opposite direction is also easy modulo Theorem 2.1: it is done by constructing a counter-examples to (A), (B) and (C) assuming that $\mathrm{ADS}^-(\kappa)$ holds for some regular cardinal.

The following reflection theorem on coloring number of graphs is also equivalent to FRP.

For a graph $G = \langle G, \mathcal{E} \rangle$ (where $\mathcal{E} \subseteq [G]^2$) *the coloring number* of G is defined by:

$$\text{(2.7)} \quad \mathrm{col}(G) = \min\{\mu : \text{there is a well-ordering } \sqsubset \text{ of } G \text{ such that } |\{y \in G : y \sqsubset x \text{ and } \{x,y\} \in \mathcal{E}\}| < \mu \text{ for all } x \in G\}.$$

(D) Any uncountable graph G has countable coloring number if all induced subgraphs of G of cardinality $\aleph_1$ have countable coloring number ([7]).

The proof of (D) from FRP is by induction on the cardinality of G. The singular cardinal case of this proof uses Shelah's Singular Compactness Theorem.

It is easy to see that the chromatic number of a graph G is less than or equal to the coloring number of G. The assertion obtained from (D) by replacing coloring number by chromatic number is false by a theorem of Erdős and Hajnal.

The statement obtained from (D) by replacing coloring number by list-chromatic number is consistent but it is independent of FRP (see [9]).

The following statement on reflection of strong separation property of topological spaces is also equivalent to FRP.

A topological space X is said to be *collectionwise Hausdorff* (*cwH*, for short) if, for any closed and discrete $D \subseteq X$, there is a family $\mathcal{U}$ of pairwise disjoint open sets such that, for all $d \in D$, there is $U \in \mathcal{U}$ with $D \cap U = \{d\}$.

A topological space X has *local density* $\leq \kappa$, if for every $p \in X$, there is a $Y \in [X]^{\leq\kappa}$ such that $p \in int(\overline{Y})$.

(E) For every countably tight topological space X of local density $\leq \aleph_1$, if X is $\leq \aleph_1$-cwH, then X is cwH ([7]).

The proof of the equivalence of the next statement with FRP is the most complicated among the equivalence proofs of the statements cited in this section.

For a Boolean algebra B, a subalgebra A of B is said to be *relatively complete* in B (notation: $A \leq_{\mathrm{rc}} B$) if $\sum^A \{a \in A : a \leq_A b\}$ exists for any $b \in B$. A Boolean algebra B is said to be *openly generated* if $\{A \in [B]^{\aleph_0} : A \leq_{\mathrm{rc}} B\}$ contains a club subset of $[B]^{\aleph_0}$. The notion of the open generatedness was first introduced by E.V. Ščepin in the context of topological spaces. Lutz Heindorf then translated this notion into Boolean algebras via Stone duality. Lutz Heindorf and Leonid Shapiro called the openly generated Boolean algebras "rc-filtered Boolean algebras" in [12].

(F) A Boolean algebra B is openly generated if the set $\{A \in [B]^{\aleph_1} : B \text{ is openly generated}\}$ contains a club subset of $[B]^{\aleph_1}$ ([8]).

The proof of (F) from FRP uses various lemmas on openly generated algebras provided in [12] and [1].

At the κth step in the induction proof of (F) from FRP where κ is a successor to a singular cardinal, we use a strengthening of Singular Cardinal Hypothesis known as Shelah's Strong Hypothesis (SSH) which is shown to follow from FRP in [8]. The proof of SSH from FRP given in [8] uses some heavy tools from Shelah's Cardinal Arithmetic. Later Hiroshi Sakai found a more direct proof of SSH from FRP ([16]).

SSH was introduced in [17] where it was simply called Strong Hypothesis. We just skip the original definition of this principle since it involves a good deal of Shelah's PCF theory which cannot be explained in short. The reader may be consider Theorem 2.2, (b) as an alternative definition of SSH,

SSH by itself is a prominent principle in terms of the reverse-mathematical criterion over ZFC. This can be seen in the following theorem:

Theorem 2.2 *The following are equivalent:*

(a) SSH*;*

(b) $cf([\kappa]^\theta, \subseteq) = \kappa$ *holds for all cardinals* κ*,* θ *with* $\theta < cf(\kappa)$ *(Saharon Shelah, see [15]);*

(c) $cf([\kappa]^{\aleph_0}, \subseteq) = \kappa^+$ *for all singular cardinals* κ*, with* $cf(\kappa) = \omega$ *(Saharon Shelah, see [15]);*

(d) *For any countably tight topological space X, if X is $< \aleph_1$-thin, then X is thin ([8]);*

(e) *For any countably tight topological space X, if X is $< \kappa$-thin for $\kappa = \max\{\aleph_1, d(X)\}$, then X is thin ([8]).*

Here a topological space X is said to be *thin* if, for any $D \subseteq X$, we have $|\,\overline{D}\,| \leq |\,D\,|^+$. For a cardinal κ, X is said to be $< \kappa$*-thin* if, for any $D \in [X]^{<\kappa}$ we have $|\,\overline{D}\,| \leq |\,D\,|^+$.

Let us finish listing assertions equivalent to FRP by citing just one more assertion equivalent to FRP, a reflection statement obtained quite recently about the non-existence of orthonormal bases of pre-Hilbert spaces.

An inner product space X over the scalar field $K = \mathbb{R}$ or $\mathbb{C}$ is also often called a *pre-Hilbert space*. An orthonormal system $\mathcal{S} \subseteq X$ is said to be an *orthonormal basis* if $\mathcal{S}$ spans a dense sub-linear space of X. Paul Halmos found in 1960's that there are pre-Hilbert spaces without any orthonormal bases. By Bessel's inequality it is easy to see that all maximal orthonormal bases of a pre-Hilbert space X have the same cardinality. This cardinality is called the *dimension* of the pre-Hilbert space X. Halmos' example of pre-Hilbert spaces X were such that the dimension of X is strictly less than the density of X. In [4] it is proved that there are also pre-Hilbert spaces without any orthonormal bases whose dimension is equal to the density.

For a cardinal κ, let

(2.8) $\quad \ell_2(\kappa) = \{f \in {}^{\kappa}K \,:\, \sum_{\alpha\in\kappa} |f(\alpha)|^2 < \infty\}.$

$\ell_2(\kappa)$ equipped with coordinatewise addition and scalar multiplication as well as the inner product defined by

(2.9) $\quad (f, g) = \sum_{\alpha\in\kappa} f(\alpha) \cdot \overline{g(\alpha)}\ \text{ for } f,\, g \in \ell_2(\kappa)$

is a/the Hilbert space with density κ.

For any pre-Hilbert space with $d(X) = \kappa$ we may assume without loss of generality that X is a dense sub-inner-product-space of $\ell_2(\kappa)$.

For $f \in \ell_2(\kappa)$, let $\mathrm{supp}(f) = \{\alpha \in \kappa \,:\, f(\alpha) \neq 0_K\}$. By the definition (2.8), $\mathrm{supp}(f)$ is a countable set for all $f \in \ell_2(\kappa)$.

For a pre-Hilbert space X with $X \underset{dense}{\subseteq} \ell_2(\kappa)$ and $S \subseteq \kappa$, let

(2.10) $\quad X \downarrow S = \{f \in \ell_2(\kappa) \,:\, f \in X \text{ and } \mathrm{supp}(f) \subseteq S\}.$

Let us call a pre-Hilbert space X *pathological* if there is no orthonormal basis of X. The following assertion is also equivalent to FRP over ZFC:

(G) For any regular $\kappa > \omega_1$ and any dense sub-inner-product-space X of $\ell_2(\kappa)$, if X is pathological then

$$S_X^{\aleph_1} = \{U \in [\kappa]^{\aleph_1} : X \downarrow U \text{ is pathological}\}$$

is stationary in $[\kappa]^{\aleph_1}$ ([4]).

The proof of the equivalence of (G) to FRP is relatively easy once the basic theory of pre-Hilbert spaces has been established. For the proof of the implication of (G) from FRP in [4], we need a singular compactness which looks slightly different from Shelah's Singular Compactness Theorem (see Theorem 2.3 below). This theorem however can be proved using practically the same ideas of the proof of Shelah's Singular Compactness Theorem given in [13]:

Theorem 2.3 ([4]) *Suppose that λ is a singular cardinal and X is a pre-Hilbert space which is a dense sub-inner-product-space of $\ell_2(\lambda)$. If X is pathological then there is a cardinal $\lambda' < \lambda$ such that*

(2.11) $\{u \in [\lambda]^{\kappa^+} : X \downarrow u$ *is a pathological pre-Hilbert space*$\}$

is stationary in $[\lambda]^{\kappa^+}$ for all $\lambda' \leq \kappa < \lambda$.

3 Fixed points by equiconsistency

The reverse-mathematical criterion can be also formulated in terms of equiconsistency.

The theory ACA_0 is regarded as very important also since it is equiconsistent with Peano Arithmetic. This equiconsistency also has strong impact in the philosophy of mathematics because of the existence of consistency proofs of Peano Arithmetic by Gentzen and Gödel (see [5] for more discussion about this).

AC is equiconsistent with $x \equiv x$ over ZF and CH also with $x \equiv x$ over ZF.

Some of the large cardinal axioms serve as more non-trivial examples of the reverse-mathematical phenomena by equiconsistency.

For example the existence of an inaccessible cardinal is known to be equiconsistent over ZFC with each of the following assertions:

– *"All sets of reals in $L(\mathbb{R})$ have the perfect set property" ((Robert Solovay and) Ernst Specker, 1957).*

– *"The negation of Kurepa hypothesis" (Jack Silver, 1970).*

– *"All sets of reals in $L(\mathbb{R})$ are measurable" ((Robert Solovay and) Saharon Shelah, 1984).*

As already mentioned in Section 2, FRP has a consistency strength of fairly large large cardinals. At present the exact consistency strength of FRP seems to be out of reach.

For a cardinal λ let FRP($<\lambda$) be the assertion

(FRP($<\lambda$)): For any regular cardinal $\kappa < \lambda$ and stationary $S \subseteq E^{\omega}_{\kappa}$ and mapping $g : S \to [\kappa]^{\leq\aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that

(3.1) $cf(I) = \omega_1$;

(3.2) $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;

(3.3) for any regressive $f : S \cap I \to \kappa$ such that $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ such that $f^{-1}{}''\{\xi^*\}$ is stationary in $\sup(I)$.

The existence of a weakly compact cardinal is equiconsistent with:

– *"For all stationary $S \subseteq [\omega_2]^{\aleph_0}$ there is $\gamma < \omega_2$ of uncountable cofinality such that $S \cap [\gamma]^{\aleph_0}$ is stationary in $[\gamma]^{\aleph_0}$" (Boban Veličković, [19]).*

In [6] it is proved that FRP($< \aleph_3$) follows from the statement above. Hence the consistency strength of FRP($< \aleph_3$) is weaker than or equal to that of a weakly compact cardinal.

Tadatoshi Miyamoto showed that the consistency strength of FRP($< \aleph_3$) is much less than a weakly compact cardinal.

Theorem 3.1 (Tadatoshi Miyamoto, 2010 [14]) *The following assertions are equiconsistent over* ZFC*:*

(a) *there is a Mahlo cardinal;*

(b) FRP($<\aleph_3$).

As already emphasized in Section 1 a host of equivalence results over ZFC by itself does not imply the truth of the assertion (FRP in our case) but merely suggests the relevance of the assertion or at most its naturalness according to the naturalness of the mathematical statements proved to be equivalent to the assertion.

In contrast to the subsystems of the second order arithmetic which is "true" as a subsystem of Zermelo's axiom system of set theory, the truth of the axioms independent of ZFC must be discussed in some other way. In case of FRP, its possible truth may be considered to be suggested by the fact that FRP is a consequence of both of the in some sense natural but mutually inconsistent axioms: Martin's Maximum (MM) and Rado's Conjecture (RC)(MM implies FRP since MM implies Axiom R. For the proof of FRP form RC see [10]).

In any case, assertions which satisfy the reverse-mathematical criterion (in terms of implication or equiconsistency) over ZFC or their negations can be considered to be prominent nodes in each of the branches of the tree of possible (consistent) extensions of ZFC.

When I made this remark about the tree of extensions in my talk at CTFM2015 meeting, somebody in the audience made the comment that we were even talking about the reverse-mathematical tree, now that the reverse-mathematical zoo (of diverse subsystems of second order arithmetic which wildly exceed the basic collection of 5) is often mentioned. I do not know how much sarcasm was intended in the comment. Nevertheless I liked the comment, since, while a zoo is (in many cases) just a random collection of samples of the wild life on the earth, a tree is a metaphor of the universe.

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