On reflection numbers under large continuum

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Abstract

We give a survey of the results on the reflection numbers which are connected to Rado's Conjecture, Galvin's Conjecture, countable chromatic number of graphs, Hamburger's Problem on metrizability or freeness of Boolean algebras. In particular we examine some models of set theory in which the continuum is very large while some of these numbers are less than or equal to the continuum.

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All additional details not included in the published version of the paper are either typeset in typewriter font or put in appendices. The numbering of the assertions is kept identical with the published version.

1 Reflection numbers

Let $\mathcal{C}$ be a class of structures of some given type or signature. We also allow here classes of structures of types of higher-order; several classes of topological spaces, for example, are also considered under such $\mathcal{C}$'s. We often identify $A \in \mathcal{C}$ with its underlying set. We assume that $\mathcal{C}$ is provided with a notion of substructures $\sqsubseteq_{\mathcal{C}}$ with appropriate properties like transitivity. For A and $\kappa \in \mathit{Card} \setminus \omega + 1$, let

$$S^{\mathcal{C}}_{<\kappa}(A) = \{B \in \mathcal{C} \,:\, B \sqsubseteq_{\mathcal{C}} A,\ |\,B\,| < \kappa\}. \tag{1.1}$$

We also assume that $S^{\mathcal{C}}_{<\kappa}(A)$ contains a club in $[A]^{<\kappa}$ for all $A \in \mathcal{C}$ and uncountable $\kappa \leq |\,A\,|$.

For such a class of structures $\mathcal{C}$ and a property P, the reflection number $\mathfrak{Refl}\,(\mathcal{C}, P)$ of the property P in $\mathcal{C}$ is defined by

$$\mathfrak{Refl}\,(\mathcal{C}, P) = \begin{cases} \min\{\kappa \in \mathit{Card} \,:\, \text{for all } A \in \mathcal{C} \text{ if } A \not\models P \text{ then there are} \\ \qquad\qquad \text{club many } B \in S^{\mathcal{C}}_{<\kappa}(A) \text{ with } B \not\models P\}, \\ \qquad\qquad\qquad \text{if } \{\kappa \in \mathit{Card} \,:\, \cdots\} \neq \emptyset\,; \\ \infty, \qquad\qquad\qquad \text{otherwise.} \end{cases} \tag{1.2}$$

If P is hereditary, that is, if $A \models P$ for $A \in \mathcal{C}$ and $B \sqsubseteq_{\mathcal{C}} A$ always imply $B \models P$, the reflection number $\mathfrak{Refl}\,(\mathcal{C}, P)$ can be represented more simply by

$$\mathfrak{Refl}\,(\mathcal{C}, P) = \begin{cases} \min\{\kappa \in \mathit{Card} \,:\, \text{for all } A \in \mathcal{C} \text{ if } A \not\models P \text{ then there is} \\ \qquad\qquad B \in S^{\mathcal{C}}_{<\kappa}(A) \text{ with } B \not\models P\}, \\ \qquad\qquad\qquad \text{if } \{\kappa \in \mathit{Card} \,:\, \cdots\} \neq \emptyset\,; \\ \infty, \qquad\qquad\qquad \text{otherwise.} \end{cases} \tag{1.3}$$

Among the instances (A) $\sim$ (E) of reflection numbers $\mathfrak{Refl}\,(\mathcal{C}, P)$ we are going to introduce in the next section, all of the properties P considered there except that in (E) are hereditary.

2 Some examples of reflection numbers

We are going to consider the following instances (A) $\sim$ (E) of $\mathcal{C}$ and $\mathcal{P}$ in the general setting of the reflection number $\mathfrak{Refl}\,(\mathcal{C}, P)$ defined in the last section.

(A) $\mathcal{C}$ = trees; P = "special".

Recall that a tree T is said to be special if T can be represented as a union of countably many antichains (i.e. pairwise incomparable sets).

The statement $\mathfrak{Refl}\,(\mathcal{C}, P) = \aleph_2$ for these $\mathcal{C}$ and P is known as the *Rado Conjecture* and studied extensively, e.g. in [15], [17], [18], [4], [12]. We shall call this reflection number *the reflection number of the Rado Conjecture* and denote it by $\mathfrak{Refl}_{\mathsf{Rado}}$.

We have $\aleph_1 < \mathfrak{Refl}_{\mathsf{Rado}} \leq \infty$. The first inequality is trivial. The equation $\mathfrak{Refl}_{\mathsf{Rado}} = \infty$ is possible: This holds for example if we have $\square_\kappa$ for class may $\kappa \in Card$ — actually $\mathsf{ADS}^-(\kappa)$ for class many regular uncountable κ is enough (see [12]).

The Rado Conjecture ($\mathfrak{Refl}_{\mathsf{Rado}} = \aleph_2$) is known to be consistent: for example, it can be forced when we start from a model of ZFC with a strongly compact cardinal κ and collapse it to $\aleph_2$ by $\mathrm{Col}(\aleph_1, \kappa)$ (Todorcĕvić [15], see also Section 3 below).

The Rado Conjecture is known to imply many interesting mathematical consequences:

- $2^{\aleph_0} \leq \aleph_2$ (Todorčević [17]).
- strong forms of Chang's Conjecture (Todorčević [17], Doebler [4], Fuchino, Sakai, Torres and Usuba [12]).
- $\mathfrak{Refl}_{\mathsf{Rado}} = \aleph_2$ implies the Fodor-type Reflection Principle (FRP) (Fuchino, Sakai, Torres and Usuba [12]) and hence all consequences of FRP like SCH (Fuchino and Rinot [10] — a direct proof of SCH from the Rado Conjecture was given in Todorďević [17]), stationarity reflection (of sets of ordinals of countable cofinality) etc.
- Semistationary Reflection Principle (SSR) of Sakai [14] (Doebler [4]).

(B) $\mathcal{C}$ = partial orderings; P = "countable union of chains".

The statement $\mathfrak{Refl}(\mathcal{C}, P) = \aleph_2$ for these $\mathcal{C}$ and P is known as the *Galvin Conjecture*. It is a longstanding open problem if the Galvin Conjecture is consistent with ZFC. We shall call $\mathfrak{Refl}(\mathcal{C}, P)$ for these $\mathcal{C}$ and P *the reflection number of the Galvin Conjecture* and denote it with $\mathfrak{Refl}_{\mathsf{Galvin}}$.

(C) $\mathcal{C}$ = graphs; P = "of countable chromatic number".

Recall that a graph Γ is of countable chromatic number if and only if it is a union of countably many subsets Γ_n, $n \in \omega$ of Γ such that each of Γ_n's contains no adjacent pairs. We shall call $\mathfrak{Refl}(\mathcal{C}, P)$ for these $\mathcal{C}$ and P *the reflection number of countable chromatic number* and denote it with $\mathfrak{Refl}_{\mathsf{chr}}$. In spite of the similar definition to that of $\mathfrak{Refl}_{\mathsf{Rado}}$, it is known that $\mathfrak{Refl}_{\mathsf{chr}}$ can never be $\aleph_2$:

Theorem 2.1 (Erdős and Hajnal [5]) $\beth_\omega \leq \mathfrak{Refl}_{\mathsf{chr}}$. ❑

Erdős and Hajnal originally proved this theorem under GCH. A modification of the proof without relying on GCH can be found e.g. in [11].

Corollary 2.2 *Assuming the consistency of* ZFC + *"there are at least two strongly compact cardinals", the inequality* $\mathfrak{Refl}_{\mathsf{Rado}} < \mathfrak{Refl}_{\mathsf{chr}} < \infty$ *is consistent.*

Proof. The model obtained by collapsing the first strongly compact cardinal κ by $\mathrm{Col}(\aleph_1, \kappa)$ will do. The inequality $\mathfrak{Refl}_{\mathsf{chr}} < \infty$ follows from Lemma 2.3 below.

□ (Corollary 2.2)

Lemma 2.3 (Todorčević [18])
$\mathfrak{Refl}_{\mathsf{Rado}} \leq \mathfrak{Refl}_{\mathsf{Galvin}} \leq \mathfrak{Refl}_{\mathsf{chr}} \leq$ *the* ω_1*-strongly compact cardinal (if it exists).*

Proof. $\mathfrak{Refl}_{\mathsf{Rado}} \leq \mathfrak{Refl}_{\mathsf{Galvin}}$: Suppose that T is a tree such that every subtrees of T of cardinality $< \kappa = \mathfrak{Refl}_{\mathsf{Galvin}}$ are special. We have to show that T is special.

Let $<_w$ be a well-ordering on T and let $<_w^T$ be the relation on T defined by

(2.1) $\quad t <_w^T t' \Leftrightarrow t$ and t' are incomparable in T and $t \restriction \alpha^* <_w t' \restriction \alpha^*$ where $\alpha^* = \min\{\alpha \in \mathrm{On} : t \restriction \alpha$ and $t' \restriction \alpha$ are incomparable$\}$.

It is easy to see that $<_w^T$ is a partial ordering on T.

For any $T' \subseteq T$,

(2.2) $\quad T'$ is an antichain if and only if T' is linearly ordered with respect to $<_w^T$.

It follows that any $T' \in [T]^{<\kappa}$ is the union of countably many linearly ordered sets with respect to $<_w^T$. Since $\kappa = \mathfrak{Refl}_{\mathsf{Galvin}}$, it follows that T itself is the union of countably many linearly ordered sets with respect to $\leq_w^T$. By (2.2), this means that T is special.

$\mathfrak{Refl}_{\mathsf{Galvin}} \leq \mathfrak{Refl}_{\mathsf{chr}}$: Suppose that $P = \langle P, \leq_P \rangle$ is a partial ordering such that every $Q \in [P]^{<\kappa}$ for $\kappa = \mathfrak{Refl}_{\mathsf{chr}}$ are the union of countably many linearly ordered sets with respect to $\leq_P$.

We have to show that P itself is also the union of countably many linearly ordered subsets with respect to $\leq_P$.

For p, $q \in P$, let

(2.3) $\quad p \, E \, q \ \Leftrightarrow \ p$ and q are incomparable with respect to $\leq_P$.

Then, for any $Q \subseteq P$,

(2.4) $\quad Q$ is linearly ordered with respect to $\leq_P \ \Leftrightarrow Q$ contains no adjacent pair with respect to E.

Thus, by the assumption on P, we have that all subsets Q of P of cardinality $< \kappa$ are countable chromatic with respect to E. By $\kappa = \mathfrak{Refl}_{\mathsf{chr}}$ it follows that $(\langle P, E \rangle)$ is also countable chromatic. The latter means by (2.4) that P is the union of countably many linearly ordered subsets with respect to $\leq_P$.

$\mathfrak{Refl}_{\mathsf{chr}} \leq$ the ω_1-strongly compact cardinal: Recall that a cardinal κ is the ω_1-strongly compact cardinal if and only if it is the smallest cardinal κ such that, for any $\mathcal{L}_{\omega_1,\omega}$-theory T, if all $T' \in [T]^{<\kappa}$ are satisfiable (that is, they have models) then T itself is also satisfiable.

Suppose that κ is ω_1-strongly compact and let $\Gamma = \langle \Gamma, R_\Gamma \rangle$ be a graph such that all $\Gamma' \in [\Gamma]^{<\kappa}$ are countable chromatic.

It is then easy to see that all subsets T' of cardinality $< \kappa$ of the following set T of formulas in $L_{\omega_1,\omega}$ are satisfiable:

$$\begin{aligned} T = &\{c_g\ R_\Gamma\ c_{g'} \,:\, g, g' \in \Gamma,\ g\ R_\Gamma\ g'\} \\ &\cup\{\neg c_g\ R_\Gamma\ c_{g'} \,:\, g, g' \in \Gamma,\ g\ \not R_\Gamma\ g'\} \\ &\cup\{\text{“}\Gamma_n(\cdot) \text{ is pairwise non adjacent with respect to } R_\Gamma\text{”} \,:\, n \in \omega\} \\ &\cup\{\forall x \left(\textstyle\bigvee\!\!\!\bigvee_{n\in\omega} \Gamma_n(x)\right)\}. \end{aligned}$$

By ω_1-strong compactness of κ, it follows that T has a model M.

Clearly

$$\{\{g \in \Gamma \,:\, M \models \Gamma_n(c_g)\} \,:\, n \in \omega\} \tag{2.5}$$

is a partition of Γ into countably many pairwise non adjacent subsets with respect to R_Γ. ❑ (Lemma 2.3)

Let us continue with some other examples of reflection numbers:

(D) $\mathcal{C}$ = first countable topological spaces; P = “metrizable”.

The first countability is added to avoid some trivial examples of non reflection:

Example 2.4 (Hajnal and Juhász [13]) *For an uncountable regular cardinal κ, let $X = \kappa + 1$ be the topological space whose open sets are generated by*

$$\{\{\alpha\} \,:\, \alpha < \kappa\} \cup \{[\alpha, \kappa + 1) \,:\, \alpha < \kappa\}. \tag{2.6}$$

All $Y \in [X]^{<\kappa}$ are discrete and hence metrizable but X is not since $\chi(\kappa, X) = \kappa$. ❑

The question about the consistency of $\mathfrak{Refl}(\mathcal{C}, P) = \aleph_2$ for these $\mathcal{C}$ and P is called *Hamburger's Problem* and is still open.

We shall call this reflection number *the reflection number of Hamburger's Problem* and denote it with $\mathfrak{Refl}_{\mathsf{HP}}$.

Lemma 2.5 (1) $\aleph_1 < \mathfrak{Refl}_{\mathsf{HP}} \leq \infty$.

(2) $\mathfrak{Refl}_{\mathsf{HP}} = \infty$ *is consistent.*

(3) $\mathfrak{Refl}_{\mathsf{HP}} \leq$ *the ω_1-strongly compact cardinal (if it exists).*

Proof. (1): ω_1 with the order topology or E^κ_ω for any cardinal of uncountable cofinality (also with the order topology) are among the examples showing the inequality $\aleph_1 < \mathfrak{Refl}_{\mathsf{HP}}$.

(2): This holds if $\square_\kappa$ holds for cofinally many κ — actually $\mathsf{ADS}^-(\kappa)$ for class many regular uncountable κ is enough (see Proposition 6.3 in [7])).

(3): Suppose that $(X, \mathcal{O})$ is a first countable topological space such that all subspaces $Y \in [X]^{<\kappa}$ are metrizable. For each $x \in X$, let $\{O_{x,n} : n \in \omega\}$ be an open neighborhood base of x.

Let T be the $\mathcal{L}_{\omega_1,\omega}$ theory in the language with the binary relation symbols $O_n(x, y)$ for all $n \in \omega$ coding "$y \in O_{x,n}$" and the binary symbols $d_q(x, y)$ for all $q \in \mathbb{Q}_{\geq 0}$ which should code "$d(x, y) \leq q$":

$$
\begin{aligned}
(2.7) \qquad T = {} & \{O_n(c_a, c_b) : a, b \in X,\ b \in O_{a,n}\} \\
& \cup \{\neg O_n(c_a, c_b) : a, b \in X,\ b \notin O_{a,n}\} \\
& \cup \{\forall x \forall y\ (d_q(x, y) \to d_q(y, x)) : q \in \mathbb{Q}_{\geq 0}\} \\
& \cup \{\forall x \forall y\ (d_q(x, y) \to d_{q'}(x, y)) : q, q' \in \mathbb{Q}_{\geq 0},\ q \leq q'\} \\
& \cup \{\forall x \forall y\ (d_0(x, y) \to x \equiv y)\} \\
& \cup \{\forall x \forall y \forall z\ (d_q(x, y) \wedge d_{q'}(y, z) \to d_{q+q'}(x, z)) : q, q' \in \mathbb{Q}_{\geq 0}\} \\
& \cup \{\forall x \textstyle\bigvee_{q \in \mathbb{Q}_{>0}} \forall y\ (d_q(x, y) \to O_n(x, y)) : n \in \omega\} \\
& \cup \{\forall x \textstyle\bigwedge_{q \in \mathbb{Q}_{>0}} \bigvee_{n \in \omega} \forall y\ (O_n(x, y) \to d_q(x, y))\}
\end{aligned}
$$

Clearly all $T' \in [T]^{<\kappa}$ are satisfiable. Since κ is ω_1-strongly compact, it follows that T is also satisfiable. Let M be a model of T. Then $d : X^2 \to \mathbb{R}$ defined by

$$(2.8) \qquad d(a, b) = \min\{q \in \mathbb{Q} : M \models d_q(c_a, c_b)\} \text{ for } a,\, b \in X$$

is a metric on X generating the topology of $(X, \mathcal{O})$. □ (Lemma 2.5)

(E) $\mathcal{C}$ = atomless Boolean algebras; P = "free".

We shall denote the corresponding reflection number by $\mathfrak{Refl}_{\mathsf{fBa}}$.

Lemma 2.6 (1) $\aleph_1 < \mathfrak{Refl}_{\mathsf{fBa}} \leq \infty$.

(2) $\mathfrak{Refl}_{\mathsf{fBa}} = \infty$ *is consistent.*

(3) $\mathfrak{Refl}_{\mathsf{fBa}} \leq$ *the first supercompact cardinal (if it exists).*

Proof. (1): $\aleph_1 < \mathfrak{Refl}_{\mathsf{fBa}}$ since all countable atomless Boolean algebras are free.

(2): By Corollary 2.5. in [8].

(3): Suppose that κ is a supercompact cardinal and B is an atomless Boolean algebra such that there is a club $\mathcal{D} \subseteq S^{\mathcal{C}}_{<\kappa}(B)$ such that all $A \in \mathcal{D}$ are free where $\mathcal{C}$ denotes the class of all atomless Boolean algebras. Without loss of generality, we may assume that $|B| \geq \kappa$. Let $j : V \overset{\prec}{\to} M$ be $|B|$-supercompact elementary embedding. Thus we have $j(\kappa) > |B|$ and ${}^{|B|}M \subseteq M$. Letting $A^* = j''B$, we have $A^* \in j(\mathcal{D})$ and $A^* \in M$. By the elementarity of j it follows that $M \models A^*$ is free. Thus $A \cong A^*$ is really free. □ (Lemma 2.6)

Problem 2.7 *Does "$\mathfrak{Refl}_{\mathsf{fBa}} \leq$ the ω_1-strongly compact cardinal" hold?*

3 A consistency proof of the Rado Conjecture

Todorčević proved the following proposition under the existence of a supercompact cardinal. Almost the same proof will do under a strongly compact cardinal.

Proposition 3.1 (Todorcěvić [15]) *Suppose that κ is strongly compact and $\mathbb{P} = \mathrm{Col}(\aleph_1, \kappa)$. Then we have*

(3.1) $\Vdash_{\mathbb{P}}$ "$\mathfrak{Refl}_{\mathsf{Rado}} = \aleph_2$".

For the proof of Proposition 3.1, we use the following lemma also due to Todorčević:

Lemma 3.2 *Suppose that T is a non-special tree and $\mathbb{P}$ a σ-closed p.o. Then $\Vdash_{\mathbb{P}}$ "T is a non-special tree".*

Proof. We prove the contraposition of the statement. Suppose that, for a tree T and a σ-closed p.o. $\mathbb{P}$, we have $\Vdash_{\mathbb{P}}$ "T is a special tree". Note that it follows that

(3.2) $ht(T) \leq \omega_1$.

Let $\underset{\sim}{f}$ be a $\mathbb{P}$-name of a witness of the specialness of T, that is,

(3.3) $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{f} : T \to \omega$ and
$\underset{\sim}{f}(t) \neq \underset{\sim}{f}(t')$ for all $t, t' \in T$ such that t and t' are comparable".

For $t \in T$ we can define $p_t \in \mathbb{P}$ and $n_t \in \omega$ by induction on $ht(t)$ such that

(3.4) p_t decides $\underset{\sim}{f}(t)$ to be n_t; and

(3.5) $p_t \leq_{\mathbb{P}} p_{t'}$ if $t \leq_T t'$.

Note that this is possible by the σ-closedness of $\mathbb{P}$ and (3.2).

The mapping $f^* : T \to \omega$ defined by

(3.6) $f^*(t) = n_t$ for $t \in T$

witnesses the specialness of T (in the ground model). $\square$ (Lemma 3.2)

Proof of Proposition 3.1: Let $\mathbb{P} = \mathrm{Col}(\aleph_1, \kappa)$ and let G be a $(V, \mathbb{P})$-generic filter. Note that

(3.7) $\quad V[G] \models \kappa = \omega_2.$

Suppose that $T = \langle T, \leq_T \rangle \in V[G]$ is a tree of size $\lambda > \aleph_1$ in $V[G]$ (so λ is a cardinal in V with $\lambda \geq \kappa$) such that

(3.8) $\quad V[G] \models$ "T is non-special".

By (1.3), it is enough to show that, in $V[G]$, there is a non-special $T' \in [T]^{\aleph_1}$.

Without loss of generality, we may assume that (the underlying set of) T is the set of ordinals λ.

Let $j : V \overset{\preceq}{\to} M$ be a λ-strongly compact embedding. That is, j is an elementary embedding such that

(3.9) $\quad \kappa = crit(j)$;

(3.10) $\quad {}^{\kappa}M \subseteq M$; and

(3.11) $\quad$ for all $X \in [M]^{\leq\lambda}$ (in V) there is $Y \in \left([M]^{<j(\kappa)}\right)^M$ with $X \subseteq Y$.

Let $\mathbb{P}^* = j(\mathbb{P})$. Then, by (3.10), we have $\mathbb{P}^* = \mathrm{Col}(\aleph_1, j(\kappa))$ (in V). Let $G^* \supseteq G$ be a $(V, \mathbb{P}^*)$-generic filter. j can be extended to $j^* : V[G] \overset{\preceq}{\to} M[G^*]$ by declaring $j^*(\underset{\sim}{a}^G) = (j(\underset{\sim}{a}))^{G^*}$ for all $\mathbb{P}$-name $\underset{\sim}{a}$.

Since $|\, j''\lambda \,| = \lambda$ (in V), there is $Y \in \left([M]^{<j(\kappa)}\right)^M$ such that $j''\lambda \subseteq Y \subseteq j(\lambda)$ by (3.11). Let $T^* = \langle Y, j^*(\leq_T) \cap Y^2 \rangle$. Then $T^* \in M[G^*]$ and, by (3.7),

(3.12) $\quad M[G^*] \models$ "T^* is a subtree of $j(T)$ of cardinality $< \aleph_2$.

In $V[G^*]$, T is embeddable in T^* and, by Lemma 3.2, we have $V[G^*] \models$ "T is non-special". It follows that $M[G^*] \models$ "T^* is non-special".

Thus

(3.13) $\quad M[G^*] \models$ "there is a non-special $T' \in [j^*(T)]^{<\aleph_2}$".

By the elementarity of j^*, it follows that

(3.14) $\quad V[G] \models$ "there is a non-special $T' \in [T]^{<\aleph_2}$".

$\square$ (Proposition 3.1)

4 Models with large continuum

The continuum can be consistently very large in many different ways. For example it can be weakly inaccessible (provided that we work under the consistency of ZFC + "there exists an inaccessible cardinal").

More extreme situations would be:

(4.1) There is an inner model M with $Card^M \cap 2^\omega = Card \cap 2^\omega +$ and 2^ω is a fairly large cardinal (e.g. strongly compact, supercompact etc.) in M; or

(4.2) $2^{\aleph_0}$ is a generic large cardinal, that is, there are generic elementary embeddings with the critical point $2^{\aleph_0}$.

We can attain the situations like in (4.1) or (4.2) e.g. by starting from a large cardinal, say a strongly compact κ, and adding κ many reals in a coherent manner.

Reflection numbers can be small under large continuum in such sense. The following result is an example of this:

Theorem 4.1 (Dow, Tall and Weiss [2]) *Suppose that κ is a strongly compact cardinal and $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$. Then $\Vdash_{\mathbb{P}}$ "$\mathfrak{Refl}_{\mathsf{HP}} \leq 2^{\aleph_0}$".* ❑

In the following we show that $\mathfrak{Refl}_{\mathsf{Rado}}$ behaves similarly in the generic extension of Theorem 4.1.

Theorem 4.2 *Suppose that κ is a strongly compact cardinal. Then, for any $\mu \geq \kappa$ and $\mathbb{P} = \mathrm{Fn}(\mu, 2)$, we have*

(4.3) $\Vdash_{\mathbb{P}}$ "$\mathfrak{Refl}_{\mathsf{Rado}} \leq 2^{\aleph_0}$".

In particular, by setting $\mu > \kappa$, we obtain the consistency of $\mathfrak{Refl}_{\mathsf{Rado}} < 2^{\aleph_0}$ (modulo a strongly compact cardinal).

Note that, in contrast, Rado's Conjecture (i.e. $\mathfrak{Refl}_{\mathsf{Rado}} = \aleph_2$) implies that $2^{\aleph_0} \leq \aleph_2 = \mathfrak{Refl}_{\mathsf{Rado}}$ (Todorčević [17]).

Mateo Viale asked if $\mathfrak{Refl}_{\mathsf{Rado}} = \aleph_3$ implies $2^{\aleph_0} \leq \aleph_3$. This is still open. More generally:

Problem 4.3 *Is it provable in ZFC that $\mathfrak{Refl}_{\mathsf{Rado}} = \aleph_n$ implies $2^{\aleph_0} \leq \aleph_n$ for all $n \geq 3$?*

For the proof of Theorem 4.2, we need the following lemma. Remember that a p.o. $\mathbb{P}$ is said to be σ-centered if it can be represented as the union of countably many centered subsets (i.e. filter bases).

Lemma 4.4 (1) *Suppose that T is a non-special tree. If $\mathbb{P}$ is a σ-centered p.o., then* $\Vdash_{\mathbb{P}}$ *"T is not special".*

(2) *Suppose that T is a non-special tree. Then, for any μ and $\mathbb{P} = \mathrm{Fn}(\mu, 2)$, we have* $\Vdash_{\mathbb{P}}$ *"T is not special".*

Proof. (1): Suppose that T is a non-special tree and $\mathbb{P} = \bigcup_{n\in\omega} C_n$ where each C_n is centered.

If $\not\Vdash_{\mathbb{P}}$ "T is not special" then there are $p \in \mathbb{P}$ and $\mathbb{P}$-names $\underset{\sim}{A}_m$, $m \in \omega$ such that

$$(4.4) \qquad p \Vdash_{\mathbb{P}} \text{“}T \text{ is a union of pairwise incomparable } \underset{\sim}{A}_m,\ m \in \omega\text{”}.$$

For each $m, n \in \omega$

$$(4.5) \qquad A_{m,n} = \{t \in T \,:\, q \Vdash_{\mathbb{P}} \text{“}t \in \underset{\sim}{A}_m\text{” for some } q \in C_n \text{ such that } q \leq_{\mathbb{P}} p\}$$

is a pairwise incomparable subset of T and $T = \bigcup\{A_{m,n} \,:\, m, n \in \omega\}$. This is a contradiction to the assumption that T is not special.

(2): Note that $\mathbb{P}$ is σ-centered if and only if $\mu \leq 2^{\aleph_0}$. Thus, if $\mu \leq 2^{\aleph_0}$, then the claim follows from (1).

Suppose that $\mu > 2^{\aleph_0}$ and $\not\Vdash_{\mathbb{P}}$ "T is not special". By the homogeneity of $\mathbb{P}$ it follows that

$$(4.6) \qquad \Vdash_{\mathbb{P}} \text{“}T \text{ is special”}.$$

Let $\mathbb{Q} = \mathrm{Col}(\aleph_1, \kappa^+)$. By (4.6) we have $\Vdash_{\mathbb{P}*\underset{\sim}{Q}}$ "T is special" where $\underset{\sim}{Q}$ is a $\mathbb{P}$-name of a p.o. such that $\mathbb{P} * \underset{\sim}{Q} \sim \mathbb{Q} * \mathbb{P}$. Thus we have

$$(4.7) \qquad \Vdash_{\mathbb{Q}*\mathbb{P}} \text{“}T \text{ is special”}.$$

Since $\Vdash_{\mathbb{Q}}$ "$\mathbb{P}$ is σ-centered", it follows that $\Vdash_{\mathbb{Q}}$ "T is special" by (1).

By Lemma 3.2, it follows that T is special. $\square$ (Lemma 4.4)

Proof of Theorem 4.2: Suppose that G is a $(V, \mathbb{P})$-generic filter and $T = \langle T, <_T\rangle$ a non-special tree in $V[G]$. Let $V[G] \models \lambda = |\,T\,|$ and let $j : V \to M$ be an elementary embedding such that $M \subseteq V$ is a transitive class $\subseteq V$, $crit(j) = \kappa$, $j(\kappa) > \lambda$ and

$$(4.8) \qquad \text{for all } X \in [M]^{\leq\lambda} \text{ (in } V\text{) there is } Y \in \left([M]^{<j(\kappa)}\right)^M \text{ with } X \subseteq Y.$$

Let $\mathbb{P}^* = j(\mathbb{P}) = \mathrm{Fn}(j(\mu), 2)$ and let G^* be a $(\mathbb{P}^*, V)$-generic filter with $G^* \supseteq G$.

Let $j^* : V[G] \overset{\preceq}{\to} M[G^*]$ be the extension of j defined by $j^*(\underset{\sim}{a}^G) = [j(\underset{\sim}{a})]^{G^*}$ for each $\mathbb{P}$-name $\underset{\sim}{a}$. By the ccc of $\mathbb{P}^*$ and (4.8), we have

(4.9) for all $X \in [M[G^*]]^{\leq\lambda}$ (in $V[G^*]$) there is $Y \in \left([M[G^*]]^{<j(\kappa)}\right)^{M[G^*]}$ with $X \subseteq Y$.

Thus, in $M[G^*]$, there is a subtree T' of $j^*(T)$ of size $< j(\kappa)$ containing $j^{*\,\prime\prime}T$. Since $V[G^*] \models j^{*\,\prime\prime}T \cong T$, and $V[G^*] \models$ "T is not special" by Lemma 4.4, (2), we have $V[G^*] \models$ "$j^{*\,\prime\prime}T$ is not special". Since $V[G^*] \models j^{*\,\prime\prime}T \subseteq T'$, it follows that $V[G^*] \models$ "T' is not special" and hence $M[G^*] \models$ "T' is not special". Thus

$$M[G^*] \models \text{“there is a non-special subtree } T' \text{ of } j^*(T) \text{ of size } < j^*(\kappa). \tag{4.10}$$

By elementarity of j^*, it follows that

$$V[G] \models \text{“there is a non-special subtree } T' \text{ of } T \text{ of size } < \kappa. \tag{4.11}$$

Since $\kappa \leq \mu \leq (2^{\aleph_0})^{V[G]}$, we have

$$V[G] \models \text{“there is a non-special subtree } T' \text{ of } T \text{ of size } < 2^{\aleph_0}. \tag{4.12}$$

❏ (Theorem 4.2)

In analogy to the common terminology in set-theoretic topology, we say a structure A is indestructibly $\neg P$ for a property P, if $\Vdash_{\mathbb{P}}$ "$A \models \neg P$" holds for any σ-closed p.o. $\mathbb{P}$.

For a class $\mathcal{C}$ of structures as in Section 1 and a property P, we define *the indestructible reflection number* $\mathfrak{Refl}^*(\mathcal{C}, P)$ *of* P *in* $\mathcal{C}$ is defined by:

$$\mathfrak{Refl}^*(\mathcal{C}, P) = \begin{cases} \min\{\kappa \in \mathit{Card} : & \text{for all } A \in \mathcal{C} \text{ if } A \text{ is indestructively } \neg P \\ & \text{then there are club many } B \in S^{\mathcal{C}}_{<\kappa}(A) \\ & \text{with } B \not\models P\}, \\ & \qquad \text{if } \{\kappa \in \mathit{Card} : \cdots\} \neq \emptyset\,; \\ \infty, & \qquad \text{otherwise.} \end{cases} \tag{4.13}$$

Let $\mathfrak{Refl}^*_{\mathsf{Rado}}$, $\mathfrak{Refl}^*_{\mathsf{Galvin}}$, $\mathfrak{Refl}^*_{\mathsf{chr}}$, $\mathfrak{Refl}^*_{\mathsf{HP}}$, $\mathfrak{Refl}^*_{\mathsf{fBa}}$ be the indestructible version of corresponding reflection numbers in (A) $\sim$ (E). By Lemma 3.2, $\mathfrak{Refl}^*_{\mathsf{Rado}} = \mathfrak{Refl}_{\mathsf{Rado}}$. Since the proof of Lemma 2.3 also holds for indestructible version we have:

$$\begin{array}{ccccc} & & \mathfrak{Refl}_{\mathsf{Galvin}} & \leq & \mathfrak{Refl}_{\mathsf{chr}} \\ & & \veebar & & \veebar \\ \mathfrak{Refl}_{\mathsf{Rado}} & \leq & \mathfrak{Refl}^*_{\mathsf{Galvin}} & \leq & \mathfrak{Refl}^*_{\mathsf{chr}}. \end{array}$$

The following Theorems can be proved similarly to Proposition 3.1 and Theorem 4.2:

Theorem 4.5 *Suppose that κ is a strongly compact cardinal and $\mathbb{P} = \mathrm{Col}(\aleph_1, \mu)$ for some $\mu \geq \kappa$. Then we have*

$$(4.14) \qquad \Vdash_{\mathbb{P}} \text{“} \mathfrak{Refl}^*_{\mathsf{Galvin}} = \mathfrak{Refl}^*_{\mathsf{chr}} = \mathfrak{Refl}^*_{\mathsf{HP}} = \aleph_2 \text{”}. \qquad \square$$

Theorem 4.6 *Suppose that κ is strongly compact. Then, for any $\mu \geq \kappa$ and $\mathbb{Q} = \mathrm{Fn}(\mu, 2)$, we have*

$$(4.15) \qquad \Vdash_{\mathbb{Q}} \text{“} \mathfrak{Refl}^*_{\mathsf{Galvin}} = \mathfrak{Refl}^*_{\mathsf{chr}} \leq 2^{\aleph_0} \text{”}. \qquad \square$$

Similarly to the theorems above $\Vdash_{\mathbb{P}}$ “$\mathfrak{Refl}^*_{\mathsf{fBa}} = \aleph_2$” and $\Vdash_{\mathbb{Q}}$ “$\mathfrak{Refl}^*_{\mathsf{fBa}} \leq 2^{\aleph_0}$” can be obtained when we start from a supercompact κ. As we see in Theorem 5.8 the last assertion can be still improved to $\Vdash_{\mathbb{Q}}$ “$\mathfrak{Refl}_{\mathsf{fBa}} \leq 2^{\aleph_0}$”.

5 Martin's axiom and large continuum

Let us remind first the following classical theorem by Kurepa, just for completeness:

Lemma 5.1 (Đ. Kurepa 1940) *For a tree, the following are equivalent:*

(a) *T is special.*

(b) *There is a strictly order preserving $f : T \to \mathbb{Q}$.*

Proof. (b) $\Rightarrow$ (a): Suppose that $f : T \to \mathbb{Q}$ is strictly order preserving. Then $f^{-1}{}''\{q\}$, $q \in \mathbb{Q}$ is a countable partition of T into pairwise incomparable sets.

(a) $\Rightarrow$ (b): Suppose that $h : T \to \omega$ is such that

(5.1) $\qquad h^{-1}{}''\{n\}$ is pairwise incomparable for all $n \in \omega$.

Let $f : T \to \mathbb{Q}$ be defined by

$$(5.2) \qquad f(t) = \sum\{2^{-(k+1)} : k \leq h(t), \text{ there is } t' \leq_T t \text{ such that } h(t') = k\}.$$

To show that this f is strictly order preserving, assume that $t <_T u$. By (5.1), we have $h(t) \neq h(u)$.

If $h(t) > h(u)$, then

$$\begin{aligned} f(t) = {} & \sum\{2^{-(k+1)} : k \leq h(u), \text{ there is } t' \leq_T t \text{ such that } h(t') = k\} \\ & + \sum\{2^{-(k+1)} : h(u) < k \leq h(t), \text{ there is } t' \leq_T t \text{ such that } h(t') = k\} \\ & < \sum\{2^{-(k+1)} : k \leq h(u), \text{ there is } t' <_T u \text{ such that } h(t') = k\} + 2^{-(h(u)+1)} \\ & = f(u). \end{aligned}$$

If $h(t) < h(u)$, then

$$
\begin{aligned}
f(t) &= \sum\{2^{-(k+1)} : k \leq h(t), \text{ there is } t' \leq_T t \text{ such that } h(t') = k\} \\
&< \sum\{2^{-(k+1)} : k \leq h(t), \text{ there is } t' \leq_T u \text{ such that } h(t') = k\} + 2^{-(h(u)+1)} \\
&\leq f(u).
\end{aligned}
$$

$\square$ (Lemma 5.1)

Let $\mathfrak{ma}$ denote the first cardinal κ for which Martin's Axiom for a family of κ many dense sets does not hold. Note that MA_κ is equivalent to $\mathfrak{ma} \geq \kappa^+$.

Theorem 5.2 (Baumgartner-Malitz-Reinhardt [1]) *For any tree T of size $< \mathfrak{ma}$ without uncountable branches, there is a mapping $f : T \to \mathbb{Q}$ such that, for any t, $t' \in T$, $t <_T t'$ implies $f(t) < f(t')$ (Note that, by Lemma 5.1, there is such a mapping f if and only if T is special).*

The Theorem 5.2 follows from Lemma 5.4 below. The following combinatorial Lemma 5.3 is the key to the Lemma 5.4:

Lemma 5.3 *Suppose that T is a tree without uncountable branches. If $S \subseteq [T]^{<\aleph_0}$ is uncountable and pairwise disjoint, then there are s, $s' \in S$ such that any $t \in s$ and $t' \in s'$ are incomparable.*

Proof. Without loss of generality we may assume that $S \subseteq [T]^n$ for some $n \in \omega$.

Assume toward a contradiction that, for all s, $s' \in S$, there are $t \in s$ and $t' \in s'$ such that t and t' are compatible.

We assume that there is some canonical linear ordering on T and with this ordering we can talk about "the kth element of s" for $s \in S$ and $k < n$. Let D be a ultrafilter over S with $\{S \setminus u : u \in [S]^{\leq\aleph_0}\} \subseteq D$. Thus, all elements of D are uncountable.

For $s \in S$, $t \in s$ and $k < n$, let

$$F^k_t = \{s' \in S : t \text{ is compatible with the } k\text{th element of } s'\}. \tag{5.3}$$

By assumption we have $S = \bigcup_{k<n, t\in s} F^k_t$. Thus, for each $s \in S$, there is $t_s \in s$ and $k_s < n$ such that $F^{k_s}_{t_s} \in D$.

Let $k^* < n$ be such that $S' = \{s \in S : k_s = k^*\}$ is uncountable.

The following claim is a contradiction to the assumption on T as desired:

Claim 5.3.1 *$\{t_{s'} : s' \in S'\}$ is linearly ordered.*

$\vdash$ Suppose s, $s' \in S'$. Then $F^{k^*}_{t_s} \cap F^{k^*}_{t_{s'}} \in D$. Since elements of D are uncountable and by the pairwise disjointness of elements of S,

$$\{\text{the } k^*\text{'th element of } u : u \in F^{k^*}_{t_s} \cap F^{k^*}_{t_{s'}}\} \tag{5.4}$$

is uncountable while $T \downarrow t_s \cap T \downarrow t_{s'}$ is countable. Hence there is $s'' \in F^{k^*}_{t_s} \cap F^{k^*}_{t_{s'}}$ such that t_s, $t_{s'} \leq_T$ the k^*th element of s''. Since T is a tree, it follows that t_s and $t_{s''}$ are compatible. $\dashv$ (Claim 5.3.1)

$\square$ (Lemma 5.3)

Lemma 5.4 *Suppose that T is a tree without uncountable branches. Then the p.o. $\langle \mathbb{P}_T, \leq_{\mathbb{P}_T} \rangle$ defined by*

$$\begin{array}{lll} & \mathbb{P}_T = \{p : p \textit{ is a finite partial function from } T \textit{ to } \omega \textit{ such that,} & \\ (5.5) & \qquad (5.5a) \;\; \textit{for any two distinct } t, t' \in \mathrm{dom}(p), & \\ & \qquad\qquad p(t) = p(t') \textit{ implies } t \textit{ and } t' \textit{ are incomparable}\} & \end{array}$$

with the ordering

(5.6) $\quad p \leq_{\mathbb{P}_T} p' \;\Leftrightarrow\; p' \subseteq p$

has the ccc.

Proof. Suppose that $A \subseteq \mathbb{P}_T$ is uncountable. We have to show that there are two compatible elements in A. By Δ-system Lemma and countability of ω, we may assume without loss of generality, that $\{\mathrm{dom}(p) : p \in A\}$ builds a Δ-system $\subseteq [T]^{<\aleph_0}$ with the root s_0 and that

(5.7) $\quad p \restriction s_0$ is the same for all $p \in S$.

By Lemma 5.3, there are p, $p' \in A$ such that any $t \in \mathrm{dom}(p) \setminus s_0$ and $t' \, \mathrm{dom}(p') \setminus s_0$ are incomparable. Let $p'' = p \cup p'$.

Claim 5.4.1 $p'' \in \mathbb{P}_T$.

$\vdash$ p'' is a partial function by (5.7). Thus it is enough to show that p'' satisfies (5.5a).

Suppose $p''(t) = p''(t')$ for two distinct t, $t' \in \mathrm{dom}(p'')$. If t, $t' \in \mathrm{dom}(p)$ or t, $t' \in \mathrm{dom}(p')$ then t and t' are incomparable since p, $p' \in \mathbb{P}_T$. If $t \in \mathrm{dom}(p) \setminus s_0$ and $t' \in \mathrm{dom}(p') \setminus s_0$ — or $t \in \mathrm{dom}(p') \setminus s_0$ and $t' \in \mathrm{dom}(p) \setminus s_0$ — then t and t' are incomparable anyway by the choice of p and p'. $\dashv$ (Claim 5.4.1)

Since $p'' \supseteq p, p'$, it follows that $p'' \leq_{\mathbb{P}_T} p, p'$. Thus p and p' are compatible in $\mathbb{P}_T$. $\square$ (Lemma 5.4)

Proof of Theorem 5.2: Suppose that T is a tree without uncountable branches and $|\,T\,| < \mathfrak{ma}$. For each $t \in T$ let

(5.8) $\quad D_t = \{p \in \mathbb{P}_T : t \in \mathrm{dom}(p)\}$.

Then each D_t, $t \in T$ is dense in T. Let $\mathcal{D} = \{D_t : t \in T\}$. Since $|\mathcal{D}| < \mathfrak{ma}$ there is a $\mathcal{D}$-generic G over $\mathbb{P}_T$.

$f_G = \bigcup G$ is a function from T to ω and witnesses the specialness of T.

❑ (Theorem 5.2)

A prominent consequence of Theorem 5.2 is that, under $\mathsf{MA}_{\aleph_1}$, every Aronszajn tree is special.

The following Lemma together with Theorem 5.2 implies that $\mathfrak{Refl}_{\mathsf{Rado}}$ is strictly larger than $\mathfrak{ma}$.

Let

$$(5.9) \qquad T_{\mathbb{R}} = \{t : t \text{ is an increasing sequence of elements of } \mathbb{R} \text{ of order-type of countable successor ordinal } \}$$

be the tree with the ordering

$$(5.10) \qquad t \leq_{T_{\mathbb{R}}} t' \ \Leftrightarrow\ t' \text{ is an end-extension of } t$$

for t, $t' \in T_{\mathbb{R}}$. Clearly $T_{\mathbb{R}}$ has no uncountable branch.

Lemma 5.5 (Todorčević [15]) *$T_{\mathbb{R}}$ is not special.*

Proof. This follows immediately from Theorem 1 in [15]. ❑ (Lemma 5.5)

Corollary 5.6 $\mathfrak{ma} < \mathfrak{Refl}_{\mathsf{Rado}}$.

Proof. $T_{\mathbb{R}}$ does not have any uncountable branch. By Theorem 5.2, it follows that all subtrees of $T_{\mathbb{R}}$ of size $< \mathfrak{ma}$ are special. Since $T_{\mathbb{R}}$ itself is not special by Lemma 5.5, $T_{\mathbb{R}}$ witnesses the inequality $\mathfrak{ma} < \mathfrak{Refl}_{\mathsf{Rado}}$. ❑ (Corollary 5.6)

The non-freeness in any variety is preserved by a ccc p.o.:

Proposition 5.7 (Fuchino [6]) *For an algebra in a variety $\mathcal{V}$, if there is a ccc p.o. $\mathbb{P}$ such that $\Vdash_{\mathbb{P}}$ "A is free" then A is really free.* ❑

Theorem 5.8 *Suppose that κ is a supercompact cardinal. If $\mathbb{P}$ is the standard p.o. forcing Martin's Axiom and $2^{\aleph_0} = \kappa$ then we have*

$$(5.11) \qquad \Vdash_{\mathbb{P}} \text{“}\mathsf{MA} \ +\ \mathfrak{Refl}_{\mathsf{fBa}} = 2^{\aleph_0} < \mathfrak{Refl}_{\mathsf{Rado}}\text{”}.$$

Proof. $\Vdash_{\mathbb{P}}$ " $\ldots = \ldots$ " follows from Proposition 5.7 and an argument similar to the proof of Theorem 4.2. $\Vdash_{\mathbb{P}}$ " $\ldots < \ldots$ " follows from Corollary 5.6. ❑ (Theorem 5.8)

If we have a strongly compact cardinal above the supercompact κ in Theorem 5.8, we can conclude that $\mathfrak{Refl}_{\mathsf{Rado}} < \infty$ and manipulate further the value of $\mathfrak{Refl}_{\mathsf{Rado}}$.

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