Strong downward Löwenheim-Skolem theorems for stationary logics, I

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Contents

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All additional details not included in the published version of the paper are either typeset in typewriter font (the font this paragraph is typeset) or put in separate appendices. The numbering of the assertions is kept identical with the published version. The most up-to-date version of the present text is available at https://fuchino.ddo.jp/papers/SDLS-x.pdf

Abstract

This note concerns the model theoretic properties of logics extending the first-order logic with monadic (weak) second-order variables equipped with the stationarity quantifier. The eight variations of the Strong Downward Löwenheim-Skolem Theorem (SDLS) down to $< \aleph_2$ for this logic with the interpretation of second-order variables as countable subsets of the structures are classified into four principles. The strongest of these four is shown to be equivalent to the conjunction of CH and the Diagonal Reflection Principle for internally clubness of S. Cox. We show that a further strengthening of this SDLS and its variations follow from the Game Reflection Principle of B. König and its generalizations.

1 Introduction and preliminaries

intro

For a first-order structure $\mathfrak{A}$, we denote with $|\mathfrak{A}|$ the underlying set and $\|\mathfrak{A}\|$ the cardinality of the underlying set. Nevertheless, if we are talking about a set A, we continue to denote the cardinality of A with $|\,A\,|$.

In the following, we assume that the signature of the structures we consider is always countable.

$\mathcal{L}^{\aleph_0,II}$ denotes the weak (monadic) second-order logic with second-order variables X, Y, Z etc. whose intended interpretation is that they run over countable subsets of the underlying set of a structure. We shall call this type of second-order variables *weak second-order variables* (in $\aleph_0$-interpretation). In this logic, in addition to the constructions of the first-order logic, we have the element relation symbol ε as a logical predicate and allow the expression "$x \mathrel{\varepsilon} X$" for a first order variable x and a weak second-order variable X to make an atomic formula. We also allow the quantification of the form "$\exists X$" (and its dual "$\forall X$") over the weak second-order variables X.

The relation symbol ε is simply interpreted as the element relation and the interpretation of the quantifier $\exists X$ in $\mathcal{L}^{\aleph_0,II}$ is defined by

$$(1.1)\qquad \begin{array}{l}\mathfrak{A} \models \exists\, X \varphi(a_0, ..., a_{m-1}, B_0, ..., B_{n-1}, X) \;\Leftrightarrow \\ \quad \text{there exists a } B \in [\ |\mathfrak{A}|\ ]^{\aleph_0} \text{ such that } \mathfrak{A} \models \varphi(a_0, ..., a_{m-1}, B_0, ..., B_{n-1}, B)\end{array}$$ intro-0

for a first-order structure $\mathfrak{A}$, an $\mathcal{L}^{\aleph_0,II}$-formula φ in the signature of the structure $\mathfrak{A}$ with $\varphi = \varphi(x_0, ..., x_{m-1}, X_0, ..., X_{n-1}, X)$ where $x_0, ..., x_{m-1}$ are first-order and X_0, ..., X_{n-1}, X second-order variables, $a_0, ..., a_{m-1} \in |\mathfrak{A}|$, and $B_0, ..., B_{n-1} \in [\ |\mathfrak{A}|\ ]^{\aleph_0}$.

If we allow the weak second-order variables in $\aleph_0$-interpretation and the logical relation symbol ε but no quantification over the weak second-order variables, the resulting logic is called $\mathcal{L}^{\aleph_0}$.

$\mathcal{L}^{\aleph_0}_{stat}$ is the logic obtained from $\mathcal{L}^{\aleph_0}$ by adding the stationarity quantifier "$stat\,X$" (and its dual "$aa\,X$" but neither the existential nor universal quantification over second-order variables). The semantics of the logic is defined by

$$\begin{aligned}(1.2)\qquad &\mathfrak{A} \models stat\,X\varphi(a_0,\ldots,a_{m-1},B_0,\ldots,B_{n-1},X) \ \Leftrightarrow \\ &\qquad \{B \in [\,|A|\,]^{\aleph_0} \ :\ \mathfrak{A} \models \varphi(a_0,\ldots,a_{m-1},B_0,\ldots,B_{n-1},B)\} \text{ is stationary}\end{aligned}$$

sdls-0

for a first-order structure $\mathfrak{A}$, an $\mathcal{L}^{\aleph_0}_{stat}$-formula φ in the signature of $\mathfrak{A}$ with $\varphi = \varphi(x_0, \ldots, x_{m-1}, X_0, \ldots, X_{n-1}, X)$, $a_0, \ldots, a_{m-1} \in |\mathfrak{A}|$ and $B_0, \ldots, B_{n-1} \in [A]^{\aleph_0}$.

The stationarity quantifier was introduced in [17] and has been studied intensively mainly in connection with the questions about the completeness of deduction systems and the compactness. The reader may consult e.g. [2] or [20] for further reference.

$\mathcal{L}^{\aleph_0,II}_{stat}$ is the weak second-order logic in which both types of quantifiers "$\exists X$" and "$stat\,X$" are allowed.

The dual quantifier to the stationarity quantifier expresses "there are club many": for $\mathcal{L} = \mathcal{L}^{\aleph_0}_{stat}$ or $\mathcal{L}^{\aleph_0,II}_{stat}$ and $\mathcal{L}$-formula φ, let $aa\,X\,\varphi$ be the abbreviation of $\neg stat\,X\neg\varphi$. By (1.2), we have

$$\begin{aligned}(1.3)\qquad &\mathfrak{A} \models aa\,X\varphi(a_0,\ldots,a_{m-1},B_0,\ldots,B_{n-1},X) \ \Leftrightarrow \\ &\{B \in [A]^{\aleph_0} \ :\ \mathfrak{A} \models \varphi(a_0,\ldots,a_{m-1},B_0,\ldots,B_{n-1},B)\} \text{ contains a club set.}\end{aligned}$$

sdls-2

!!

The first order quantifier Q where $Qx\,\varphi$ is to be interpreted as "there are uncountably many x such that φ" is expressible using the stationarity quantifier as

$$(1.4)\qquad stat\,X\,\exists x\,(x \not\varepsilon\ X \wedge \varphi).$$

sdls-2-0

For one of the logics $\mathcal{L}$ as above, and structures $\mathfrak{A}$, $\mathfrak{B}$ of the same signature with $\mathfrak{B} \subseteq \mathfrak{A}$, we say that $\mathfrak{B}$ is $\mathcal{L}$-elementary submodel of $\mathfrak{A}$ (notation: $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$) if, for any formula $\varphi(x_0, \ldots, X_0, \ldots)$ in $\mathcal{L}$ of the signature where $x_0, \ldots$ are first order and $X_0, \ldots$ weak second-order variables, for any $b_0, \ldots \in |\mathfrak{B}|$ and for any countable subsets $B_0, \ldots$ of $|\mathfrak{B}|$, we have !!

$$(1.5)\qquad \mathfrak{B} \models \varphi(b_0, \ldots, B_0, \ldots) \text{ holds if and only if } \mathfrak{A} \models \varphi(b_0, \ldots, B_0, \ldots).$$

sdls-3

$\mathfrak{B}$ is a weakly $\mathcal{L}$-elementary submodel of $\mathfrak{A}$ (notation: $\mathfrak{B} \prec^{-}_{\mathcal{L}} \mathfrak{A}$), if

$$(1.6)\qquad \mathfrak{B} \models \varphi(b_0, \ldots, b_{n-1}) \text{ holds if and only if } \mathfrak{A} \models \varphi(b_0, \ldots, b_{n-1}) \text{ holds}$$

sdls-4

for all formulas $\varphi = \varphi(x_0, \ldots)$ in $\mathcal{L}$ without free weak second-order variables, and for all $b_0, \ldots, b_{n-1} \in |B|$.

The Strong Downward Löwenheim-Skolem Theorem[1)] for (formulas of a language) $\mathcal{L}$ down to $<\kappa$ is the assertion defined by

$\mathsf{SDLS}(\mathcal{L}, <\kappa)$: For any structure $\mathfrak{A}$ of countable signature there is $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$ of cardinality $< \kappa$.

We also consider the Strong Downward Löwenheim-Skolem Theorem with respect to the weak $\mathcal{L}$-elementary submodel relation:

$\mathsf{SDLS}^-(\mathcal{L}, <\kappa)$: For any structure $\mathfrak{A}$ of countable signature, there is $\mathfrak{B} \prec^-_{\mathcal{L}} \mathfrak{A}$ of cardinality $< \kappa$.

We shall call the cardinal κ as above the *reflection cardinal* or *Löwenheim-Skolem cardinal* of the respective Strong Downward Löwenheim-Skolem Theorem.

Since all logics considered here contain full first order logic, we may restrict ourselves to structures of relational signature, i.e. of such a signature that contains only constant and relation symbols. We shall call such structures simply as *relational structures*.

With the four logics introduced above and the two types of Strong Downward Löwenheim-Skolem theorems with reflection cardinal "$<\aleph_2$", we obtain 8 possible principles:

$\mathsf{SDLS}(\mathcal{L}^{\aleph_0}, <\aleph_2)$, $\mathsf{SDLS}(\mathcal{L}^{\aleph_0,II}, <\aleph_2)$, $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$, $\mathsf{SDLS}(\mathcal{L}^{\aleph_0,II}_{stat}, <\aleph_2)$, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}, <\aleph_2)$, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0,II}, <\aleph_2)$, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0,II}_{stat}, <\aleph_2)$.

In Theorem 1.1 below, we show that these principles are classified into 4 equivalence classes (over ZFC) and each of them is equivalent to one of well known principles.

Some of these Strong Downward Löwenheim Skolem Theorems are very strong combinatorial principles: in [16], M. Magidor noticed that the reflection of uncountable coloring number of a graph down to a subgraph of cardinality $\aleph_1$ follows from $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$. Since this reflection statement about coloring number of graphs is equivalent to Fodor-type Reflection Principle (FRP, see [7], [8]), we actually have the implication

(1.7) $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ implies FRP. sdls-4-a-a

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1) The adjective "strong" is added to indicate that $\mathfrak{B}$ in the statement of the property is not merely elementarily equivalent to but also elementary submodel of $\mathfrak{A}$.

FRP implies the total failure of square principle ([7]). Thus this principle is placed very high in the hierarchy of consistency strength measured by large large cardinals. Actually, all the known consistency proofs of FRP require at least the existence of a strongly compact cardinal.

FRP also implies SCH ([10]) and shown to be equivalent to many "mathematical" reflection statements ([7], [8], [9], [10]).

It is also easy to see that $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ is strictly stronger than FRP: it is easy to see that $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ implies the reflection principle on stationarity of subsets of $[\lambda]^{\aleph_0}$ which is called RP in [14]. RP implies FRP ([7]). This implication is strict: FRP is known to be consistent with arbitrarily large continuum (see [7]) while RP implies $2^{\aleph_0} \leq \aleph_2$ (Todorčević, see Theorem 37.18 in [14]).

For the Diagonal Reflection Principle $\mathsf{DRP}(\mathsf{IC}_{\aleph_0})$ mentioned in the next Theorem see Section 3.

main-thm

Theorem 1.1 (1) $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}, <\aleph_2)$ *is a theorem in* ZFC*.*

(2) *Each of* $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}, <\aleph_2)$*,* $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0,II}, <\aleph_2)$ *and* $\mathsf{SDLS}(\mathcal{L}^{\aleph_0,II}, <\aleph_2)$ *is equivalent to* CH *(over* ZFC*).*

(3) $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *is equivalent to* $\mathsf{DRP}(\mathsf{IC}_{\aleph_0})$ *(over* ZFC*).*

(4) *Each of* CH + $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$*,* $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0,II}_{stat}, <\aleph_2)$*,* $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *and* $\mathsf{SDLS}(\mathcal{L}^{\aleph_0,II}_{stat}, <\aleph_2)$ *is equivalent to* CH + $\mathsf{DRP}(\mathsf{IC}_{\aleph_0})$ *(over* ZFC*).*

(1) is trivial: $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}, <\kappa)$ is a reformulation of the usual Downward Löwenheim Skolem Theorem for first-order logic down to an elementary structure of cardinality $< \kappa$. In particular, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}, <\aleph_2)$ is also a theorem in ZFC.

(2) of Theorem 1.1 is proved in the following Section 2. The rest of the proof will be given in Section 3.

In Section 4, we consider a generalization of Game Reflection Principle by Bernhard König and show that this principle unifies the picture of reflection properties (including the Löwenheim-Skolem-Theorem type statements considered here).

In the continuation of the present paper [13], we will discuss the possibility of Strong Löwenheim-Skolem Theorems of stationary logics down to continuum (i.e. with the reflection cardinal either "$< 2^{\aleph_0}$" or "$\leq 2^{\aleph_0}$") under very large continuum.

Before finishing this section we cite some basic facts about club and stationary sets used repeatedly in the following sections.

L-basics-a

Lemma 1.2 *Suppose that* κ *is a regular uncountable cardinal and* X*,* Y *are sets with* $X \subseteq Y$*. For* $A \subseteq [X]^{<\kappa}$*, let* !!

$$\tilde{A} = \{b \in [Y]^{<\kappa} : b \cap X \in A\}. \tag{1.8}$$ basics-0-0

Then we have;

(1) *A contains a club in $[X]^{<\kappa}$ if and only if $\tilde{A}$ contains a club in $[Y]^{<\kappa}$.*

(2) *A is stationary in $[X]^{<\kappa}$ if and only if $\tilde{A}$ is stationary in $[Y]^{<\kappa}$.*

Proof. (1): Suppose that $C \subseteq A$ is a club in $[X]^{<\kappa}$. Then

(ℵ1.1) $\tilde{C} = \{b \in [Y]^{<\kappa} : b \cap X \in C\}$

is a club in $[Y]^{<\kappa}$ and $\tilde{C} \subseteq \tilde{A}$.

Suppose now that $\tilde{C} \subseteq \tilde{A}$ is a club in $[Y]^{<\kappa}$. We show that A contains a club in $[X]^{<\kappa}$. Let θ be a sufficiently large regular cardinal and let

(ℵ1.2) $\mathfrak{A} = \langle \mathcal{H}(\theta), \in, \lhd, X, Y, \kappa, \tilde{C} \rangle$ basics-1

where $\lhd$ is a well-ordering on $\mathcal{H}(\theta)$ (we shall denote the corresponding relation symbol also with $\lhd$) and X, Y, κ, $\tilde{C}$ are interpretations of constant symbols $\underset{\sim}{X}$, $\underset{\sim}{Y}$, $\underset{\sim}{\kappa}$, $\underset{\sim}{\tilde{C}}$ respectively.

For $a \in \mathcal{P}(\mathcal{H}(\theta))$, we shall denote with $sk_*^{\mathfrak{A}}(a)$ the $\subseteq$-minimal subset c of $\mathcal{H}(\theta)$ such that

(ℵ1.3) $a \subseteq c$; basics-1-0

(ℵ1.4) $\kappa \cap c < \kappa$; basics-1-1

(ℵ1.5) c is closed with respect to all definable functions in $\mathfrak{A}$. basics-1-2

$sk_*^{\mathfrak{A}}(a)$ is well-defined since κ is regular.

(ℵ1.6) $\mathfrak{A} \restriction sk^*_{\mathfrak{A}}(a) \prec \mathfrak{A}$ basics-1-3

by (ℵ1.5) and since $\mathfrak{A}$ has definable Skolem functions because of $\lhd$.

Thus,

(ℵ1.7) $b \subseteq sk_*^{\mathfrak{A}}(a)$ holds for all $b \in sk_*^{\mathfrak{A}}(a) \cap [\mathcal{H}(\theta)]^{<\kappa}$. basics-2

C-basics-0

Claim 1.2.1 *For any $a \in [\mathcal{H}(\theta)]^{<\kappa}$, $Y \cap sk_*^{\mathfrak{A}}(a) \in \tilde{C}$.*

$\vdash$ By (ℵ1.7) and the elementarity (ℵ1.6), we have $\bigcup(\tilde{C} \cap sk_*^{\mathfrak{A}}(a)) = \bigcup \underset{\sim}{\tilde{C}}^{sk_*^{\mathfrak{A}}(a)} = Y \cap sk_*^{\mathfrak{A}}(a)$. Since $\tilde{C} \cap sk_*^{\mathfrak{A}}(a) = \underset{\sim}{\tilde{C}}^{sk_*^{\mathfrak{A}}(a)}$ is directed by the elementarity (ℵ1.6): [we have $sk_*^{\mathfrak{A}}(a) \models$ "$\underset{\sim}{\tilde{C}}$ is directed" since $\mathfrak{A} \models$ "$\underset{\sim}{\tilde{C}}$ is directed"], we also have $\bigcup(\tilde{C} \cap sk_*^{\mathfrak{A}}(a)) \in \tilde{C}$. $\dashv$ (Claim 1.2.1)

Let

(ℵ1.8) $C = \{a \in [X]^{<\kappa} : sk^{\mathfrak{A}}(a) \cap X = a\}$. basics-3

Clearly C is a club $\subseteq [X]^{<\kappa}$. Thus the following Claim concludes the proof.

C-basics-1

Claim 1.2.2 $C \subseteq A$.

$\vdash$ For $a \in C$, we have $a = sk^{\mathfrak{A}}(a) \cap X = (sk^{\mathfrak{A}}(a) \cap Y) \cap X$. $sk^{\mathfrak{A}}(a) \cap Y \in \tilde{C} \subseteq \tilde{A}$ by Claim 1.2.1. Thus, by the definition (1.8) of $\tilde{A}$, it follows that $a \in A$. $\dashv$ (Claim 1.2.2)

(2): Suppose that A is not stationary in $[X]^{<\kappa}$ then there is a club $C \subseteq [X]^{<\kappa}$ disjoint from A. $\tilde{C} = \{\tilde{c} \in [Y]^{<\kappa} : \tilde{c} \cap X \in C\}$ is then a club disjoint from $\tilde{A}$. Thus $\tilde{A}$ is not stationary in $[Y]^{<\kappa}$.

Suppose now that $\tilde{A}$ is not stationary in $[Y]^{<\kappa}$ and let $\tilde{C}$ be a club disjoint from $\tilde{A}$. Let $\mathfrak{A}$ be the structure defined as in ($\aleph$1.2) for this $\tilde{C}$.

$$(\aleph 1.9) \quad C = \{a \in [X]^{<\kappa} : a = sk^{\mathfrak{A}}_{*}(a) \cap X\}$$ basics-4

is then a club $\subseteq [X]^{<\kappa}$. For each $a \in C$, we have $sk^{\mathfrak{A}}_{*}(a) \cap Y \in \tilde{C}$ by Claim 1.2.1. Thus $sk^{\mathfrak{A}}_{*}(a) \cap Y \notin \tilde{A}$. It follows that $a = (sk^{\mathfrak{A}}_{*}(a) \cap Y) \cap X \notin A$. Thus C is disjoint from A and A is not stationary. ❑ (Lemma 1.2)

L-basics-0

Lemma 1.3 *Suppose that κ, X, Y are as in Lemma 1.2 with $|X| \geq \kappa$. Suppose $A^* \subseteq [Y]^{<\kappa}$ and let*

$$(1.9) \qquad A = \{a \in [X]^{<\kappa} : a = b \cap X \text{ for some } b \in A^*\}.$$ basics-4-0

(1) *If A^* contains a club $\subseteq [Y]^{<\kappa}$ then A also contains a club $\subseteq [X]^{<\kappa}$.*

(2) *If A^* is stationary in $[Y]^{<\kappa}$ then A is stationary in $[X]^{<\kappa}$.*

Proof.

Let $\tilde{A}$ be defined by (1.8) for our A. Note that we have $A^* \subseteq \tilde{A}$.

(1): If A^* contains a club in $[Y]^{<\kappa}$ then $\tilde{A}$ also contains a club. By Lemma 1.2, (1), it follows that A contains a club in $[X]^{<\kappa}$.

(2): If A is not stationary in $[X]^{<\kappa}$ then $\tilde{A}$ is not stationary in $[Y]^{<\kappa}$ by Lemma 1.2, (2). Thus A^* is neither stationary in $[Y]^{<\kappa}$. ❑ (Lemma 1.3)

2 Proof of Theorem 1.1, (2)

prelim

Suppose that $\mathcal{L}$ is one of the four logics $\mathcal{L}^{\aleph_0}$, $\mathcal{L}^{\aleph_0,II}$, $\mathcal{L}^{\aleph_0}_{stat}$, $\mathcal{L}^{\aleph_0,II}_{stat}$. For a cardinal κ, we also consider the following strengthenings of the Strong Downward Löwenheim-Skolem Theorems introduced in the last section:

$\mathsf{SDLS}_+(\mathcal{L},<\kappa)$: For any structure $\mathfrak{A}=\langle A,...\rangle$ of countable signature with $|A|\geq\kappa$, there are *stationarily many* $M\in[A]^{<\kappa}$ such that $\mathfrak{A}\restriction M\prec_{\mathcal{L}}\mathfrak{A}$.

$\mathsf{SDLS}^-_+(\mathcal{L},<\kappa)$: For any structure $\mathfrak{A}=\langle A,...\rangle$ of countable signature with $|A|\geq\kappa$, there are *stationarily many* $M\in[A]^{<\kappa}$ such that $\mathfrak{A}\restriction M\prec^-_{\mathcal{L}}\mathfrak{A}$.

For $\kappa=\aleph_2$ the +-version of the Strong Downward Löwenheim-Skolem Theorems are equivalent to the corresponding Strong Downward Löwenheim-Skolem Theorems without +.

P-sdls-0

Lemma 2.1 *Suppose $\mathcal{L}$ is one of the four logics as above. Then*

(1) $\mathsf{SDLS}_+(\mathcal{L},<\aleph_2)$ *and* $\mathsf{SDLS}(\mathcal{L},<\aleph_2)$ *are equivalent and*

(2) $\mathsf{SDLS}^-_+(\mathcal{L},<\aleph_2)$ *and* $\mathsf{SDLS}^-(\mathcal{L},<\aleph_2)$ *are equivalent.*

Proof. $\mathsf{SDLS}^-_+(\mathcal{L}^{\aleph_0},<\aleph_2)$ is a consequence of corresponding Löwenheim-Skolem Theorem for first-order logic and hence a theorem in ZFC as is the case with $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0},<\aleph_2)$.

In the following, we show the equivalences "$\mathsf{SDLS}_+(\mathcal{L}^{\aleph_0},<\aleph_2)\Leftrightarrow\mathsf{SDLS}(\mathcal{L}^{\aleph_0},<\aleph_2)$" and
"$\mathsf{SDLS}^-_+(\mathcal{L}^{\aleph_0}_{stat},<\aleph_2)\Leftrightarrow\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat},<\aleph_2)$". Other cases can be proved similarly.

In both of the cases, the direction "$\Rightarrow$" is clear. So we only show the implication "$\Leftarrow$".

To prove the implication "$\mathsf{SDLS}_+(\mathcal{L}^{\aleph_0},<\aleph_2)\Leftarrow\mathsf{SDLS}(\mathcal{L}^{\aleph_0},<\aleph_2)$", assume that $\mathsf{SDLS}(\mathcal{L}^{\aleph_0},<\aleph_2)$ holds. Suppose that $\mathfrak{A}=\langle A,...\rangle$ is a structure of countable signature and of cardinality $\geq\aleph_2$ and $\mathcal{D}\subseteq[A]^{\aleph_1}$ is a club. We want to show that there is $B\in\mathcal{D}$ such that $\mathfrak{A}\restriction B\prec_{\mathcal{L}^{\aleph_0}}\mathfrak{A}$.

Without loss of generality, we may assume that $\mathfrak{A}$ is relational.

Let λ be a regular cardinal such that $\mathfrak{A}\in\mathcal{H}(\lambda)$. Note that we have $A\in\mathcal{H}(\lambda)$ and hence $A\subseteq\mathcal{H}(\lambda)$.

Let

$$\tilde{\mathfrak{A}}=\langle\mathcal{H}(\lambda),A,\mathfrak{A},\mathcal{D},\in\rangle \tag{2.1}$$ sdls-5

where we assume $A=\underset{\sim}{A}^{\tilde{\mathfrak{A}}}$ for a unary relation symbol $\underset{\sim}{A}$, $\mathfrak{A}=\underset{\sim}{\mathfrak{A}}^{\tilde{\mathfrak{A}}}$ and $\mathcal{D}=\underset{\sim}{\mathcal{D}}^{\tilde{\mathfrak{A}}}$ for constant symbols $\underset{\sim}{\mathfrak{A}}$ and $\underset{\sim}{\mathcal{D}}$.

By $\mathsf{SDLS}(\mathcal{L}^{\aleph_0},<\aleph_2)$, there is a $\tilde{\mathfrak{B}}\prec_{\mathcal{L}^{\aleph_0}}\tilde{\mathfrak{A}}$ such that $|\tilde{B}|\leq\aleph_1$ for $\tilde{B}=|\tilde{\mathfrak{B}}|$.

Cl-sdls-0

Claim 2.1.1 *$\omega_1\subseteq\tilde{B}$.*

$\vdash$ By elementarity (in the first-order logic), we have $\omega_1\cap\tilde{B}\leq\omega_1$. For any $U\in[\omega_1\cap\tilde{B}]^{\aleph_0}$, we have

$$\tilde{\mathfrak{A}} \models \text{“}\exists x\,(x \in \omega_1 \wedge \forall y\,(y \mathrel{\varepsilon} U \to y \in x))\text{”}. \tag{2.2}$$ sdls-6

It follows that

$$\tilde{\mathfrak{B}} \models \text{“}\exists x\,(x \in \omega_1 \wedge \forall y\,(y \mathrel{\varepsilon} U \to y \in x))\text{”}. \tag{2.3}$$ sdls-7

Since U is an arbitrary countable set with $U \in [\omega_1 \cap \tilde{B}]^{\aleph_0}$, this implies that $\omega_1 \subseteq \tilde{B}$.

$\dashv$ (Claim 2.1.1)

Let $B = |\mathfrak{A}| \cap \tilde{B}$ and $\mathfrak{B} = \mathfrak{A} \restriction B$.

Cl-sdls-1

Claim 2.1.2 *$B \in \mathcal{D}$.*

$\vdash$ For $D \in \mathcal{D} \cap \tilde{B}$, we have $D \subseteq B$ by Claim 2.1.1 and elementarity. Also by elementarity, $\mathcal{D} \cap \tilde{B}$ is directed and cofinal in $[A]^{\aleph_1}$. It follows that $B = \bigcup(\mathcal{D} \cap \tilde{B}) \in \mathcal{D}$. $\dashv$ (Claim 2.1.2)

Cl-sdls-0-0

Claim 2.1.3 (1) *$[\tilde{B}]^{\aleph_0} \subseteq \tilde{B}$.*

(2) *For an $\mathcal{L}^{\aleph_0}$-formula $\varphi = \varphi(x_0, ..., X_0, ...)$ in the signature of $\mathfrak{A}$, $a_0, ... \in B$ and $U_0, ... \in [B]^{\aleph_0}$, we have*

$$\tilde{\mathfrak{B}} \models \text{“}\underset{\sim}{\mathfrak{A}} \models \varphi(a_0, ..., U_0, ...)\text{”} \quad \Leftrightarrow \quad \mathfrak{B} \models \varphi(a_0, ..., U_0, ...). \tag{2.4}$$ sdls-8

$\vdash$ (1): Suppose $U \in [\tilde{B}]^{\aleph_0}$. Then we have $\tilde{\mathfrak{A}} \models \text{“}\exists x \forall y(y \mathrel{\varepsilon} U \leftrightarrow y \underset{\sim}{\in} x)\text{”}$. By elementarity we also have $\tilde{\mathfrak{B}} \models \text{“}\exists x \forall y(y \mathrel{\varepsilon} U \leftrightarrow y \underset{\sim}{\in} x)\text{”}$ this means that $U \in \tilde{B}$.

(2): This follows from (1). $\dashv$ (Claim 2.1.3)

The following Claim together Claim 2.1.2 shows that our B is as desired.

Cl-sdls-0-1

Claim 2.1.4 *$\mathfrak{B} \prec_{\mathcal{L}^{\aleph_0}} \mathfrak{A}$.*

$\vdash$ Suppose that $\varphi = \varphi(x_0, ..., X_0, ...)$ is an $\mathcal{L}^{\aleph_0}$-formula in the signature of $\mathfrak{A}$, $a_0, ... \in B$ and $U_0, ... \in [B]^{\aleph_0}$. Then we have

$$\begin{aligned} \mathfrak{A} \models \varphi(a_0, ..., U, ...) &\Leftrightarrow \tilde{\mathfrak{A}} \models \text{“}\underset{\sim}{\mathfrak{A}} \models \varphi(a_0, ..., U, ...)\text{”} \\ &\Leftrightarrow \tilde{\mathfrak{B}} \models \text{“}\underset{\sim}{\mathfrak{A}} \models \varphi(a_0, ..., U, ...)\text{”} \quad ; \text{by } \mathfrak{B} \prec_{\mathcal{L}^{\aleph_0}} \mathfrak{A} \\ &\Leftrightarrow \mathfrak{B} \models \varphi(a_0, ..., U, ...) \qquad ; \text{by Claim 2.1.3, (2).} \end{aligned} \tag{2.5}$$ sdls-9

$\dashv$ (Claim 2.1.4)

To prove the implication $\mathsf{SDLS}^-_+(\mathcal{L}^{\aleph_0}_{stat}, {<}\,\aleph_2) \Leftarrow \mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, {<}\,\aleph_2)$, assume that $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, {<}\,\aleph_2)$ holds. Let $\mathfrak{A} = \langle A, ...\rangle$ be a structure of cardinality $\geq \aleph_2$ and $\mathcal{D} \subseteq [A]^{\aleph_1}$ is a club.

We want to show that there is $B \in \mathcal{D}$ such that $\mathfrak{A} \restriction B \prec^-_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$. Without loss of generality, we may assume that $\mathfrak{A}$ is relational.

Let $\tilde{\mathfrak{A}}$ be defined as in (2.1). By $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, {<}\,\aleph_2)$, there is $\tilde{\mathfrak{B}} \prec^-_{\mathcal{L}^{\aleph_0}_{stat}} \tilde{\mathfrak{A}}$ such that $|\,\tilde{B}\,| \leq \aleph_1$ for $\tilde{B} = |\tilde{\mathfrak{B}}|$.

We can show again that $\omega_1 \subseteq \tilde{B}$ since $\omega_1 \cap \tilde{B}$ is uncountable by the elementarity $\tilde{\mathfrak{B}} \prec^-_{\mathcal{L}^{\aleph_0}_{stat}} \tilde{\mathfrak{A}}$ (remember (1.4)). Hence we can repeat the argument of Claim 2.1.2 to show $B \in \mathcal{D}$.

Thus the following Claim implies that our B is as desired.

Cl-sdls-2

Claim 2.1.5 *For an $\mathcal{L}^{\aleph_0}_{stat}$-formula $\varphi = \varphi(x_0, ..., X_0, ...)$ in the signature of the structure $\mathfrak{A}$, $a_0, ... \in B$ and $U_0, ... \in [B]^{\aleph_0} \cap \tilde{B}$, we have*

$$\tilde{\mathfrak{B}} \models \text{“}\,\underset{\sim}{\mathfrak{A}} \models \varphi(a_0, ..., U_0, ...)\text{”} \quad \Leftrightarrow \quad \mathfrak{B} \models \varphi(a_0, ..., U_0, ...). \tag{2.6}$$ sdls-10

$\vdash$ By induction on φ. The critical step is when

$$\varphi(x_0, ..., X_0, ...) = \mathit{stat}\, X\, \psi(x_0, ..., X_0, ..., X) \tag{2.7}$$

and (2.6) holds for ψ. (2.6) can be shown in this case by

$$\begin{aligned}
&\mathfrak{A} \models \varphi(a_0, ..., U_0, ...)\\
&\Leftrightarrow \tilde{\mathfrak{A}} \models \text{“}\,\underset{\sim}{\mathfrak{A}} \models \varphi(a_0, ..., U_0, ...)\text{”}\\
&\Leftrightarrow \tilde{\mathfrak{B}} \models \text{“}\,\underset{\sim}{\mathfrak{A}} \models \varphi(a_0, ..., U_0, ...)\text{”}\\
&\Leftrightarrow \tilde{\mathfrak{B}} \models \text{“}\,\underset{\sim}{\mathfrak{A}} \models \mathit{stat}\, X, \psi(a_0, ..., U_0, ..., X)\text{”}\\
&\Leftrightarrow \tilde{\mathfrak{B}} \models \text{“}\,\mathit{stat}\, X\, \exists x\, (x \equiv X \cap \underset{\sim}{A} \ \wedge\ \underset{\sim}{\mathfrak{A}} \models \psi(a_0, ..., U_0, ..., x))\text{”} \quad ;\ \text{(a)}\\
&\Leftrightarrow \{a \in [B]^{\aleph_0} \cap \tilde{B} \,:\, \mathfrak{B} \models \psi(a_0, ..., U_0, ..., a)\} \text{ is stationary} \quad ;\ \text{(b)}\\
&\Leftrightarrow \mathfrak{B} \models \mathit{stat}\, X\, \psi(a_0, ..., U_0, ..., X)\\
&\Leftrightarrow \mathfrak{B} \models \varphi(a_0, ..., U_0, ...).
\end{aligned}$$

The equivalence of (a) and the line above it holds by the elementarity $\tilde{\mathfrak{B}} \prec^-_{\mathcal{L}^{\aleph_0}_{stat}} \tilde{\mathfrak{A}}$ and since the corresponding equivalence also holds in $\tilde{\mathfrak{A}}$. The equivalence of (b) and the line above it holds by induction hypothesis. $\dashv$ (Claim 2.1.5)

□ (Lemma 2.1)

The following Proposition 2.2 together with Lemma 2.1 implies Theorem 1.1, (2):

P-sdls-1

Proposition 2.2 *Suppose that κ is a cardinal with $cf(\kappa) \geq \omega_2$. Then the following are equivalent:*

(a) *$\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$;* (b) *$\mathsf{SDLS}_+(\mathcal{L}^{\aleph_0}, {<}\,\kappa)$;* (c) *$\mathsf{SDLS}^-_+(\mathcal{L}^{\aleph_0,II}, {<}\,\kappa)$;*
(d) *$\mathsf{SDLS}_+(\mathcal{L}^{\aleph_0,II}, {<}\,\kappa)$.*

Proof. (d) $\Rightarrow$ (b) and (d) $\Rightarrow$ (c) are clear.

(b) $\Rightarrow$ (a): Assume $\mathsf{SDLS}_+(\mathcal{L}^{\aleph_0}, <\kappa)$ and $\mu < \kappa$. We want to show that $\mu^{\aleph_0} < \kappa$. Let $\mathfrak{A} = \langle \kappa \cup \mathcal{P}(\mu), \mu, E\rangle$ where $E = \{\langle \alpha, a\rangle \,:\, \alpha \in \mu,\, a \in \mathcal{P}(\mu),\, \alpha \in a\}$ and, μ and E are interpretations of the unary and binary relation symbols $\underline{\mu}$ and $\underline{E}$, respectively. Let $\mathfrak{B} = \langle B, \underline{\mu}^{\mathfrak{B}}, \underline{E}^{\mathfrak{B}}\rangle$ be an $\mathcal{L}^{\aleph_0}$-elementary submodel of $\mathfrak{A}$ of cardinality $< \kappa$ such that $\mu \subseteq B$ (there is such B since $\{D \in [A]^{<\kappa} \,:\, \mu \subseteq D\}$ is club in $[A]^{<\kappa}$). Consider the $\mathcal{L}^{\aleph_0}$-formula $\varphi(X) = \exists x \forall y (\underline{\mu}(y) \rightarrow (y \,\varepsilon\, X \leftrightarrow \underline{E}(y, x)))$. For all $U \in [\underline{\mu}^{\mathfrak{B}}]^{\aleph_0}$, we have $\mathfrak{A} \models \varphi(U)$. By elementarity, it follows that $\mathfrak{B} \models \varphi(U)$. This implies that $U \in |\mathcal{B}|$. Thus $[\underline{\mu}^{\mathfrak{B}}]^{\aleph_0} \subseteq |\mathfrak{B}|$. Since $\underline{\mu}^{\mathfrak{B}} = \mu$, $\mu^{\aleph_0} \leq \|\mathfrak{B}\| < \kappa$.

(c) $\Rightarrow$ (a): Assume $\mathsf{SDLS}^-_+(\mathcal{L}^{\aleph_0,II}, <\kappa)$ and $\mu < \kappa$. Let $\mathfrak{A}$ be as in the proof of "(b) $\Rightarrow$ (a)" Consider the $\mathcal{L}^{\aleph_0,II}$-sentence in L:

$$(2.8) \qquad \psi = \forall X(\text{“}X \subseteq \underline{\mu}\text{”} \rightarrow \exists x \forall y\, (\underline{\mu}(y) \rightarrow (y \,\varepsilon\, X \leftrightarrow \underline{E}(y, x))))$$

where "$X \subseteq \underline{\mu}$" is an abbreviation of $\forall x\, (x \,\varepsilon\, X \rightarrow \underline{\mu}(x))$. Clearly we have $\mathfrak{A} \models \psi$.

Let $\mathfrak{B} \prec^-_{\mathcal{L}^{\aleph_0,II}} \mathfrak{A}$ be of cardinality $< \kappa$ with $\mu \subseteq |\mathfrak{B}|$. Since $\mathfrak{B} \models \psi$ by the elementarity, $[\underline{\mu}^{\mathfrak{B}}]^{\aleph_0} \subseteq |\mathfrak{B}|$. Thus we again have $\mu^{\aleph_0} \leq \|\mathfrak{B}\| < \kappa$.

(a) $\Rightarrow$ (d): Assume (a) and let $\mathfrak{A}$ be structure in countable signature with $\|\mathfrak{A}\| \geq \kappa$. Let $A = |\mathfrak{A}|$ and let $\mathcal{D} \subseteq [A]^{<\kappa}$ be a club in $[A]^{<\kappa}$. We want to show that there is $M \in \mathcal{D}$ such that $\mathfrak{A} \restriction M \prec_{\mathcal{L}^{\aleph_0,II}} \mathfrak{A}$.

Without loss of generality, we may assume that the signature of the structure is relational. Let θ be a regular cardinal such that $\mathfrak{A} \in \mathcal{H}(\theta)$. Let

$$(2.9) \qquad \tilde{\mathfrak{A}} = \langle \mathcal{H}(\theta), A, \mathfrak{A}, \in\rangle$$ sdls-4-0

where we assume that $A = \underline{A}^{\tilde{\mathfrak{A}}}$ for a unary predicate symbol $\underline{A}$ and $\mathfrak{A} = \underline{\mathfrak{A}}^{\tilde{\mathfrak{A}}}$ for a constant symbol $\underline{\mathfrak{A}}$.

The following claim can be proved by induction on $\mathcal{L}^{\aleph_0,II}$-formula φ:

C-sdls-a-0

Claim 2.2.1 *Suppose that $\varphi(x_0, ..., x_{m-1}, Y_0, ..., Y_{n-1})$ is an $\mathcal{L}^{\aleph_0,II}$-formula in L. Suppose further that $\tilde{\mathfrak{C}} \prec \tilde{\mathfrak{A}}$ with $C = \underline{A}^{\tilde{\mathfrak{C}}} = A \cap |\tilde{\mathfrak{C}}|$ is such that $[C]^{\aleph_0} \subseteq |\tilde{\mathfrak{C}}|$.*

Then for any $a_0, ..., a_{m-1} \in C$ and $U_0, ..., U_{n-1} \in [C]^{\aleph_0}$,

$$(2.10) \qquad \tilde{\mathfrak{C}} \models \text{“}\underline{\mathfrak{A}} \models \varphi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1})\text{”} \\ \Leftrightarrow\ \mathfrak{A} \restriction C \models \varphi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1}). \qquad \dashv$$

Let $\tilde{B} \in [\mathcal{H}(\theta)]^{<\kappa}$ be such that, letting $\tilde{\mathfrak{B}} = \tilde{\mathfrak{A}} \restriction \tilde{B}$,

$$(2.11) \qquad \tilde{\mathfrak{B}} \prec \tilde{\mathfrak{A}},$$ sdls-11-a-0

$$(2.12) \qquad A \cap \tilde{B} \in \mathcal{D} \text{ and}$$ sdls-11-a-1

$$(2.13) \qquad [\tilde{B}]^{\aleph_0} \subseteq \tilde{B}.$$ sdls-4-1

Note that $\tilde{B}$ as above can be obtained as the union of increasing chain $\langle B_\alpha : \alpha < \omega_1\rangle$ of (underlying sets of) elementary submodels of $\tilde{\mathfrak{A}}$ of cardinality $<\kappa$ together with an increasing sequence $\langle D_\alpha : \alpha < \omega_1\rangle$ of elements of $\mathcal{D}$ such that $A \cap B_\alpha \subseteq D_\alpha$, and $[B_\alpha]^{\aleph_0}$, $D_\alpha \subseteq B_{\alpha+1}$ for all $\alpha < \omega_1$. This is possible by (a) and $cf(\kappa) \geq \omega_2$. The union $\tilde{B} = \bigcup_{\alpha<\omega_1} B_\alpha = \bigcup_{\alpha<\omega_1} D_\alpha$ then satisfies (2.11), (2.12), (2.13). $|\,\tilde{B}\,| < \kappa$ and $\tilde{B} \in \mathcal{D}$ by $cf(\kappa) \geq \omega_2$. !!

Let $\mathfrak{B} = \mathfrak{A} \restriction \underset{\sim}{A}^{\tilde{\mathfrak{B}}} = \mathfrak{A} \restriction (A \cap \tilde{B})$. Then we have $\|\mathfrak{B}\| < \kappa$ and thus $X \subseteq |\mathfrak{B}|$.

We claim $\mathfrak{B} \prec_{\mathcal{L}^{\aleph_0,II}} \mathfrak{A}$:

Let $B = |\mathfrak{B}|$ and suppose $\varphi = \varphi(x_0, ..., x_{m-1}, Y_0, ..., Y_{n-1})$ is an $\mathcal{L}^{\aleph_0,II}$-formula. For $a_0, ..., a_{m-1} \in B$ and $U_0, ..., U_{n-1} \in [B]^{\aleph_0}$, we have

$$\begin{aligned}
(2.14)\quad & \mathfrak{A} \models \varphi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1}) && \\
&\Leftrightarrow\ \tilde{\mathfrak{A}} \models \text{“}\underset{\sim}{\mathfrak{A}} \models \varphi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1})\text{”} && \text{by Claim 2.2.1,}\\
&\Leftrightarrow\ \tilde{\mathfrak{B}} \models \text{“}\underset{\sim}{\mathfrak{A}} \models \varphi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1})\text{”} && \text{by elementarity,}\\
&\Leftrightarrow\ \mathfrak{B} \models \varphi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1}) && \text{by Claim 2.2.1.}
\end{aligned}$$ sdls-4-3

❏ (Proposition 2.2)

3 Diagonal Reflection Principle

DRP

Let us first recall The Diagonal Reflection Principle introduced by S. Cox in [1].

Let κ be a regular uncountable cardinal. For a class $\mathcal{C}$ of sets of size $<\kappa$ and a cardinal $\theta > \kappa$ the Diagonal Reflection Principle for κ, θ, and $\mathcal{C}$ is the following statement:

$\mathsf{DRP}(<\kappa, \theta, \mathcal{C})$: There are stationarily many $M \in [\mathcal{H}((\theta^{\aleph_0})^+)]^{<\kappa}$ such that

(1) $M \cap \mathcal{H}(\theta) \in \mathcal{C}$; and

(2) for all $R \in M$ such that R is a stationary subset of $[\theta]^{\aleph_0}$, $R \cap [\theta \cap M]^{\aleph_0}$ is stationary in $[\theta \cap M]^{\aleph_0}$.

We shall call M as above a *reflection point* of $\mathsf{DRP}(<\kappa, \theta, \mathcal{C})$.

For a cardinal μ, let

(3.1) $\mathsf{IU}_\mu = \{X : [X]^\mu \cap X \text{ is cofinal in } [X]^\mu\}$; DRP-0

(3.2) $\mathsf{IS}_\mu = \{X : [X]^\mu \cap X \text{ is stationary } [X]^\mu\}$; DRP-1

(3.3) $\mathsf{IC}_\mu = \{X : [X]^\mu \cap X \text{ contains a subset which is club in } [X]^\mu\}$. DRP-3

Elements of IU_μ, IS_μ, IC_μ are said to be *internally unbounded*, *internally stationary* and *internally club* (with respect to subsets of cardinality μ) respectively.

The Diagonal Reflection Principle with reflection cardinal κ for internally clubness is the statement:

$\mathsf{DRP}(<\kappa, \mathsf{IC}_{\aleph_0})$: $\mathsf{DRP}(<\kappa, \theta, \mathsf{IC}_{\aleph_0})$ holds for all cardinals $\theta \geq \kappa$.

$\mathsf{DRP}(<\kappa, \mathsf{IU}_{\aleph_0})$ and $\mathsf{DRP}(<\kappa, \mathsf{IS}_{\aleph_0})$ are defined similarly. Finally if $\kappa = \aleph_2$, we drop the mention on κ and write $\mathsf{DRP}(\mathsf{IU}_{\aleph_0})$, $\mathsf{DRP}(\mathsf{IS}_{\aleph_0})$ and $\mathsf{DRP}(\mathsf{IC}_{\aleph_0})$ instead of $\mathsf{DRP}(<\aleph_2, \mathsf{IU}_{\aleph_0})$, $\mathsf{DRP}(<\aleph_2, \mathsf{IS}_{\aleph_0})$ and $\mathsf{DRP}(<\aleph_2, \mathsf{IC}_{\aleph_0})$, respectively.

$\mathsf{DRP}(\mathsf{IC}_{\aleph_0})$ is one of the Diagonal Reflection Principles Cox considered in [1][2)]. We shall call $\mathsf{DRP}(\mathsf{IU}_{\aleph_0})$ ($\mathsf{DRP}(\mathsf{IS}_{\aleph_0})$ or, $\mathsf{DRP}(\mathsf{IC}_{\aleph_0})$ respectively) the *Diagonal Reflection Principle for internally unboundedness* (*for internally stationarity* or, *for internal clubness*, respectively).

We can also consider the following variations of the reflection principle which is simply called RP in [14]:

For a class $\mathcal{C}$ as above and cardinals κ and λ, let

$\mathsf{RP}(<\kappa, \theta, \mathcal{C})$: For any stationary $S \subseteq [\theta]^{\aleph_0}$ there are stationarily many $M \in [\mathcal{H}(\theta)]^{<\kappa}$ such that

(1) $M \in \mathcal{C}$; and

(2) $S \cap [\theta \cap M]^{\aleph_0}$ is stationary in $[\theta \cap M]^{\aleph_0}$.

Let

$\mathsf{RP}_{\mathsf{IC}_{\aleph_0}}$: $\mathsf{RP}(<\aleph_2, \theta, \mathsf{IC}_{\aleph_0})$ holds for all regular $\theta \geq \aleph_2$.

Let $\mathsf{RC}_{\mathsf{IU}_{\aleph_0}}$ and $\mathsf{RC}_{\mathsf{IS}_{\aleph_0}}$ be defined similarly to $\mathsf{RC}_{\mathsf{IC}_{\aleph_0}}$. $\mathsf{RC}_{IU_{\aleph_0}}$ is equivalent to the !!
principle known as Axiom R of Fleissner (see [12]).

The following implications are trivial possibly except the ones in left and right ends; the leftmost implication is proved in [7] while the proofs of rightmost (horizontal) implications are to be found in [1] and [14]:

$$\begin{array}{ccccccccc} & & \mathsf{DRP}(\mathsf{IU}_{\aleph_0}) & \leftarrow & \mathsf{DRP}(\mathsf{IS}_{\aleph_0}) & \leftarrow & \mathsf{DRP}(\mathsf{IC}_{\aleph_0}) & \leftarrow & \mathsf{MA}^{+\omega_1}(\sigma\text{-closed}) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathsf{FRP} & \leftarrow & \mathsf{RP}_{\mathsf{IU}_{\aleph_0}} & \leftarrow & \mathsf{RP}_{\mathsf{IS}_{\aleph_0}} & \leftarrow & \mathsf{RP}_{\mathsf{IC}_{\aleph_0}} & \leftarrow & \mathsf{MA}^{+}(\sigma\text{-closed}) \end{array}$$

The local versions $\mathsf{DRP}(<\kappa, \theta, \mathcal{C})$ of the Diagonal Reflection Principles (as well as Reflection Principles $\mathsf{RP}(<\kappa, \theta, \mathcal{C})$) enjoy the following type of downward transfer property:

P-DRP-0

Lemma 3.1 *Suppose that $\mathcal{C}$ is one of $\mathsf{IU}_{\aleph_0}$, $\mathsf{IS}_{\aleph_0}$, $\mathsf{IC}_{\aleph_0}$, κ is a regular cardinal and θ, θ' are cardinals with $\kappa \leq \theta < \theta'$. Then $\mathsf{DRP}(<\kappa, \theta', \mathcal{C})$ implies $\mathsf{DRP}(<\kappa, \theta, \mathcal{C})$.*

Proof. Assume that $\mathsf{DRP}(<\kappa, \theta', \mathcal{C})$ holds and suppose that $\mathcal{D}$ is a club set in $[\mathcal{H}((\theta^{\aleph_0})^+)]^{<\kappa}$. We have to show that there is $M \in \mathcal{D}$ such that

(1) $M \cap \mathcal{H}(\theta) \in \mathcal{C}$; and

(2) for all $R \in M$ such that R is a stationary subset of $[\theta]^{\aleph_0}$, $R \cap [\theta \cap M]^{\aleph_0}$ is stationary in $[\theta \cap M]^{\aleph_0}$.

Let

$$\mathcal{D}' = \{N \in [\mathcal{H}((\theta'^{\,\aleph_0})^+)]^{<\kappa} : \text{(a)}\ N \cap \mathcal{H}((\theta^{\aleph_0})^+) \in \mathcal{D},\ \text{(b)}\ \kappa, \theta \in N, \text{ and (c)}\ N \prec \mathcal{H}((\theta'^{\,\aleph_0})^+)\}. \tag{3.4}$$ DRP-4

Then $\mathcal{D}'$ contains a club set in $[\mathcal{H}((\theta'^{\,\aleph_0})^+)]^{<\kappa}$ by Lemma 1.2. Thus, by $\mathsf{DRP}(<\kappa, \theta', \mathcal{C})$, there is $M' \in \mathcal{D}'$ such that

(1)′ $M' \cap \mathcal{H}(\theta') \in \mathcal{C}$; and

(2)′ for all $R \in M'$ such that R is a stationary subset of $[\theta']^{\aleph_0}$, $R \cap [\theta' \cap M']^{\aleph_0}$ is stationary in $[\theta' \cap M']^{\aleph_0}$.

Let $M = M' \cap \mathcal{H}((\theta^{\aleph_0})^+)$. Then $M \in \mathcal{D}$ by (3.4), (a). By Lemma 1.2 and Lemma 1.3, (2)′ above implies that M satisfies (2).

By Lemma 1.3, the following Claim implies that M satisfies (1).

Cl-DRP-0

Claim 3.1.1 *For any $a \in M'$, if $|\,a\,| < \theta$, we have $a \cap \mathcal{H}(\theta) \in M$.*

$\vdash$ $\mathcal{H}(\theta)$ is definable in $\mathcal{H}(\theta')$ with the parameter θ. Let $\chi(x, \theta)$ be a formula defining $\mathcal{H}(\theta)$ in $\mathcal{H}(\theta')$. That is, for all $a \in \mathcal{H}(\theta')$ we have $\mathcal{H}(\theta') \models \chi(a, \theta)$ if and only if $a \in \mathcal{H}(\theta)$. By (3.4), (b) and (c), and by elementarity, $\chi(x, \theta)$ defines $M \cap \mathcal{H}(\theta)$ in M'. Since $\mathcal{H}((\theta'^{\aleph_0})^+) \models$ "$\forall x \exists y \forall z\, (z \in y \leftrightarrow z \in x \wedge \chi(z, \theta))$", it follows that $M' \models$ "$\forall x \exists y \forall z\, (z \in y \leftrightarrow z \in x \wedge \chi(z, \theta))$". Thus, for $a \in M'$ we have $a \cap \mathcal{H}(\theta) \in M'$. If $|\,a\,| < \theta$, we have $a \cap \mathcal{H}(\theta) \in \mathcal{H}(\theta)$ and hence $a \cap \mathcal{H}(\theta) \in M' \cap \mathcal{H}(\theta) \subseteq M$.

$\dashv$ (Claim 3.1.1)

□ (Lemma 3.1)

Let $\mathcal{C}$ be one of $\mathsf{IU}_{\aleph_0}$, $\mathsf{IS}_{\aleph_0}$, $\mathsf{IC}_{\aleph_0}$. Let κ and λ be cardinals with $\kappa \leq \lambda$. We later show that the global version of following principle characterizes the (global version of the) Diagonal Reflection Principles.

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2) In [1], the definition of DRP refers only regular θ but by Lemma 3.1 this does not make any difference.

$(*)^{\mathcal{C}}_{<\kappa,\lambda}$: For any countable expansion $\tilde{\mathfrak{A}}$ of $\langle \mathcal{H}(\lambda), \in \rangle$ and sequence $\langle S_a : a \in \mathcal{H}(\lambda) \rangle$ such that S_a is a stationary subset of $[\mathcal{H}(\lambda)]^{\aleph_0}$ for all $a \in \mathcal{H}(\lambda)$, there is an $M \in [\mathcal{H}(\lambda)]^{<\kappa}$ such that

(1) $M \in \mathcal{C}$;

(2) $\tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}}$ and

(3) $S_a \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$ for all $a \in M$.

In the following, we also consider two further variations of this principle. The first one can be considered as the "internal" version of $(*)^{\mathcal{C}}_{<\kappa,\lambda}$. This variation will play an important roll in the sequel [13] of the present paper. The relation of the second variation to $(*)^{\mathcal{C}}_{<\kappa,\lambda}$ parallels to the relation of $\mathsf{SDLS}^-_+(\cdots)$ to $\mathsf{SDLS}^-(\cdots)$.

$(*)^{int\ \mathcal{C}}_{<\kappa,\lambda}$: For any countable expansion $\tilde{\mathfrak{A}}$ of $\langle \mathcal{H}(\lambda), \in \rangle$ and sequence $\langle S_a : a \in \mathcal{H}(\lambda) \rangle$ such that S_a is a stationary subset of $[\mathcal{H}(\lambda)]^{\aleph_0}$ for all $a \in \mathcal{H}(\lambda)$, there is an $M \in [\mathcal{H}(\lambda)]^{<\kappa}$ such that

(1) $M \in \mathcal{C}$;

(2) $\tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}}$ and

(3)′ $S_a \cap M$ is stationary in $[M]^{\aleph_0}$ for all $a \in M$.

$(*)^{+\mathcal{C}}_{<\kappa,\lambda}$: For any countable expansion $\tilde{\mathfrak{A}}$ of $\langle \mathcal{H}(\lambda), \in \rangle$ and sequence $\langle S_a : a \in \mathcal{H}(\lambda) \rangle$ such that S_a is a stationary subset of $[\mathcal{H}(\lambda)]^{\aleph_0}$ for all $a \in \mathcal{H}(\lambda)$, there are stationarily many $M \in [\mathcal{H}(\lambda)]^{<\kappa}$ such that

(1) $M \in \mathcal{C}$;

(2) $\tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}}$ and

(3) $S_a \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$ for all $a \in M$. (3) modified after the submission.

P-DRP-1

Lemma 3.2 *Suppose that κ is a regular cardinal and λ a cardinals such that $\aleph_1 < \kappa \leq \lambda$.*

(1) *If $\mathcal{C}$ is one of $\mathsf{IU}_{\aleph_0}$, $\mathsf{IS}_{\aleph_0}$, then $(*)^{int\ \mathcal{C}}_{<\kappa,\lambda}$ implies $(*)^{\mathcal{C}}_{<\kappa,\lambda}$.*

(2) *$(*)^{int\ \mathsf{IC}_{\aleph_0}}_{<\kappa,\lambda}$ is equivalent to $(*)^{\mathsf{IC}_{\aleph_0}}_{<\kappa,\lambda}$.*

(3) *If $\mathcal{C}$ is one of $\mathsf{IU}_{\aleph_0}$, $\mathsf{IS}_{\aleph_0}$, $\mathsf{IC}_{\aleph_0}$ then $(*)^{+\mathcal{C}}_{<\aleph_2,\lambda}$ is equivalent to $(*)^{\mathcal{C}}_{<\aleph_2,\lambda}$.*

Proof. (1) is immediate from definitions.

(2) follows from the fact that, if M is internally club, then for any stationary $S \subseteq [M]^{\aleph_0}$, $S \cap M$ is also stationary in $[M]^{\aleph_0}$.

(3) can be proved similarly to Lemma 2.1. ❑ (Lemma 3.2)

P-DRP-2

Lemma 3.3 *Suppose that $\mathcal{C}$ is one of $\mathsf{IU}_{\aleph_0}$, $\mathsf{IS}_{\aleph_0}$, $\mathsf{IC}_{\aleph_0}$. For a regular κ and cardinals λ, λ' with $\aleph_1 < \kappa \leq \lambda < \lambda'$, $(*)^{\mathcal{C}}_{<\kappa,\lambda'}$ implies $(*)^{\mathcal{C}}_{<\kappa,\lambda}$, $(*)^{int\ \mathcal{C}}_{<\kappa,\lambda'}$ implies $(*)^{int\ \mathcal{C}}_{<\kappa,\lambda}$ and $(*)^{+\mathcal{C}}_{<\kappa,\lambda'}$ implies $(*)^{+\mathcal{C}}_{<\kappa,\lambda}$.*

Proof. We prove the first implication. The other implications can proved similarly.

Assume that κ, λ, λ' are as above and $(*)^{\mathcal{C}}_{<\kappa,\lambda'}$ holds. Suppose that $\mathfrak{A}$ is a countable expansion of the structure $\langle \mathcal{H}(\lambda), \in \rangle$ and $\langle S_a : a \in \mathcal{H}(\lambda) \rangle$ a family of stationary subsets of $[\mathcal{H}(\lambda)]^{\aleph_0}$.

Let $\tilde{\mathfrak{A}} = \langle \mathcal{H}(\lambda'), \lambda, \underbrace{\mathcal{H}(\lambda), ...,}_{=\mathfrak{A}} \in \rangle$ and let $\langle \tilde{S}_a : a \in \mathcal{H}(\lambda') \rangle$ be defined by:

$$(\aleph 3.1) \quad \tilde{S}_a = \begin{cases} \{x \in [\mathcal{H}(\lambda')]^{\aleph_0} : x \cap \mathcal{H}(\lambda) \in S_a\}, & \text{if } a \in \mathcal{H}(\lambda); \\ [\mathcal{H}(\lambda')]^{\aleph_0}, & \text{otherwise.} \end{cases}$$ sdls-6-0

Note that $\tilde{S}_a$ is stationary in $[\mathcal{H}(\lambda')]^{\aleph_0}$ for all $a \in \mathcal{H}(\lambda')$ by Lemma 1.2.

By $(*)^{\mathcal{C}}_{<\kappa,\lambda'}$ there is $N \in [\mathcal{H}(\lambda')]^{<\kappa}$ such that $N \in \mathcal{C}$, $\tilde{\mathfrak{A}} \restriction N \prec \tilde{\mathfrak{A}}$ and $\tilde{S}_a \cap [N]^{\aleph_0}$ is stationary in $[N]^{\aleph_0}$ for all $a \in N$.

Let $M = \mathcal{H}(\lambda) \cap N$. Then $\mathfrak{A} \restriction N \prec \mathfrak{A}$ and $N \in \mathcal{C}$. The latter follows from Lemma 1.3 and a claim corresponding to Claim 3.1.1.

For $a \in M$, $\tilde{S}_a \cap [N]^{\aleph_0}$ is stationary by the choice of N. By Lemma 1.3, it follows that $S_a \cap [M]^{\aleph_0} = (\tilde{S}_a \cap [N]^{\aleph_0}) \cap [M]^{\aleph_0}$ is also stationary in $[M]^{\aleph_0}$.

This shows that $(*)^{\mathcal{C}}_{<\kappa,\lambda}$ holds for the structure $\mathfrak{A}$. ❑ (Lemma 3.3)

L-sdls-0-1

Lemma 3.4 *Suppose that $\mathcal{C}$ is one of $\mathsf{IU}_{\aleph_0}$, $\mathsf{IS}_{\aleph_0}$, $\mathsf{IC}_{\aleph_0}$. For a regular cardinal $\kappa > \aleph_1$, $\mathsf{DRP}(<\kappa, \mathcal{C})$ holds if and only if* !!

$$(3.5) \quad (*)^{+\mathcal{C}}_{<\kappa,\lambda} \text{ holds for all cardinal } \lambda \geq \kappa.$$ sdls-11-0

Proof. By Lemma 3.1 and Lemma 3.3, it is enough to prove the following:

Claim 3.4.1 *For a regular cardinal κ and a cardinal θ such that*

$$(3.6) \quad \theta = 2^{<\theta} \text{ and}$$ sdls-12

$$(3.7) \quad cf(\theta) \geq \kappa,$$ sdls-13

$(*)^{+\mathcal{C}}_{<\kappa,\theta}$ is equivalent to $\mathsf{DRP}(<\kappa, \theta, \mathcal{C})$.

$\vdash$ Note that (3.6) and (3.7) imply that $\theta^{<\kappa} = \theta$ and $\theta = 2^{<\theta} = |\mathcal{H}(\theta)|$.

Assume first that $\mathsf{DRP}(<\kappa, \theta, \mathcal{C})$ holds. Let $\mathfrak{A}$ be a countable expansion of $\mathcal{H}(\theta)$, $\langle S_a : a \in \mathcal{H}(\theta)\rangle$ a family of stationary subsets of $[\mathcal{H}(\theta)]^{\aleph_0}$ and $\mathcal{D} \subseteq [\mathcal{H}(\theta)]^{<\kappa}$ a club. Let $g : \mathcal{H}(\theta) \to \theta$ be a bijection. Let

$$\tilde{\mathfrak{A}} = \langle \mathcal{H}((\theta^{\aleph_0})^+), \underbrace{\underset{\sim}{H}^{\tilde{\mathfrak{A}}}, ...,}_{=\mathfrak{A}} \underset{\sim}{S}^{\tilde{\mathfrak{A}}}, \underset{\sim}{g}^{\tilde{\mathfrak{A}}}, \in\rangle \tag{3.8}$$

where $\underset{\sim}{H}$ is a unary relation symbol and $\underset{\sim}{S}$, $\underset{\sim}{g}$ unary function symbols with

$$\underset{\sim}{H}^{\tilde{\mathfrak{A}}} = \mathcal{H}(\theta), \tag{3.9}$$

$$\underset{\sim}{S}^{\tilde{\mathfrak{A}}} = \{\langle a, S_a\rangle : a \in \mathcal{H}(\theta)\} \cup \{\langle a, \emptyset\rangle : a \in \mathcal{H}((\theta^{\aleph_0})^+) \setminus \mathcal{H}(\theta)\} \text{ and} \tag{3.10}$$

$$\underset{\sim}{g}^{\tilde{\mathfrak{A}}} \supseteq g. \tag{3.11}$$

Let

$$\tilde{\mathcal{D}} = \{M \in [\mathcal{H}((\theta^{\aleph_0})^+)]^{<\kappa} : (1)\ M \cap \mathcal{H}(\theta) \in \mathcal{D},\ (2)\ \theta \in M,\ (3)\ \tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}},\ (4)\ \kappa \cap M < \kappa\}. \tag{3.12}$$ sdls-6-1-a

Then $\tilde{\mathcal{D}}$ contains a club in $[\mathcal{H}((\theta^{\aleph_0})^+)]^{<\kappa}$ by Lemma 1.3. Thus, by $\mathsf{DRP}(<\kappa, \theta, \mathcal{C})$, there is

$$M^+ \in \tilde{\mathcal{D}} \tag{3.13}$$ sdls-6-1-0

such that

$$M^+ \cap \mathcal{H}(\theta) \in \mathcal{C} \text{ and} \tag{3.14}$$ sdls-6-2

(3.15) for all $R \in M^+$ such that R is a stationary subset of $[\theta]^{\aleph_0}$, $R \cap [\theta \cap M]^{\aleph_0}$ is stationary. sdls-6-3

Let $M = \mathcal{H}(\theta) \cap M^+$. By (3.13) and (3.12),(3), we have $\tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}}$. $M \in \mathcal{D}$ by (3.12),(1), and $M \in \mathcal{C}$ by (3.14). By virtue of $\underset{\sim}{g}^{\tilde{\mathfrak{A}}\restriction M^+}$ and $\underset{\sim}{S}^{\tilde{\mathfrak{A}}\restriction M^+}$, and by elementarity, (3.15) implies that $S_a \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$ for all $a \in M$. Thus, M witnesses $(*)^{+\mathcal{C}}_{<\kappa,\theta}$ for the structure $\mathfrak{A}$ and the sequence $\langle S_a : a \in \mathcal{H}(\theta)\rangle$.

Assume now that $(*)^{+\mathcal{C}}_{<\kappa,\theta}$ holds and suppose that $\mathcal{D} \subseteq [\mathcal{H}((\theta^{\aleph_0})^+)]^{<\kappa}$ is a club. We have to find $M \in \mathcal{D}$ satisfying (1) and (2) in the definition of $\mathsf{DRP}(<\kappa, \theta, \mathcal{C})$. Let $\mathfrak{A} = \langle \mathcal{H}((\theta^{\aleph_0})^+), \underset{\sim}{D}^{\mathfrak{A}}, \in\rangle$ where $\underset{\sim}{D}$ is a unary predicate symbol and $\underset{\sim}{D}^{\mathfrak{A}} = \mathcal{D}$.

Let $N \in [\mathcal{H}((\theta^{\aleph_0})^+)]^{\theta}$ be such that $\mathfrak{A} \restriction N \prec \mathfrak{A}$ $\mathcal{H}(\theta) \subseteq N$ and $[N]^{<\kappa} \subseteq N$. We can find such N by (3.6) and (3.7).

$\mathcal{D} \cap N$ is then a club in $[N]^{<\kappa}$.

Let $g : N \to \mathcal{H}(\theta)$ and $h : \mathcal{H}(\theta) \to \theta$ be bijections such that $g \restriction [\theta]^{\leq\omega} = id_{[\theta]^{\leq\omega}}$. Let

$$\mathfrak{A}_0 = \langle \mathcal{H}(\theta), \in, \underbrace{\ \ldots\ }_{\substack{\text{the structure } \mathfrak{A}\restriction N \\ \text{translated by } g}}, \underset{\sim}{h}^{\mathfrak{A}_0}\rangle \tag{3.16}$$ sdls-6-4

where $\underset{\sim}{h}$ is a unary function symbol with $\underset{\sim}{h}^{\mathfrak{A}_0} = h$ and let

$$S_a = \begin{cases} \{x \in [\mathcal{H}(\theta)]^{\aleph_0} \,:\, x \cap \theta \in g^{-1}(a)\}, \\ \qquad\qquad \text{if } g^{-1}(a) \text{ is a stationary subset of } [\theta]^{\aleph_0}; \\ \mathcal{H}(\theta), \\ \qquad\qquad \text{otherwise} \end{cases} \tag{3.17}$$ sdls-6-5

for $a \in \mathcal{H}(\theta)$.

By $(*)^{+\mathcal{C}}_{<\kappa,\theta}$, there is $M_0 \in [\mathcal{H}(\theta)]^{<\kappa}$ such that $M_0 \in \mathcal{C}$, $\mathfrak{A}_0 \restriction M_0 \prec \mathfrak{A}_0$, $\kappa \cap M_0 < \kappa$ and $S_a \cap [M_0]^{\aleph_0}$ is stationary in $[M_0]^{\aleph_0}$ for all $a \in M_0$. Let $M = g^{-1}{}''M_0$. Then $M \in \mathcal{D}$ (this can be shown similarly to Claim 2.1.2) and M satisfies (1) and (2) in the definition of $\mathsf{DRP}(<\kappa, \theta, \mathcal{C})$. ⊣ (Claim 3.4.1)

□ (Lemma 3.4)

P-DRP-3

Lemma 3.5 *Suppose that $\kappa > \aleph_1$ is a regular cardinal.*

(1) $\mathsf{SDLS}^-_+(\mathcal{L}^{\aleph_0}_{stat}, <\kappa) \;\Leftrightarrow\; \mathsf{DRP}(<\kappa, \mathsf{IC}_{\aleph_0})$.

(2) (f) $\mathsf{SDLS}_+(\mathcal{L}^{\aleph_0,II}_{stat}, <\kappa) \;\Leftrightarrow\;$ (a) $\mathsf{DRP}(<\kappa, \mathsf{IC}_{\aleph_0})$ *and $\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$* !!
$\Leftrightarrow$ (a′) $\mathsf{DRP}(<\kappa, \mathsf{IU}_{\aleph_0})$ *and $\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$*
$\Leftrightarrow$ (b) $\mathsf{DRP}(<\kappa, \mathsf{IC}_{\aleph_0})$ *and $2^{\aleph_0} < \kappa$.* (b′) $\mathsf{DRP}(<\kappa, \mathsf{IU}_{\aleph_0})$ *and $2^{\aleph_0} < \kappa$.*

Proof. (1): Suppose first that $\mathsf{SDLS}^-_+(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$ holds. By Lemma 3.4, it is enough to show that $(*)^{+\mathsf{IC}_{\aleph_0}}_{<\kappa,\lambda}$ holds for all $\lambda \geq \kappa$. Let $\lambda \geq \kappa$. Let $\tilde{\mathfrak{A}}$ be a countable expansion of $\langle \mathcal{H}(\lambda), \in\rangle$, $\langle S_a \,:\, a \in \mathcal{H}(\lambda)\rangle$ a sequence of stationary subsets of $[\mathcal{H}(\lambda)]^{\aleph_0}$ and $\mathcal{D} \subseteq [\mathcal{H}(\lambda)]^{<\kappa}$ a club.

Let

$$\tilde{\mathfrak{A}}^* = \langle \underbrace{\mathcal{H}(\lambda), \ldots, \in}_{\tilde{\mathfrak{A}}}, \underset{\sim}{\vec{S}}^{\tilde{\mathfrak{A}}^*}\rangle \tag{3.18}$$ sdls-13-0

where $\underset{\sim}{\vec{S}}$ is a binary relation symbol and

$$\underset{\sim}{\vec{S}}^{\tilde{\mathfrak{A}}^*} = \{\langle a, s\rangle \in (\mathcal{H}(\lambda))^2 \,:\, s \in S_a\}. \tag{3.19}$$ sdls-13-1

Let $M \in [\mathcal{H}(\lambda)]^{<\kappa}$ be such that

$$M \in \mathcal{D} \text{ and} \tag{3.20}$$ sdls-13-1-0

$$\tilde{\mathfrak{A}}^* \restriction M \prec^-_{\mathcal{L}^{\aleph_0}_{stat}} \tilde{\mathfrak{A}}^*. \tag{3.21}$$ sdls-13-2

We have

$$(3.22)\quad \tilde{\mathfrak{A}}^* \restriction M \models aa\, X\, \exists y \forall z\, (z\ \varepsilon\ X \leftrightarrow z \in y)$$

!!

by (3.21) and since apparently the same sentence holds in $\tilde{\mathfrak{A}}^*$. Thus $M \in \mathsf{IC}_{\aleph_0}$.

Similarly $\tilde{\mathfrak{A}}^* \restriction M \models \forall x\, \mathit{stat}\, X\, \exists y\, (\underset{\sqcup}{\vec{S}}(x, y) \land \forall z\, (z\ \varepsilon\ X \leftrightarrow z \in y))$ holds and hence, for all $a \in M$, $S_a \cap M$ is stationary in $[M]^{\aleph_0}$.

Suppose now that $(*)^{+\mathsf{IC}_{\aleph_0}}_{<\kappa,\lambda}$ holds for all $\lambda \geq \kappa$. Let $\mathfrak{A} = \langle A, ...\rangle$ be a structure in countable signature and of cardinality $\geq \kappa$, and $\mathcal{D} \subseteq [A]^{<\kappa}$ a club. Without loss of generality, we may assume that $\mathfrak{A}$ is a relational structure. Let λ be a regular cardinal such that $\mathfrak{A} \in \mathcal{H}(\lambda)$. In particular, we have $A \subseteq \mathcal{H}(\lambda)$.

Let $\tilde{\mathfrak{A}} = \langle \mathcal{H}(\lambda), \underbrace{\underset{\sqcup}{A}^{\tilde{\mathfrak{A}}}, ...}_{=\mathfrak{A}}, \in\rangle$ where $\underset{\sqcup}{A}$ is a unary relation symbol and $\underset{\sqcup}{A}^{\tilde{\mathfrak{A}}} = A$.

For each $a \in \mathcal{H}(\lambda)$, let

$$(3.23)\quad S_a = \begin{cases} \{U \in [\mathcal{H}(\lambda)]^{\aleph_0} : |\, U \cap A\,| = \aleph_0, \\ \qquad\qquad \mathfrak{A} \models \psi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1}, U \cap A)\}, \\ \qquad \text{if } \psi = \psi(x_0, ..., x_{m-1}, Y_0, ..., Y_{n-1}, X) \text{ is an } \mathcal{L}^{\aleph_0}_{\mathit{stat}}\text{-formula} \\ \qquad \text{in the signature of } \mathfrak{A},\ a_0, ..., a_{m-1} \in A,\ U_0, ..., U_{n-1} \in [A]^{\aleph_0}, \\ \qquad \mathfrak{A} \models \mathit{stat}\, X\, \psi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1}, X) \text{ and} \\ \qquad a = \langle \psi, a_0, ..., a_{m-1}, U_0, ..., U_{n-1}\rangle; \\ [\mathcal{H}(\lambda)]^{\aleph_0}, \\ \qquad \text{otherwise.} \end{cases}$$

x-sdls-7

Let

$$(3.24)\quad \tilde{\mathcal{D}} = \{U \in [\mathcal{H}(\lambda)]^{<\kappa} : U \cap A \in \mathcal{D}\}.$$

x-sdls-7-0

$\tilde{\mathcal{D}}$ contains a club in $[\mathcal{H}(\lambda)]^{<\kappa}$ by Lemma 1.2.

By $(*)^{+\mathsf{IC}_{\aleph_0}}_{<\kappa,\lambda}$, there is an internally club $M \in [\mathcal{H}(\lambda)]^{<\kappa}$ such that

$$(3.25)\quad M \in \tilde{\mathcal{D}},$$

x-sdls-7-1-0

$$(3.26)\quad \tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}} \ \text{ and}$$

x-sdls-7-a

$$(3.27)\quad S_a \cap [M]^{\aleph_0} \text{ is stationary in } [M]^{\aleph_0} \text{ for all } a \in M.$$

x-sdls-7-a-0

Let $\tilde{\mathfrak{B}} = \tilde{\mathfrak{A}} \restriction M$ and let $B = \underset{\sqcup}{A}^{\tilde{\mathfrak{B}}} = A \cap M$ and $\mathfrak{B} = \mathfrak{A} \restriction B$. Denoting the underlying set of $\tilde{\mathfrak{B}}$ by $\tilde{B}$, we have $\tilde{B} = M$.

$B \in \mathcal{D}$ by (3.25) and the definition (3.24) of $\tilde{\mathcal{D}}$.

By the elementarity $\tilde{\mathfrak{B}} \prec \tilde{\mathfrak{A}}$ (3.26), the following Claim implies $\mathfrak{B} \prec^{-}_{\mathcal{L}^{\aleph_0}_{\mathit{stat}}} \mathfrak{A}$.

Claim 3.5.1 *For any $\mathcal{L}^{\aleph_0}_{\mathit{stat}}$-formula $\varphi(x_0, ..., x_{m-1}, Y_0, ..., Y_{n-1})$ in the signature of the structures $\mathfrak{A}$, $a_0, ..., a_{m-1} \in B$ and $U_0, ..., U_{n-1} \in [B]^{\aleph_0} \cap \tilde{B}$, we have*

$$(3.28)\quad \tilde{\mathfrak{B}} \models \text{“}\mathfrak{A} \models \varphi(a_0, ..., U_0, ...)\text{”} \;\Leftrightarrow\; \mathfrak{B} \models \varphi(a_0, ..., U_0, ...).$$ x-sdls-8

$\vdash$ By induction on φ. The crucial step in the induction is when φ is of the form *stat* $X\psi$ and (3.28) holds for ψ:

Suppose first that $\tilde{\mathfrak{B}} \models$ “$\mathfrak{A} \models \varphi(a_0, ..., U_0, ...)$” holds. Then, by elementarity and by the definition of $\tilde{\mathfrak{A}}$, we have $\mathfrak{A} \models \mathit{stat}\, X\psi(a_0, ..., U_0, ..., U_{n-1}, X)$. Thus, letting $a = \langle \varphi, a_0, ..., a_{m-1}, U_0, ..., U_{n-1}\rangle$, we have $a \in \tilde{B}$ and

$$(3.29)\quad S_a = \{U \in [\mathcal{H}(\lambda)]^{\aleph_0} \,:\, |\,U \cap A\,| = \aleph_0,\ \mathfrak{A} \models \psi(a_0, ..., U_0, ..., U_{n-1}, U \cap A)\}$$ x-sdls-10

by the definition (3.23) of S_a.

By (3.27), $S_a \cap [\tilde{B}]^{\aleph_0}$ is stationary in $[\tilde{B}]^{\aleph_0}$. By the choice of $\tilde{\mathfrak{B}}$, $[\tilde{B}]^{\aleph_0} \cap \tilde{B}$ contains a club. Thus $S_a \cap [\tilde{B}]^{\aleph_0} \cap \tilde{B}$ is stationary. It follows that

$$\begin{aligned}(3.30)\quad & \{U \cap B \,:\, |\,U \cap B\,| = \aleph_0, U \in S_a \cap [\tilde{B}]^{\aleph_0} \cap \tilde{B}\} \\ & = \{U \cap B \,:\, |\,U \cap B\,| = \aleph_0,\ B \cap U \in \tilde{B}, \\ & \qquad \tilde{\mathfrak{B}} \models \text{“}\mathfrak{A} \models \psi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1}, U \cap B)\text{”}\} \\ & \hfill \text{(by (3.29))} \\ & \subseteq \{V \in [B]^{\aleph_0} \,:\, \mathfrak{B} \models \psi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1}, V)\} \\ & \hfill \text{(by induction hypothesis)}\end{aligned}$$ x-sdls-11

is stationary. Thus $\mathfrak{B} \models \mathit{stat}\, X\, \psi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1}, X)$, that is, $\mathfrak{B} \models \varphi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1})$.

Suppose now that $\tilde{\mathfrak{B}} \not\models$ “$\mathfrak{A} \models \varphi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1})$”. Then we have

$$(3.31)\quad \tilde{\mathfrak{B}} \models \text{“there is a club } \mathcal{C} \subseteq [\underset{\sim}{A}]^{\aleph_0} \text{ such that } \mathfrak{A} \models \neg\psi(a_0, ..., U_0, ..., U_{n-1}, x) \text{ for all } x \in \mathcal{C}\text{”}.$$ x-sdls-12

By elementarity, there is a $\mathcal{C}_0 \in \tilde{B}$ such that $\mathcal{C}_0$ is a club $\subseteq [A]^{\aleph_0}$ and

$$(3.32)\quad \tilde{\mathfrak{B}} \models \text{“}\mathfrak{A} \models \neg\psi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1}, V)\text{” for all } V \in \mathcal{C}_0 \cap \tilde{B}.$$ x-sdls-13

Since $[B]^{\aleph_0} \cap \tilde{B}$ contains a club by the choice of $\tilde{\mathfrak{B}}$,

$$(3.33)\quad \{V \cap B \,:\, |\,V \cap B\,| = \aleph_0,\ V \in \mathcal{C}_0 \cap ([B]^{\aleph_0} \cap \tilde{B}) = \mathcal{C}_0 \cap \tilde{B}\}$$ x-sdls-14

contains a club. By (3.32), this means that there are club many $V \in \mathcal{C}_0 \cap \tilde{B}$ such that $\tilde{\mathfrak{B}} \models$ “$\mathfrak{A} \models \neg\psi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1}, V)$” which in turn means by induction hypothesis that

$$(3.34)\quad \text{there are club many } V \in \mathcal{C}_0 \cap \tilde{B} \text{ such that } \mathfrak{B} \models \neg\psi(a_0, ..., U_0, ..., U_{n-1}, V).$$ x-sdls-16

Thus $\mathfrak{B} \models \neg stat\, X\, \psi(a_0, ..., U_0, ..., U_{n-1}, X)$, i.e. $\mathfrak{B} \not\models \varphi(a_0, ..., U_0, ...)$. $\dashv$ (Claim 3.5.1)

(2): (f) $\Rightarrow$ (a) holds by (1) and Proposition 2.2. The implications (a) $\Rightarrow$ (a′) $\Rightarrow$ (b′) and (a) $\Rightarrow$ (b) $\Rightarrow$ (b′) are obvious. !!

To show the implication (b′) $\Rightarrow$ (f), suppose that $(*)^{+\mathsf{IU}_{\aleph_0}}_{<\kappa,\lambda}$ holds for all $\lambda \geq \kappa$ (see Lemma 3.4) and $\mu = 2^{\aleph_0} < \kappa$.

For a structure $\mathfrak{A}$ in countable signature and of cardinality $\geq \kappa$, let λ, $\tilde{\mathfrak{A}}$, $\langle S_a : a \in \mathcal{H}(\lambda)\rangle$ be as in the proof of (1). By $(*)^{+\mathsf{IU}_{\aleph_0}}_{<\kappa,\lambda}$, there is an internally unbounded $M \in [\mathcal{H}(\lambda)]^{<\kappa}$ such that

(3.35) $\mu \subseteq M$, x-sdls-17

(3.36) $\tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}}$ and x-sdls-18

(3.37) $S_a \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$ for all $a \in M$. x-sdls-19

Let $\tilde{\mathfrak{B}} = \tilde{\mathfrak{A}} \restriction M$ and let $B = \underset{\smile}{A}^{\tilde{\mathfrak{B}}} = A \cap M$ and $\mathfrak{B} = \mathfrak{A} \restriction B$.

Since M is internally unbounded, (3.35) and (3.36) imply that

(3.38) $[M]^{\aleph_0} \subseteq M$. x-sdls-20

We have $\mathfrak{B} \prec^-_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$ (this can be seen as in the proof of (1)). Since all weak second-order objects are internal in $\mathfrak{B}$ by (3.38), this implies $\mathfrak{B} \prec_{\mathcal{L}^{\aleph_0,II}_{stat}} \mathfrak{A}$.

$\square$ (Lemma 3.5)

Theorem 1.1, (3) follows from Lemma 3.5, (1) and Lemma 2.1. Theorem 1.1, (4) follows from Corollary 3.6 below and Lemma 2.1.

P-DRP-4

Corollary 3.6 *Suppose that κ is a regular cardinal $> \aleph_1$. Then the following are equivalent:* !!

(a) $\mathsf{DRP}(<\kappa, \mathsf{IC}_{\aleph_0})$ + $\mu^{\aleph_0} < \kappa$ *for all* $\mu < \kappa$;

(a′) $\mathsf{DRP}(<\kappa, \mathsf{IU}_{\aleph_0})$ + $\mu^{\aleph_0} < \kappa$ *for all* $\mu < \kappa$;

(b) $\mathsf{DRP}(<\kappa, \mathsf{IC}_{\aleph_0})$ + $2^{\aleph_0} < \kappa$;

(b′) $\mathsf{DRP}(<\kappa, \mathsf{IU}_{\aleph_0})$ + $2^{\aleph_0} < \kappa$;

(c) $\mathsf{SDLS}^-_+(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$ + $2^{\aleph_0} < \kappa$;

(d) $\mathsf{SDLS}^-_+(\mathcal{L}^{\aleph_0,II}_{stat}, <\kappa)$;

(e) $\mathsf{SDLS}_+(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$;

(f) $\mathsf{SDLS}_+(\mathcal{L}^{\aleph_0,II}_{stat}, <\kappa)$.

Proof. The equivalence "(a) $\Leftrightarrow$ (a′) $\Leftrightarrow$ (b) $\Leftrightarrow$ (b′) $\Leftrightarrow$ (f)" holds by Lemma 3.5, (2). "(b) $\Leftrightarrow$ (c)" follows from Lemma 3.5, (1). Clearly (f) implies (d) and (e), Each of (d) and (e) implies (c) by Proposition 2.2. $\square$ (Corollary 3.6)

4 Game Reflection Principle and Rado's Conjecture

GRP

In this section, we consider generalizations ($\mathsf{GRP}^{<\mu}(<\kappa)$ for uncountable regular cardinals $\mu < \kappa$) of the Game Reflection Principle introduced by B. König [15] [3)]. We show that the Downward Löwenheim-Skolem theorems $\mathsf{SDLS}_+(\mathcal{L}^{\aleph_0,II}_{stat}, <\kappa)$ for uncountable regular κ follow from a weakening $\mathsf{GRP}^{\omega}(<\kappa)$ of $\mathsf{GRP}^{<\mu}(<\kappa)$ — under certain cardinal arithmetical assumptions on κ if $\kappa > \aleph_\omega$ (Theorem 4.7) see also the remark before Theorem 4.7, Lemma 4.1 and Lemma 4.8.

At first glance, it might seem that these generalized Game Reflection Principles are rather artificial requirements, while Downward-Löwenheim-Skolem-theorem-type reflection statements are natural generalizations of the Downward Löwenheim-Skolem theorem of the first-order logic. However, the characterization of the Game Reflection Principles in terms of generically large cardinals (Theorem 4.13) suggests that the naturalness of the Game Reflection Principles can be also discussed, and that these principles are among the strongest possible reflection statements available. In the sequel [13] of the present paper, we continue the line of research we begin in this section, and show that existential statements of certain type of generically large cardinals serve as a delimitation for various reflection principles including variations of Downward Löwenheim-Skolem theorems for stationary logics.

Most of the ideas in this section are (at least implicitly) present in [15]. However, there are some technical details explained only here (in particular, Lemma 4.1 as well as (4.29) and (4.33), (3) and their usage in the proof of Theorem 4.13). We do not know if the proof of Theorem 4.13 below or the proof of the corresponding theorem in [15] goes through without the details around (4.29) and (4.33), (3).

The Game Reflection Principle of B. König [15] (Strong Game Reflection in his terminology and denoted here as $\mathsf{GRP}^{<\omega_1}(<\aleph_2)$) is a reflection statement concerning the non-existence of winning strategy for one of the players in the following game:

For any uncountable set A and $\mathcal{A} \subseteq {}^{\omega_1>}A$, $\mathcal{G}^{{}^{\omega_1>}A}(\mathcal{A})$ is the following game of length ω_1 for players I and II. A match in $\mathcal{G}^{{}^{\omega_1>}A}(\mathcal{A})$ looks like:

I	a_0		a_1		a_2		$\cdots$	a_ξ		$\cdots$
II		b_0		b_1		b_2	$\cdots$		b_ξ	$\cdots$

$(\xi < \omega_1)$

[3)] $\mathsf{GRP}^{<\omega_1}(<\omega_2)$ in our notation is what König calls the Strong Game Reflection Principle in [15].

where a_ξ, $b_\xi \in A$ for $\xi < \omega_1$.

II wins this game if

$$(4.1) \qquad \langle a_\xi, b_\xi : \xi < \eta\rangle \in \mathcal{A} \text{ and } \langle a_\xi, b_\xi : \xi < \eta\rangle^\frown\langle a_\eta\rangle \notin \mathcal{A} \text{ for some } \eta < \omega_1; \text{ or } \langle a_\xi, b_\xi : \xi < \omega_1\rangle \in [\mathcal{A}]$$

where $\langle a_\xi, b_\xi : \xi < \eta\rangle$ denotes the sequence $f \in {}^{2\cdot\eta}A$ such that $f(2\cdot\xi) = a_\xi$ and $f(2\cdot\xi+1) = b_\xi$ for all $\xi < \eta$ and

$$(4.2) \qquad [\mathcal{A}] = \{f \in {}^{\omega_1}A : f \restriction \alpha \in \mathcal{A} \text{ for all } \alpha < \omega_1\}.$$

For a regular cardinals $\mu < \kappa$ and a set A, $\mathcal{C} \subseteq [A]^{<\kappa}$ is *μ-club* if

(4.3) $\mathcal{C}$ is cofinal in $[A]^{<\kappa}$ with respect to $\subseteq$ and we have $\bigcup_{\alpha<\nu} c_\alpha \in \mathcal{C}$ for any $\subseteq$-increasing sequence $\langle c_\alpha \in \mathcal{C} : \alpha < \nu\rangle$ in $\mathcal{C}$ with $\mu \leq cf(\nu) < \kappa$. grp-a-a-0

For a regular cardinal $\kappa > \aleph_1$, let

$\mathsf{GRP}^{<\omega_1}(<\kappa)$: For any set A of regular cardinality $\geq \kappa$ and ω_1-club $\mathcal{C} \subseteq [A]^{<\kappa}$, if the player II has no winning strategy in $\mathcal{G}^{\omega_1>A}(\mathcal{A})$ for some $\mathcal{A} \subseteq {}^{\omega_1>}A$, there is $B \in \mathcal{C}$ such that II has no winning strategy in $\mathcal{G}^{\omega_1>B}(\mathcal{A} \cap {}^{\omega_1>}B)$.

We also consider the following weakening of $\mathsf{GRP}^{<\omega_1}(<\kappa)$:

For uncountable set A and $\mathcal{A} \subseteq {}^{\omega\geq}A$, $\mathcal{G}^{\omega\geq A}(\mathcal{A})$ is the game of length ω played between players I and II. A match in $\mathcal{G}^{\omega\geq A}(\mathcal{A})$ looks like :

I	a_0		a_1		a_2		$\cdots$	a_n		$\cdots$
II		b_0		b_1		b_2	$\cdots$		b_n	$\cdots$

$(n < \omega)$

where a_n, $b_n \in A$ for $n < \omega$.

II wins this game if $\langle a_n, b_n : n < m\rangle \in \mathcal{A}$ and $\langle a_n, b_n : n < m\rangle^\frown\langle a_m\rangle \notin \mathcal{A}$ for some $m \in \omega$; or $\langle a_n, b_n : n < w\rangle \in \mathcal{A}$ for all $w \leq \omega$.

For a regular cardinal $\kappa > \aleph_1$, let

$\mathsf{GRP}^{\omega}(<\kappa)$: For a set A of regular cardinality $\geq \kappa$ and ω_1-club $\mathcal{C} \subseteq [A]^{<\kappa}$, if the player II has no winning strategy in $\mathcal{G}^{\omega\geq A}(\mathcal{A})$ for some $\mathcal{A} \subseteq {}^{\omega\geq}A$, then there is $B \in \mathcal{C}$ such that II has no winning strategy in $\mathcal{G}^{\omega\geq B}(\mathcal{A} \cap {}^{\omega\geq}B)$.

The difference between the games $\mathcal{G}^{\omega_1>A}(\widetilde{\mathcal{A}})$ for $\widetilde{\mathcal{A}} \subseteq {}^{\omega_1>}A$ and the games $\mathcal{G}^{\omega\geq A}(\mathcal{A})$ for $\mathcal{A} \subseteq {}^{\omega\leq}A$ is quite subtle. In $\mathcal{G}^{\omega_1>A}(\widetilde{\mathcal{A}})$, the player II wins if the outcome of the game as an ω_1-sequence is a *branch* in $\widetilde{\mathcal{A}}$, while the player II wins in $\mathcal{G}^{\omega\geq A}(\mathcal{A})$ if the outcome of the game as an ω-sequence is an *element* of $\mathcal{A}$. The symbol '$<$' in the superscript of the notation $\mathsf{GRP}^{<\omega_1}(<\kappa)$ and its absence in the !!

superscript of $\mathsf{GRP}^{\omega}(<\kappa)$ allude this subtle difference of the games involved. In spite of this difference, we can prove that $\mathsf{GRP}^{\omega}(<\kappa)$ is a special case of the reflection $\mathsf{GRP}^{<\omega_1}(<\kappa)$. This fact for the case $\kappa = \aleph_2$ was used several times in König [15] without any explicit mention:

L-sdls-a

Lemma 4.1 *For a regular cardinal $\kappa > \aleph_1$, $\mathsf{GRP}^{<\omega_1}(<\kappa)$ implies $\mathsf{GRP}^{\omega}(<\kappa)$.*

Proof. For an arbitrary set A and $\mathcal{A} \subseteq {}^{\omega\geq}A$, let

$$\widetilde{\mathcal{A}}^{A} = \{t \in {}^{\omega_1>}A \,:\, t \restriction \alpha \in \mathcal{A} \text{ for all } \alpha < \min\{\omega+1, \ell(t)\}\}. \tag{4.4}$$ sdls-4-a

The following is obvious:

Cl-sdls-a

Claim 4.1.1 *The player II has a winning strategy in $\mathcal{G}^{\,\omega\geq A}(\mathcal{A})$ if and only if he has a winning strategy in $\mathcal{G}^{\,\omega_1>A}(\widetilde{\mathcal{A}}^{A})$.* ⊣

Assume $\mathsf{GRP}^{<\omega_1}(<\kappa)$ holds. Suppose that A is uncountable and $\mathcal{A} \subseteq {}^{\omega\geq}A$ is such that the player II does not have any winning strategy in $\mathcal{G}^{\,\omega\geq A}(\mathcal{A})$. By Claim 4.1.1, the player II does not have any winning strategy in $\mathcal{G}^{\,\omega_1>A}(\widetilde{\mathcal{A}}^{A})$.

Let $\mathcal{C} \subseteq [A]^{\aleph_1}$ be an ω_1-club.

By $\mathsf{GRP}^{<\omega_1}(<\kappa)$ there is $B \in \mathcal{C}$ such that the player II does not have any winning strategy in $\mathcal{G}^{\,\omega_1>B}(\widetilde{\mathcal{A}}^{A} \cap {}^{\omega_1>}B)$. Since $\widetilde{\mathcal{A}}^{A} \cap {}^{\omega_1>}B = \widetilde{\mathcal{A} \cap {}^{\omega\geq}B}^{B}$, it follows, again by Claim 4.1.1, that the player II does not have any winning strategy in $\mathcal{G}^{\,\omega\geq B}(\mathcal{A} \cap {}^{\omega\geq}B)$. □ (Lemma 4.1)

L-sdls-a-0

Lemma 4.2 *For a regular cardinal $\kappa > \aleph_1$, $\mathsf{GRP}^{\omega}(<\kappa)$ implies $2^{\aleph_0} < \kappa$.*

Proof. Assume that $\mathsf{GRP}^{\omega}(<\kappa)$ holds and, toward a contradiction, assume also that $2^{\aleph_0} \geq \kappa$. Let $B \subseteq {}^{\omega}2$ be a Bernstein set[4] of cardinality $2^{\aleph_0}$ and let B^* be an !! arbitrary subset of B of cardinality κ. Note that B^* is also a Bernstein set. Let $\langle r_\xi \,:\, \xi < \kappa\rangle$ be a 1-1 enumeration of B^*.

Let $A = {}^{\omega>}2 \cup \kappa$ and let $\mathcal{A} \subseteq {}^{\omega\geq}A$ be defined by $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$ where

$$\mathcal{A}_0 = \{\langle a_n, b_n \,:\, n < k\rangle, \langle a_n, b_n \,:\, n < k\rangle^\frown\langle a_k\rangle \,:\, k < \omega,\ a_k \in \kappa\} \tag{4.5}$$ sdls-14-0

$$\mathcal{A}_1 = \{\langle a_n, b_n \,:\, n \in \omega\rangle \,:\, a_n \in \kappa \text{ for all } n \in \omega,\ \langle b_n \,:\, n \in \omega\rangle \text{ is an increasing sequence in } {}^{\omega>}2, \text{ and } \textstyle\bigcup_{n\in\omega} b_n = r_\xi \text{ for some } \xi > \sup\{a_n \,:\, n \in \omega\}\}. \tag{4.6}$$ sdls-4-a-0

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[4] A set $B \subseteq {}^{\omega}2$ is a Bernstein set if it does not contain any perfect set. Since there are only $2^{\aleph_0}$ many perfect subsets of ${}^{\omega}2$, a Bernstein set of cardinality $2^{\aleph_0}$ can be obtained easily by a diagonal construction.

Let $\mathcal{C} = \{C \in [A]^{<\kappa} : {}^{\omega>}2 \subseteq C\}$. $\mathcal{C}$ is an ω_1-club $\subseteq [A]^{<\kappa}$. The player II has a winning strategy in $\mathcal{G}^{\omega\geq C}(\mathcal{A} \cap {}^{\omega\geq}C)$ for any $C \in \mathcal{C}$ (he can play $\langle b_n : n \in \omega\rangle$ such that $\bigcup_{n\in\omega} b_n = r_\xi$ for a $\xi > \sup(\kappa \cap C)$). By $\mathsf{GRP}^\omega(<\kappa)$ it follows that the player II has a winning strategy σ in $\mathcal{G}^{\omega\geq A}(\mathcal{A})$.

Now let $\langle \ell_n : n \in \omega\rangle$ and $\langle t_s : s \in {}^{\omega>}2\rangle$ be sequences such that

(4.7) $\langle \ell_n : n \in \omega\rangle$ is a strictly increasing sequence in ω; sdls-4-a-1

(4.8) $t_s \in {}^{2\ell_n}A$ for all $s \in {}^n2$; sdls-4-a-2

(4.9) for each $s \in {}^n2$, $t_s = \langle a_k, b_k : k < \ell_n\rangle$ is a partial match in $\mathcal{G}^{\omega\geq A}(\mathcal{A})$ in which player II's moves are chosen according to σ, and player I chooses her moves avoiding her sudden death; sdls-4-a-3

(4.10) for s, $s' \in {}^{\omega>}2$ with $s \subseteq s'$ we have $t_s \subseteq t_{s'}$; sdls-4-a-4

For $s \in {}^n2$ and $t_s = \langle a_k, b_k : k < \ell_n\rangle$, let $u_s = \langle b_k : k < \ell_n\rangle$. By (4.10), we have $u_s \subseteq u_{s'}$ for any s, $s' \in {}^{\omega>}2$ with $s \subseteq s'$, and,

(4.11) for all $s \in {}^{\omega>}2$, $u_{s^\frown 0}$ and $u_{s^\frown 1}$ are distinct (hence incompatible by (4.8)). sdls-4-a-5

The condition (4.11) is realizable since σ is a winning strategy: if σ would suggest the same outputs, from some partial match on, independently of the moves of the player I, the player I would be able to predict the outcome $\bigcup_{n\in\omega} b_n$ and could choose her move easily so that she should win.

Now since $\bigcup_{n\in\omega} t_{f\restriction n}$ for each $f \in {}^\omega 2$ is a match in $\mathcal{G}^{\omega\geq A}(\mathcal{A})$ in which player II's moves are chosen according to σ, we have $\bigcup_{n\in\omega} u_{f\restriction n} \in B^*$. This is a contradiction since $\{\bigcup_{n\in\omega} u_{f\restriction n} : f \in {}^\omega 2\}$ is a perfect set by the construction of u_t's. ❑ (Lemma 4.2)

Recall that Rado's Conjecture (RC) can be formulated in terms of reflection of non-specialty of trees ([19]): A tree is special if T is a union of countably many antichains; an antichain of a tree is a pairwise incomparable subset of T (which is an antichain in the sense of the forcing p.o. T with the reverse tree ordering). Thus T is special if and only if there is a mapping $f : T \to \omega$ such that $f^{-1}{}''\{n\}$ is pairwise incomparable for all $n \in \omega$. We shall call such a mapping f a *good coloring* of T.

RC: For any tree T, if T is not special then there is $B \in [T]^{<\aleph_2}$ such that B is not special.

We consider here the following generalization of RC: For a regular cardinal $\kappa > \aleph_1$,

$\mathsf{RC}(<\kappa)$: For any tree T, if T is not special then there is $B \in [T]^{<\kappa}$ such that B is not special.

Thus the original Rado's Conjecture (RC) is RC($<\aleph_2$).

Theorem 4.3 ([15]) *Suppose that $\kappa > \aleph_1$ is a regular cardinal. Then,* $\mathsf{GRP}^{<\omega_1}(<\kappa)$ *implies* $\mathsf{RC}(<\kappa)$. T-sdls-a

The theorem above will be proved after the following two lemmas. (2) of the second lemma will play an essential roll in the characterization theorem and in the consistency proof of $\mathsf{GRP}^{<\omega_1}(<\kappa)$ where κ is a successor of a regular cardinal (modulo a supercompact cardinal).

Lemma 4.4 ([19]) *If T is a non-special tree and $\mathbb{P}$ is a σ-closed p.o., then* $\Vdash_{\mathbb{P}}$ "*T is not special*". L-sdls-a-0-a-0

Proof. `See [https://fuchino.ddo.jp/notes/math-notes-11.pdf], Section 12.` ❑ (Lemma 4.4)

L-sdls-a-0-a-1

Lemma 4.5 *Suppose that $\mathbb{P}$ is a σ-closed p.o., A a set and $\mathcal{A} \subseteq {}^{\omega_1>}A$.*

(1) *If σ is a winning strategy of the player II in $\mathcal{G}^{\omega_1>A}(\mathcal{A})$, then* $\Vdash_{\mathbb{P}}$ "$\check{\sigma}$ is a winning strategy of the player II in $\mathcal{G}^{\omega_1>\check{A}}(\check{\mathcal{A}})$".

(2) *If* $\Vdash_{\mathbb{P}}$ "the player II has a winning strategy in $\mathcal{G}^{\omega_1>\check{A}}(\check{\mathcal{A}})$", *then the player II has a winning strategy in $\mathcal{G}^{\omega_1>A}(\mathcal{A})$ (in V).*

Proof. (1): In $\mathsf{V}^{\mathbb{P}}$, the initial segment of any match is a ground model set by the σ-closedness of $\mathbb{P}$. Thus, the player II can apply σ to remain in $\mathcal{A}$ in a match (as far as the player I does not throw up). Thus σ is still a winning strategy for the player II in $\mathsf{V}^{\mathbb{P}}$.

(2): Let $\underset{\sim}{\sigma}$ be a $\mathbb{P}$-name of a winning strategy of the player II. For each initial segment $m = \langle a_\alpha, b_\alpha : \alpha < \xi\rangle^\frown\langle a_\xi\rangle$ of a match in $\mathcal{G}^{\omega_1>A}(\mathcal{A})$ we assign $\mathbb{p}_m \in \mathbb{P}$ and b_m such that

(4.12) If $m \subseteq m'$ then $\mathbb{p}_{m'} \leq_{\mathbb{P}} \mathbb{p}_m$ for any initial segments m m' of a match; grp-a

(4.13) for $m = \langle a_\alpha, b_\alpha : \alpha < \xi\rangle^\frown\langle a_\xi\rangle$, $\mathbb{p}_m \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\sigma}(\langle a_\alpha, b_\alpha : \alpha < \xi\rangle^\frown\langle a_\xi\rangle) = b_m$". grp-a-0

Then the strategy σ of the player II defined by $\sigma(m) = b_m$ is a winning for the player II. ❑ (Lemma 4.5)

Proof of Theorem 4.3: Assume that $\mathsf{GRP}^{<\omega_1}(<\kappa)$ holds. Suppose that T is a non special tree.

Let $A = T \cup \omega$ and $\mathcal{A} \subseteq {}^{\omega_1>}A$ be defined by $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$ where

(4.14) $$\mathcal{A}_0 = \{\langle a_\alpha, b_\alpha : \alpha < \xi\rangle^\frown\langle a_\xi\rangle : \xi < \omega_1,\ \langle a_\alpha, b_\alpha : \alpha < \xi\rangle \in {}^{\xi}A \text{ and } a_\xi \in T\}, \text{ and}$$ grp-a-1-0

(4.15) $\mathcal{A}_1 = \{\langle a_\alpha, b_\alpha : \alpha < \xi\rangle : \xi < \omega_1,\ a_\alpha \in T,\ b_\alpha \in \omega,$ and $a_\alpha \mapsto b_\alpha$ is a good coloring$\}$. grp-a-1-1

Cl-grp-0-0

Claim 4.3.1 *For any $T' \subseteq T$, if T' is special then the player II has a winning strategy in $\mathcal{G}^{{}^{\omega_1>}B}(\mathcal{A} \cap {}^{\omega_1>}B)$ for $B = T' \cup \omega$.*

$\vdash$ Suppose that T' is special and let $f : T' \to \omega$ be such that $f^{-1}{}''\{n\}$ is pairwise incomparable for all $n \in \omega$. Then the player II wins if he simply chooses $b_\alpha = f(a_\alpha)$ answering the α-th move a_α of the player I as far as $a_\alpha \in T$. $\dashv$ (Claim 4.3.1)

Cl-grp-0-1

Claim 4.3.2 *The player II does not have any winning strategy in $\mathcal{G}^{{}^{\omega_1>}A}(\mathcal{A})$.*

$\vdash$ Suppose otherwise and let σ be a winning strategy of the player II. Let $\mathbb{P}$ be a σ-closed p.o. collapsing the cardinality of T to $\aleph_1$. By Lemma 4.5, σ is still a winning strategy for the player II in $\mathcal{G}^{{}^{\omega_1>}A}(\mathcal{A})$ in $\mathsf{V}^{\mathbb{P}}$. In $\mathsf{V}^{\mathbb{P}}$ the player I can enumerate all elements of T in her moves a_α, $\alpha < \omega_1$. Hence if the player II apply σ to such moves of the player I, the match delivers a good coloring of T. This is a contradiction since, by the choice of T and by Lemma 4.4, we have $\mathsf{V}^{\mathbb{P}} \models$ "T is non-special".

$\dashv$ (Claim 4.3.2)

Note that $\{B \in [A]^{<\kappa} : \omega \subseteq B\}$ is ω_1-club. By $\mathsf{GRP}^{<\omega_1}(<\kappa)$, there is $T' \in [T]^{<\kappa}$ such that, for $B = T' \cup \omega$, the player II does not have any winning strategy in $\mathcal{G}^{{}^{\omega_1>}B}(\mathcal{A} \cap {}^{\omega_1>}B)$.

By Claim 4.3.1, T' is non-special. ❑ (Theorem 4.3)

L-grp-2

Proposition 4.6 *Suppose that κ is an uncountable regular cardinal such that*

(4.16) $\mu^{\aleph_0} < \kappa$ *for all* $\mu < \kappa$ *holds.* grp-a-2

Then $\mathsf{GRP}^{\omega}(<\kappa)$ implies $(*)^{+\mathsf{IC}_{\aleph_0}}_{<\kappa,\lambda}$ for all $\lambda \geq \kappa$.

Proof. Let $\langle S_a : a \in \mathcal{H}(\lambda)\rangle$ be as in the statement of $(*)^{+\mathsf{IC}_{\aleph_0}}_{<\kappa,\lambda}$. Let $A = \mathcal{H}(\lambda)$ and let $\mathcal{A} \subseteq {}^{\omega\geq}A$ be such that

(4.17) ${}^{\omega>}A \subseteq \mathcal{A}$ and grp-0

(4.18) $\langle a_n, b_n : n \in \omega\rangle \in \mathcal{A}$ if and only if $\{a_n : n \in \omega\} \cup \{b_n : n \in \omega \setminus 1\} \notin S_{b_0}$. grp-17

Cl-grp-1

Claim 4.6.1 *The player II does not have any winning strategy in $\mathcal{G}^{{}^{\omega\geq}A}(\mathcal{A})$.*

$\vdash$ Suppose that σ is a strategy of the player II. We show that σ is not a winning strategy for the player II.

For an arbitrary $a_0 \in A$ let $b_0 \in A$ be player II's answer to a_0 according to σ. Let $N \prec \mathcal{H}(\lambda)$ be countable such that a_0, b_0, $\sigma \in N$ and $N \in S_{b_0}$ (there is such an N since S_{b_0} is stationary in $[\mathcal{H}(\lambda)]^{\aleph_0}$). Let $\langle a_n, b_n : n \in \omega\rangle$ be a match in $\mathcal{G}^{\omega\geq A}(\mathcal{A})$ where a_0 and b_0 are just the a_0 and the b_0 chosen above, player II's moves are chosen according to σ and $\{a_n : n \in \omega\}$ enumerates N. Since $\sigma \in N$, we have

$$(4.19)\quad \{a_n : n \in \omega\} \cup \{b_n : n \in \omega \setminus 1\} = N \in S_{b_0}.$$ grp-18

Thus the player II loses this match.

This shows that σ is not a winning strategy for the player II. $\dashv$ (Claim 4.6.1)

Now let $\tilde{\mathfrak{A}}$ be an arbitrary countable expansion of $\langle \mathcal{H}(\lambda), \in\rangle$ and $\mathcal{D}$ be a club subset of $[\mathcal{H}(\lambda)]^{<\kappa}$. Let

$$(4.20)\quad \mathcal{D}^* = \{M \in [\mathcal{H}(\lambda)]^{<\kappa} : M \in \mathcal{D},\ \tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}},\ [M]^{\aleph_0} \subseteq M\}.$$ grp-19

Since we have $2^{\aleph_0} < \kappa$ (4.16), $\mathcal{D}^*$ is an ω_1-club. By $\mathsf{GRP}^\omega(<\kappa)$ there is an $M \in \mathcal{D}^*$ such that the player II does not have any winning strategy in $\mathcal{G}^{\omega\geq M}(\mathcal{A}\cap {}^{\omega\geq}M)$. We show that this M is as in the assertion of $(*)^{+\mathsf{IC}_{\aleph_0}}_{\kappa,\lambda}$.

By the definition of $\mathcal{D}^*$, we have $M \in \mathcal{D}$, $\tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}}$ and M is internally club. Thus the following Claim finishes the proof of (1):

Cl-grp-2

Claim 4.6.2 *$S_a \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$ for all $a \in M$.*

$\vdash$ Suppose that $S_{a^*} \cap [M]^{\aleph_0}$ were not stationary in $[M]^{\aleph_0}$ for some $a^* \in M$. Then the player II would have the following winning strategy in $\mathcal{G}^{\omega\geq M}(\mathcal{A}\cap {}^{\omega\geq}M)$: Let $\mathcal{C} \subseteq [M]^{\aleph_0}$ be a club disjoint from S_{a^*}. II can win if he chooses $b_0 = a^*$ and an increasing sequence $\langle C_k : k \in \omega\rangle$ in $\mathcal{C}$ along with his moves b_n such that $\{a_n, b_n : n < k\} \subseteq C_k$ and organizes his moves b_n, $n \in \omega$ such that they gradually enumerate the set $\bigcup_{k\in\omega} C_k \in \mathcal{C}$. $\dashv$ (Claim 4.6.2)

□ (Proposition 4.6)

Note that, by Lemma 4.2, the condition (4.16) below holds under $\mathsf{GRP}^\omega(<\kappa)$ if we have e.g. $\kappa < \aleph_\omega$.

T-grp-a

Theorem 4.7 *Suppose that κ is a regular uncountable cardinal such that*

(4.16) *$\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$ holds.*

Then $\mathsf{GRP}^\omega(<\kappa)$ implies $\mathsf{SDLS}_+(\mathcal{L}^{\aleph_0,II}_{stat}, <\kappa)$.

Proof. By Corollary 3.6, Lemma 3.4 and Proposition 4.6. □ (Theorem 4.7)

Instead of games in $\mathcal{G}^{\omega_1>A}(\mathcal{A})$ of length ω_1, we can also consider the same kind of games of length μ for arbitrary regular uncountable cardinal μ and corresponding game reflection principles.

For any set A and $\mathcal{A} \subseteq {}^{\mu>}A$, $\mathcal{G}^{\mu>A}(\mathcal{A})$ is the following game of length μ for players I and II. A match in $\mathcal{G}^{\mu>A}(\mathcal{A})$ looks like:

$$\begin{array}{c|cccccccccc} \mathrm{I} & a_0 & & a_1 & & a_2 & & \cdots & a_\xi & & \cdots \\ \hline \mathrm{II} & & b_0 & & b_1 & & b_2 & \cdots & & b_\xi & \cdots \end{array} \qquad (\xi < \mu)$$

where a_ξ, $b_\xi \in A$ for $\xi < \mu$.

II wins this match if

(4.21) $\langle a_\xi, b_\xi : \xi < \eta\rangle \in \mathcal{A}$ and $\langle a_\xi, b_\xi : \xi < \eta\rangle^\frown\langle a_\eta\rangle \notin \mathcal{A}$ for some $\eta < \mu$; or $\langle a_\xi, b_\xi : \xi < \mu\rangle \in [\mathcal{A}]$

where $\langle a_\xi, b_\xi : \xi < \eta\rangle$ and $[\mathcal{A}]$ are defined similarly as before.

For uncountable regular cardinals μ, κ with $\mu < \kappa$,

$\mathsf{GRP}^{<\mu}(<\kappa)$: For any set A of regular cardinality $\geq \kappa$ and μ-club $\mathcal{C} \subseteq [A]^{<\kappa}$, if the player II has no winning strategy in $\mathcal{G}^{\mu>A}(\mathcal{A})$ for some $\mathcal{A} \subseteq {}^{\mu>}A$, there is $B \in \mathcal{C}$ such that the player II has no winning strategy in $\mathcal{G}^{\mu>B}(\mathcal{A}\cap {}^{\mu>}B)$.

Note that the original definition of $\mathsf{GRP}^{<\omega_1}(<\kappa)$ coincides with the $\mathsf{GRP}^{<\mu}(<\kappa)$ for $\mu = \omega_1$ in the sense of the extended game reflection given above.

For uncountable regular cardinals μ, κ and an ordinal α with $\alpha \leq \mu < \kappa$, let

$\mathsf{GRP}^{\alpha,<\mu}(<\kappa)$: For any set A of regular cardinality $\geq \kappa$ and μ-club $\mathcal{C} \subseteq [A]^{<\kappa}$, if the player II has no winning strategy in $\mathcal{G}^{\mu>A}(\mathcal{A})$ for some $\mathcal{A} \subseteq {}^{\mu>}A$ such that $t \in \mathcal{A}$, for all $t \in {}^{\mu>}A$ with $\alpha \leq \ell(t) \leq \mu$, there is $B \in \mathcal{C}$ such that the player II has no winning strategy in $\mathcal{G}^{\mu>B}(\mathcal{A}\cap {}^{\mu>}B)$.

The proof of Lemma 4.1 shows that $\mathsf{GRP}^{\omega}(<\kappa)$ is equivalent to $\mathsf{GRP}^{\omega+1,<\omega_1}(<\kappa)$. More generally, we have the following:

Lemma 4.8 (1) *For any uncountable regular cardinals μ, κ with $\mu < \kappa$, $\mathsf{GRP}^{\mu,<\mu}(<\kappa)$ is equivalent to $\mathsf{GRP}^{<\mu}(<\kappa)$.* L-grp-3

(2) *For any uncountable regular cardinals μ, κ and a ordinal α with $\alpha \leq \mu < \kappa$, $\mathsf{GRP}^{<\mu}(<\kappa)$ implies $\mathsf{GRP}^{\alpha,<\mu}(<\kappa)$.*

(3) *For any uncountable regular cardinals μ_0 μ, κ with $\mu_0 \leq \mu < \kappa$ $\mathsf{GRP}^{<\mu}(<\kappa)$ implies $\mathsf{GRP}^{\mu_0,<\mu}(<\kappa)$ and $\mathsf{GRP}^{\mu_0,<\mu}(<\kappa)$ implies $\mathsf{GRP}^{<\mu_0}(<\kappa)$.*

Proof. (1), (2): Clear by definition.

(3): The first implication is just a special case of (2). For the second one, note that, if $\mathcal{D} \subseteq [A]^{<\kappa}$ is μ_0-club then it is also μ-club (see the definition (4.3)).

□ (Lemma 4.8)

Lemma 4.5 can be straight-forwardly generalized to the following:

L-grp-4

Lemma 4.9 *Suppose that κ is an uncountable regular cardinal, A a set, $\mathcal{A} \subseteq {}^{\kappa>}A$ and $\mathbb{P}$ a $<\kappa$-closed p.o.*

(1) *If σ is a winning strategy of the player II in $\mathcal{G}^{\kappa>A}(\mathcal{A})$ then $\Vdash_{\mathbb{P}}$ "$\check{\sigma}$ is a winning strategy of the player II in $\mathcal{G}^{\check{\kappa}>\check{A}}(\check{\mathcal{A}})$".*

(2) *If $\Vdash_{\mathbb{P}}$ "the player II has a winning strategy in $\mathcal{G}^{\check{\kappa}>\check{A}}(\check{\mathcal{A}})$", then the player II has a winning strategy in $\mathcal{G}^{\kappa>A}(\mathcal{A})$ (in V).* ❑

A cardinal κ is said to be *generically supercompact by $<\mu$-closed forcing* for a regular cardinal $\mu < \kappa$, if, for any regular λ, there is a $<\mu$-closed p.o. $\mathbb{P}$ such that, for any $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$, there is an inner model M of $\mathsf{V}[\mathbb{G}]$ and an elementary embedding $j : \mathsf{V} \overset{\preceq}{\to} M$ such that $crit(j) = \kappa$, $j(\kappa) > \lambda$ and $j''\lambda \in M$. The following can be proved similarly to Theorem 11 in [15].

L-grp-6

Lemma 4.10 *If κ is a supercompact and $\mu < \kappa$ is an uncountable regular cardinal then for $\mathbb{P} = \mathrm{Col}(\mu, \kappa)$ and $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$, we have $\mathsf{V}[\mathbb{G}] \models \kappa = \mu^+$ and κ is generically supercompact by $<\mu$-closed forcing.*

Proof. Note that we are using here Kanamori's notation:

$$\mathrm{Col}(\lambda, S) = \{\mathbb{p} : \mathbb{p} \text{ is a function } \wedge\ dom(\mathbb{p}) \subseteq S \times \lambda\ \wedge\ |\mathbb{p}| < \lambda \\ \wedge \text{ for all } \langle \alpha, \xi \rangle \in dom(\mathbb{p}),\ \mathbb{p}(\langle \alpha, \xi \rangle) = 0 \text{ or } \mathbb{p}(\langle \alpha, \xi \rangle \in \alpha)\}$$

For a regular $\lambda > \kappa$, let $j : \mathsf{V} \overset{\preceq}{\to} M$ be such that $crit(j) = \kappa$, $j(\kappa) > \lambda$ and

$$(\aleph 4.1) \quad [M]^\lambda \subseteq M. \qquad \text{grp-19-a-3}$$

Let $\mathbb{P}^* = j(\mathbb{P})$. By elementarity we have $M \models \mathbb{P}^* = \mathrm{Col}(\mu, j(\kappa))$. By ($\aleph$4.1), it follows that $\mathbb{P}^* = \mathrm{Col}(\mu, j(\kappa))^{\mathsf{V}}$. Hence $\mathsf{V}[\mathbb{G}] \models \mathbb{P}^* = \mathrm{Col}(\mu, j(\kappa))$ by $<\mu$-closedness of $\mathbb{P}$.

We have $\mathbb{P} \lessdot \mathbb{P}^*$ and $\mathbb{P}^*/\mathbb{G} \approx \mathrm{Col}(\mu, j(k))^{\mathsf{V}[G]}$. Let $\mathbb{G}^*$ be $(\mathsf{V}[\mathbb{G}], \mathbb{P}^*/\mathbb{G})$-generic filter. We have $M[\mathbb{G} * \mathbb{G}^*] \subseteq \mathsf{V}[\mathbb{G} * \mathbb{G}^*] = (\mathsf{V}[\mathbb{G}])[\mathbb{G}^*]$. The mapping

$$(\aleph 4.2) \quad j^* : \mathsf{V}[\mathbb{G}] \to M[\mathbb{G} * \mathbb{G}^*];\ \underset{\sim}{a}[\mathbb{G}] \mapsto j(\underset{\sim}{a})[\mathbb{G} * \mathbb{G}^*] \qquad \text{grp-19-a-4}$$

is then an elementary embedding witnessing the λ-generically supercompactness of κ by $<\mu$-closed forcing in $V[\mathbb{G}]$. ❑ (Lemma 4.10)

P-GRP-0

Lemma 4.11 *For a regular uncountable cardinal κ, if κ^+ is generically supercompact for $<\kappa$-closed forcing then $\mathsf{GRP}^{<\kappa}(<\kappa^+)$ holds.*

Proof. Suppose that $\theta \geq \kappa^+$ is a regular cardinal and $\mathcal{A} \subseteq {}^{\kappa>}\theta$. Suppose further that $\mathcal{D} \subseteq [\theta]^{<\kappa^+}$ is a κ-club such that,

$$\text{for every } B \in \mathcal{D}, \text{ the player II has a winning strategy in } \mathcal{G}^{\kappa>B}(\mathcal{A} \cap {}^{\kappa>}B). \tag{4.22}$$ grp-19-a

We have to show that the player II then has a winning strategy in $\mathcal{G}^{\kappa>\theta}(\mathcal{A})$.

Let $\mathbb{P}$ be a $<\kappa$-closed p.o. such that, for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are classes j, M in $\mathsf{V}[\mathbb{G}]$ such that

$$\mathsf{V}[\mathbb{G}] \models \text{“} M \text{ is a transitive class”}, \tag{4.23}$$ grp-19-0

$$\mathsf{V}[\mathbb{G}] \models \text{“} j : \mathsf{V} \overset{\prec}{\to} M \text{” and} \tag{4.24}$$ grp-19-1

$$\mathsf{V}[\mathbb{G}] \models \text{“} crit(j) = (\kappa^+)^{\mathsf{V}},\ j((\kappa^+)^{\mathsf{V}}) > \theta^\kappa \text{ and } j''\theta^\kappa \in M\text{”}. \tag{4.25}$$ grp-19-2

Let

$$B = j''\theta. \tag{4.26}$$ grp-19-3

Note that we have $B \in M$ by (4.25) and $B = j''\theta^\kappa \cap \sup(j''\theta)$.

Cl-grp-3

Claim 4.11.1 *In $\mathsf{V}[\mathbb{G}]$, we have:* (1) $([B]^{<\kappa})^{\mathsf{V}[\mathbb{G}]} = ([B]^{<\kappa})^M$ *and* $({}^{\kappa>}B)^{\mathsf{V}[\mathbb{G}]} = ({}^{\kappa>}B)^M$,

(2) $j''\mathcal{A} = j(\mathcal{A}) \cap ({}^{\kappa>}B)^M$ *and*

(3) $j''\mathcal{D} = j(f)''(j''\theta^\kappa)$ *where* $f \in \mathsf{V}$ *with* $f : \theta^\kappa \to \mathcal{D}$ *is a surjection.*

$\vdash$ (1): Since $([B]^{<\kappa})^{\mathsf{V}[\mathbb{G}]} \supseteq ([B]^{<\kappa})^M$ is trivial, we prove the other inclusion. In $\mathsf{V}[\mathbb{G}]$, suppose that $s \in ([B]^{<\kappa})^{\mathsf{V}[\mathbb{G}]}$. Then $\mathsf{V}[\mathbb{G}] \models$ “$j^{-1}{}''s \in [\theta]^{<\kappa}$”. By $<\kappa$-closedness of $\mathbb{P}$, it follows that $j^{-1}{}''s \in \mathsf{V}$ and $s = j(j^{-1}{}''s) \in M$.

The second equation can be also proved similarly.

(2): For $t \in \mathcal{A}$, $j(t) \in j(\mathcal{A})$ by elementarity. Let $\mu = dom(t)$. Since $\mu < \kappa < \kappa^+$, we have $j(\mu) = \mu$. Thus, by elementarity, we have $j(t) : \mu \to j(\theta)$. For each $\xi < \mu$, since $t(\xi) \in \theta$, $j(t)(\xi) = j(t(\xi)) \in j''\theta$. Thus $M \models$ “$j(t) : \mu \to j''\theta$”. This proves $j''\mathcal{A} \subseteq j(\mathcal{A}) \cap ({}^{\kappa>}B)^M$.

Suppose now that $t^* \in j(\mathcal{A}) \cap ({}^{\kappa>}B)^M$. By $<\kappa$-closedness of $\mathbb{P}$, $t^* \in V$. Let $\mu = dom(t^*)$. Then $\mu < \kappa < \kappa^+$ and hence $\mu = j(\mu)$. For each $\xi < \mu$, $t^*(\xi) \in B = j''\theta$. Hence there is a (unique) $\eta_\xi \in \theta$ such that $j(\eta_\xi) = t^*(\xi)$. Let $t : \mu \to \theta$; $\xi \mapsto \eta_\xi$. Note that $j^{-1} \restriction range(t^*) \in V$ by $<\kappa$-closedness of $\mathbb{P}$, and hence we can actually construct such t in V. Thus $t \in V$ and $j(t) = t^*$. By elementarity $t \in \mathcal{A}$. This proves $j''\mathcal{A} \supseteq j(\mathcal{A}) \cap ({}^{\kappa>}B)^M$.

(3): Trivial. $\dashv$ (Claim 4.11.1)

Note that, by (4.25), Claim 4.11.1, (2) and (3), we have

(4.27) $\quad j''\mathcal{A} \in M$ and grp-19-4

(4.28) $\quad j''\mathcal{D} \in M.$ grp-19-5

This can be seen much easier by using Lemma 2.5 in [13].

Cl-grp-4

Claim 4.11.2 $B \in j(\mathcal{D})$.

$\vdash$ We have $M \models$ "$\kappa \leq |B| = |\theta| < j((\kappa^+)^{\mathsf{V}}) = \kappa^+$". Thus $M \models$ "$|B| = \kappa$". Since $M \models$ "$\bigcup j''\mathcal{D} = B$", we can construct an increasing sequence $\langle u_\alpha : \alpha < \kappa \rangle$ in M with $u_\alpha \in j''\mathcal{D}$ such that $\bigcup_{\alpha<\kappa} u_\alpha = B$ (note that all initial segments of the sequence has j^{-1} image by the $<\kappa$-closedness of $\mathbb{P}$ and hence we can continue the inductive construction in M at limit steps).

By elementarity, we have $M \models$ "$j(\mathcal{D})$ is κ-club". Hence $M \models$ "$B = \bigcup_{\alpha<\kappa} u_\alpha \in j(\mathcal{D})$". $\dashv$ (Claim 4.11.2)

Now by elementarity and (4.22), we have
$M \models$ "the player II has a winning strategy in $\mathcal{G}^{\kappa>B}(j(\mathcal{A}) \cap {}^{\kappa>}B)$".

By Claim 4.11.1, (1), it follows that
$\mathsf{V}[\mathbb{G}] \models$ "the player II has a winning strategy in $\mathcal{G}^{\kappa>B}(j(\mathcal{A}) \cap {}^{\kappa>}B)$".

Since $\mathsf{V}[\mathbb{G}] \models$ "$\langle \underbrace{j(\mathcal{A}) \cap {}^{\kappa>}B}_{=j''\mathcal{A}}, \underbrace{{}^{\kappa>}B}_{\kappa(j''\theta)} \rangle \cong \langle \mathcal{A}, {}^{\kappa>}\theta \rangle$" by Claim 4.11.1, (2), (3), it follows that $\mathsf{V}[\mathbb{G}] \models$ "the player II has a winning strategy in $\mathcal{G}^{\kappa>\theta}(\mathcal{A})$".

By Lemma 4.9, (2), it follows that $\mathsf{V} \models$ "the player II has a winning strategy in $\mathcal{G}^{\kappa>\theta}(\mathcal{A})$".

$\square$ (Lemma 4.11)

A cardinal κ is said to be *generically measurable by $<\mu$-closed forcing* for a regular cardinal $\mu < \kappa$ if there is a $<\mu$-closed p.o. $\mathbb{P}$ such that for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are an inner model $M \subseteq \mathsf{V}[\mathbb{G}]$ and elementary embedding $j : V \to M$ such that $crit(j) = \kappa$. Clearly, every generically supercompact cardinal by $<\mu$-closed forcing is generically measurable by $<\mu$-closed forcing. The following is easy to prove:

L-grp-7

Lemma 4.12 *Suppose that κ^+ is a generically measurable by $<\kappa$-closed forcing. Then $2^{<\kappa} = \kappa$ holds.*

Proof. Suppose otherwise. Let $\mu < \kappa$ be such that $2^\mu > \kappa$ and let $\lambda = 2^\mu$. We have $\lambda \geq \kappa^+$.

Let $\mathbb{P}$ be a $<\kappa$-closed p.o. such that, for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are an inner model $M \subseteq \mathsf{V}[\mathbb{G}]$ and an elementary embedding $j : \mathsf{V} \overset{\preccurlyeq}{\to} M$ with $crit(j) = \kappa^+$.

In V, let $f : \lambda \to \mathcal{P}(\mu)$ be a bijection. By elementarity, $M \models$ "$j(f) : j(\lambda) \to \mathcal{P}(\mu)$ is a bijection".

We have $\mathcal{P}(\mu)^{\mathsf{V}} = \mathcal{P}(\mu)^{\mathsf{V}[\mathbb{G}]} \supseteq \mathcal{P}(\mu)^M$ by $<\kappa$-closedness of $\mathbb{P}$.

Thus

(ℵ4.3) $M \models$ "$j(f) : j(\lambda) \to \mathcal{P}(\mu)^{\mathsf{V}}$ is a bijection".

Since $j''\lambda \subsetneqq j(\lambda)$ ($\kappa^+ \in j(\lambda) \setminus j''\lambda$) and $j(f)''(j''\lambda) = f''\lambda = \mathcal{P}(\mu)^{\mathsf{V}}$, this is a contradiction. ❑ (Lemma 4.12)

For regular cardinals κ, λ with $\kappa \leq \lambda$, let

$$D_{\kappa,\lambda} = \{u \in [\lambda]^\kappa \,:\, \kappa \subseteq u\}, \tag{4.29}$$ grp-19-6

$$\mathcal{F}_{\kappa,\lambda} = \{f \,:\, f : D_{\kappa,\lambda} \to \lambda,\ f(A) \in A \text{ for all } A \in D_{\kappa,\lambda}\}, \tag{4.30}$$ grp-20

$$A_{\kappa,\lambda} = \mathcal{F}_{\kappa,\lambda} \cup \lambda \tag{4.31}$$ grp-21

and let $\mathcal{A}_{\kappa,\lambda} \subseteq {}^{\kappa>}A_{\kappa,\lambda}$ be defined as $\mathcal{A}_{\kappa,\lambda} = \mathcal{A}^0_{\kappa,\lambda} \cup \mathcal{A}^1_{\kappa,\lambda}$ where

$$\mathcal{A}^0_{\kappa,\lambda} = \{\langle a_\alpha, b_\alpha \,:\, \alpha < \xi\rangle ^\frown \langle a_\xi\rangle \in {}^{\kappa>}A \,:\, a_\xi \in \mathcal{F}_{\kappa,\lambda}\}, \text{ and} \tag{4.32}$$ grp-22

$$\begin{aligned}\mathcal{A}^1_{\kappa,\lambda} = \{\langle a_\alpha, b_\alpha \,:\, \alpha < \xi\rangle \in {}^{\kappa>}A \,:\, \xi < \kappa,\ & (1)\ \ a_\alpha \in \mathcal{F}_{\kappa,\lambda},\\ & (2)\ \ b_\alpha \in \lambda \text{ and}\\ & (3)\ \ \bigcup {a_\alpha}^{-1}{}''\{b_\alpha\} = \lambda\\ & \quad\ \text{for all } \alpha < \xi, \text{ and}\\ & (4)\ \ |\textstyle\bigcap_{\alpha<\xi} {a_\alpha}^{-1}{}''\{b_\alpha\}\,| > 2\}.\end{aligned} \tag{4.33}$$ grp-23

In the game $\mathcal{G}^{{}^{\kappa>}A_{\kappa,\lambda}}(\mathcal{A}_{\kappa,\lambda})$, the player II tries to construct a filter base $\{a^{-1}{}''\{b_\alpha\} : \alpha < \kappa\}$ over $D_{\kappa,\lambda}$ while the player I challenges by demanding the normality of the filter for the regressive functions a_α, $\alpha < \kappa$. The role of the condition (3) in (4.33) will become clear in the proof of Claim 4.13.2 below.

T-grp-0

Theorem 4.13 *For a regular uncountable κ, the following are equivalent:*

(a) $2^{<\kappa} = \kappa$ *and* $\mathsf{GRP}^{<\kappa}(<\kappa^+)$ *holds.*

(b) *The player II has a winning strategy in* $\mathcal{G}^{{}^{\kappa>}A_{\kappa,\lambda}}(\mathcal{A}_{\kappa,\lambda})$ *for all regular* λ.

(c) κ^+ *is generically supercompact for* $<\kappa$*-closed forcing.*

Proof. (c) ⇒ (a) follows from Lemma 4.11 and Lemma 4.12.

(a) ⇒ (b): Assume that $\mathsf{GRP}^{<\kappa}(<\kappa^+)$ holds and $\kappa^{<\kappa} = \kappa$.

Let θ be a sufficiently large regular cardinal and

$$\mathcal{C} \subseteq \{A_{\kappa,\lambda} \cap M \,:\, M \prec \mathcal{H}(\theta),\ |\,M\,| = \kappa,\ [M]^{<\kappa} \subseteq M, \kappa, \lambda \in M\} \tag{4.34}$$ grp-24

be κ-club. Note that the assumption $2^{<\kappa} = \kappa$ is needed here for the existence of such $\mathcal{C}$.

For $A_{\kappa,\lambda} \cap M \in \mathcal{C}$, the player II wins in a game $\langle a_\alpha, b_\alpha : \alpha < \kappa \rangle$ if he chooses $b_\alpha = a_\alpha(\lambda \cap M)$ as his αth move for all $\alpha < \kappa$. Note that $b_\alpha \in \lambda \cap M$ by regressiveness of $a_\alpha \in \mathcal{F}_{\kappa,\lambda}$ (4.30) and hence $b_\alpha \in A_{\kappa,\lambda} \cap M$. Note also that the conditions (3), (4) in (4.33) are satisfied by elementarity.

By $\mathsf{GRP}^{<\kappa}(<\kappa^+)$, it follows that the player II has a winning strategy in $\mathcal{G}^{\kappa > A_{\kappa,\lambda}}(\mathcal{A}_{\kappa,\lambda})$.

(b) $\Rightarrow$ (c): Let $\lambda \geq \kappa$ be a regular cardinal and let σ be a winning strategy of the player II in $\mathcal{G}^{\kappa > A_{\kappa,\lambda}}(\mathcal{A}_{\kappa,\lambda})$. Let

$$\mathbb{P} = \mathrm{Fn}(\kappa, |\,\mathcal{F}_{\kappa,\lambda}\,|, <\kappa) \tag{4.35}$$ grp-25

and let $\mathbb{G}$ be a $(\mathsf{V}, \mathbb{P})$-generic filter. Note that $\mathbb{P}$ is $<\kappa$-closed and $\mathsf{V}[\mathbb{G}] \models |\,(\mathcal{F}_{\kappa,\lambda})^{\mathsf{V}}\,| = \kappa$.

In $\mathsf{V}[\mathbb{G}]$, let

(4.36) $\{f_\xi : \xi < \kappa\}$ be an enumeration of $(\mathcal{F}_{\kappa,\lambda})^{\mathsf{V}}$. grp-26

By Lemma 4.9, (1), σ is still a winning strategy of the player II in $\mathcal{G}^{\kappa > (A_{\kappa,\lambda})^{\mathsf{V}}}((\mathcal{A}_{\kappa,\lambda})^{\mathsf{V}})$ in $\mathsf{V}[\mathbb{G}]$. Let α_ξ, $\xi < \kappa$ be the moves of the player II in a match where the player I takes f_ξ, $\xi < \kappa$ as her moves and the player II chooses his moves according to σ:

I	f_0		f_1		f_2		$\cdots$	f_ξ		$\cdots$	
II		α_0		α_1		α_2	$\cdots$		α_ξ	$\cdots$	$(\xi < \kappa)$

Let $\mathcal{U}_0 = \{{f_\xi}^{-1}{}''\{\alpha_\xi\} : \xi < \kappa\}$. The player II wins the match since he played according to σ, $\mathcal{U}_0$ has the finite intersection property. Let $\mathcal{U} \subseteq (\mathcal{P}(D_{\kappa,\lambda}))^{\mathsf{V}}$ be the upward closure of $\mathcal{U}_0$ (with respect to $\subseteq$).

Cl-grp-5

Claim 4.13.1 *$D_{\kappa,\lambda} \in \mathcal{U}$.*

$\vdash$ ${f_0}^{-1}{}''\{\alpha_0\} \subseteq D_{\kappa,\lambda}$. $\dashv$ (Claim 4.13.1)

Cl-grp-5-0

Claim 4.13.2 *For any $\beta \in \lambda$, we have $\{u \in D_{\kappa,\lambda} : \beta \in u\} \in \mathcal{U}$.*

$\vdash$ (In V) let $\delta \in 2$ be such that $\delta \neq \beta$ and $f \in \mathcal{F}_{\kappa,\lambda}$ be defined by

$$f(u) = \begin{cases} \beta, & \text{if } \beta \in u; \\ \delta, & \text{otherwise} \end{cases}$$

for $u \in D_{\kappa,\lambda}$. In $V[\mathbb{G}]$, there is $\xi < \kappa$ such that $f_\xi = f$. At his ξth move, the player II is forced to take β as his move because of (3) in (4.33). $\dashv$ (Claim 4.13.2)

Claim 4.13.3 *For any U, $V \in \mathcal{U}$ we have $U \cap V \in \mathcal{U}$* Cl-grp-6

$\vdash$ (In V) let $f \in \mathcal{F}_{\kappa,\lambda}$ be defined by

$$f(u) = \begin{cases} 0, & \text{if } u \notin U; \\ 1, & \text{if } u \in U \setminus V; \\ 2, & \text{if } u \in U \cap V \end{cases}$$

for $u \in D_{\kappa,\lambda}$. (In $\mathsf{V}[\mathbb{G}]$) let $\xi < \kappa$ be such that $f = f_\xi$. Since $\mathcal{U}_0$ should have intersection property, we should have $\alpha_\xi = 2$. Thus $U \cap V \in \mathcal{U}_0 \subseteq \mathcal{U}$.

$\dashv$ (Claim 4.13.3)

The following two Claims can be proved with arguments similar to those of previous Claims:

Claim 4.13.4 *For any $U \in \mathsf{V}$ with $U \subseteq D_{\kappa,\lambda}$, either $U \in \mathcal{U}$ or $D_{\kappa,\lambda} \setminus U \in \mathcal{U}$.* Cl-grp-7

$\vdash$ Consider $f \in \mathcal{F}_{\kappa,\lambda}$ defined by

$$f(u) = \begin{cases} 1, & \text{if } u \in U; \\ 0, & \text{otherwise} \end{cases}$$

for $u \in D_{\kappa,\lambda}$. $\dashv$ (Claim 4.13.4)

By the Claims above, it follows that $\mathcal{U}$ is a V-ultrafilter over $D_{\kappa,\lambda}$.

Claim 4.13.5 *For any sequence $\langle U_\alpha : \alpha < \kappa \rangle$ ($\in \mathsf{V}$), if $\{U_\alpha : \alpha < \kappa\} \subseteq \mathcal{U}$ then $\bigcap\{U_\alpha : \alpha < \kappa\} \in \mathcal{U}$.* Cl-grp-8

$\vdash$ Re-enumerate $\langle U_\alpha : \alpha < \kappa \rangle$ as $\langle U_\alpha : \alpha < \kappa \setminus 1 \rangle$ and consider $f \in \mathcal{F}_{\kappa,\lambda}$ defined by

$$f(u) = \begin{cases} 0, & \text{if } u \in \bigcap_{\alpha<\kappa} U_\alpha; \\ \min\{\alpha < \kappa : u \notin U_\alpha\}, & \text{otherwise} \end{cases}$$

for $u \in D_{\kappa,\lambda}$. $\dashv$ (Claim 4.13.5)

By Claim 4.13.5, $\mathcal{U}$ is V-κ-complete filter. From Claim 4.13.5, it follows that $\mathcal{U}$ is $<\kappa$-complete in the sense of $\mathsf{V}[\mathbb{G}]$, since, by the $<\kappa$-closedness of $\mathbb{P}$, every sequence in $\mathsf{V}[\mathbb{G}]$ of elements of V are actually in V. Note that this stronger completeness is needed in the proof of the well-foundedness of $\in_{\mathcal{U}}$ in Claim 4.13.6 below.

Let

$$(4.37) \quad {}^{D_{\kappa,\lambda}}\mathsf{V}/\mathcal{U} = \{[f]_{\mathcal{U}} : f \in \mathsf{V},\ f \text{ is a mapping with } dom(f) = D_{\kappa,\lambda}\}$$ grp-27

where $[f]_{\mathcal{U}}$ denotes the equivalence class of f with respect to the equivalence relation

(4.38) $f \sim_{\mathcal{U}} g \;\Leftrightarrow\; \{u \in D_{\kappa,\lambda} \,:\, f(u) = g(u)\} \in \mathcal{U}.$ grp-28

For $[f]_{\mathcal{U}}, [g]_{\mathcal{U}} \in {}^{D_{\kappa,\lambda}}\mathsf{V}/\mathcal{U}$

(4.39) $[f]_{\mathcal{U}} \in_{\mathcal{U}} [g]_{\mathcal{U}} \;\Leftrightarrow\; \{u \in D_{\kappa,\lambda} \,:\, f(u) \in g(u)\} \in \mathcal{U}.$ grp-29

Since $\mathcal{U}$ is a filter, it is easy to show by a standard argument that $\sim_{\mathcal{U}}$ is an equivalence relation[5)] and $\in_{\mathcal{U}}$ is well-defined.

The following is also easy and can be shown by standard arguments:

Cl-grp-8-0

Claim 4.13.6 (1) *$\in_{\mathcal{U}}$ is a set-like, extensional and well-founded class relation on ${}^{D_{\kappa,\lambda}}\mathsf{V}/\mathcal{U}$.*

(2) *Łoś's Theorem (see e.g. [3]).* ⊣

By Claim 4.13.6, (1), there is the Mostowski collapse

(4.40) $m : \langle {}^{D_{\kappa,\lambda}}\mathsf{V}/\mathcal{U}, \in_{\mathcal{U}}\rangle \overset{\cong}{\rightarrow} \langle M, \in\rangle$

for a uniquely determined transitive class M. By Łoś's Theorem,

(4.41) $j : V \rightarrow M;\; a \mapsto m([i_a]_{\mathcal{U}})$

is an elementary embedding where $i_a : D_{\kappa,\lambda} \rightarrow \mathsf{V};\; u \mapsto a$. As usual we identify $[f]_{\mathcal{U}} \in {}^{D_{\kappa,\lambda}}\mathsf{V}/\mathcal{U}$ with $m([f]_{\mathcal{U}})$ and simply write $[f]_{\mathcal{U}}$ to denote its value by m.

For $\alpha < \kappa^+$ we have $j(\alpha) = \alpha$ by Claim 4.13.5. Since $j(\kappa^+) > [d_{\kappa^+}]_{\mathcal{U}} > j(\alpha)$ for all $\alpha < \kappa^+$ where $d_{\kappa^+} : D_{\kappa,\lambda} \rightarrow \mathsf{V};\; u \mapsto \sup(u \cap \kappa^+)$. $crit(j) = \kappa^+$. [`Since` $\{u \in D_{\kappa,\lambda} \,:\, \kappa^+ = i_{\kappa^+}(u) > d_{\kappa^+}(u)\} = D_{\kappa,\lambda} \in \mathcal{U}$, $j(\kappa^+) = [i_{\kappa^+}]_{\mathcal{U}} > [d_{\kappa^+}]_{\mathcal{U}}$ `by Łoś's Theorem.`

$\{u \in D_{\kappa,\lambda} \,:\, \sup(u \cap \kappa^+) > \alpha\} \supseteq \underbrace{\{u \in D_{\kappa,\lambda} \,:\, \alpha + 1 \in u\} \in \mathcal{U}}_{\texttt{by Claim 4.13.2}}$. `This implies` $[d_{\kappa^+}]_{\mathcal{U}} > [id_\alpha]_{\mathcal{U}} = j(\alpha)$ `by Łoś's Theorem.`]

$j''\lambda \in M$ follows from the next Claim:

Cl-grp-9

Claim 4.13.7 $[id]_{\mathcal{U}} = j''\lambda$ *where* $id : D_{\kappa,\lambda} \rightarrow \mathsf{V};\; u \mapsto u$.

⊢ Suppose $\beta < \lambda$. Then Claim 4.13.2 and the definition of $\in_{\mathcal{U}}$ imply $j(\beta) = [i_\beta]_{\mathcal{U}} \in [id]_{\mathcal{U}}$.

Suppose now $[f]_{\mathcal{U}} \in [id]_{\mathcal{U}}$. Without loss of generality, we may assume that $f(u) \in u$ for all $u \in D_{\kappa,\lambda}$. But then, in $\mathsf{V}[\mathbb{G}]$, there is a $\xi < \kappa$ such that $f_\xi = f$. For $\beta = \alpha_\xi$, we have $[f]_{\mathcal{U}} = [i_\beta]_{\mathcal{U}} = j(\beta)$. ⊣ (Claim 4.13.7)

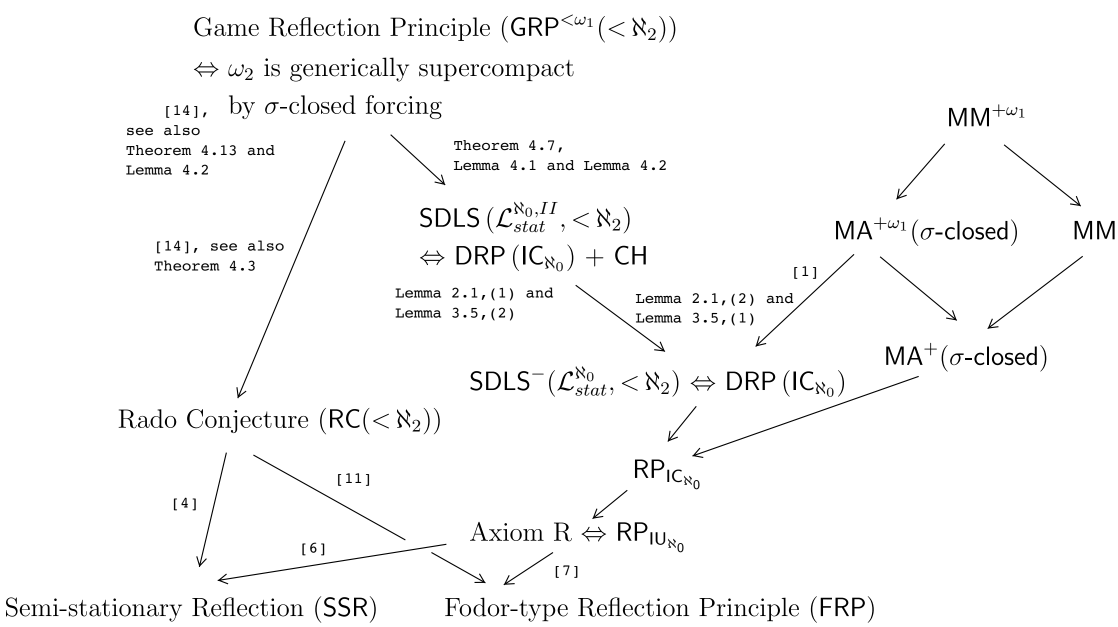
❑ (Theorem 4.13)

[5)] Strictly speaking, we have to insert here a modification of the definition of the ultrapower to make equivalence classes of $\sim_{\mathcal{U}}$ sets, but we simply drop this well-known detail.

5 Conclusion

summary

As we have seen in the previous sections, Strong Downward Löwenheim-Skolem Theorems for stationary logic (in $\aleph_0$ interpretation of the weak second-order variables) and its variations fit nicely in the web of implications of reflection axioms. In case of statements with the reflection cardinal $<\aleph_2$, this can be summarized in the following diagram:



Note that $\mathsf{GRP}^{<\omega_1}(<\aleph_2)$ implies CH while MM implies $2^{\aleph_0} = \aleph_2$. In the sequel [13] of the present paper, we shall show among other things that there is a natural Löwenheim-Skolem theorem type statement with reflection cardinal $< 2^{\aleph_0}$ which implies that the continuum is very large (e.g. weakly Mahlo and more).

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