

Some remarks on generically large cardinals

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Abstract

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This is an extended version of the paper with the same title to appear in the RIMS Kôkyûroku volume on RIMS set theory workshop 2020.

All additional details not to be contained in the submitted version of the paper are either typeset in dark electric blue (the color in which this paragraph is typeset) or put in separate appendices. The numbering of the assertions is kept identical with the submitted version.

The most up-to-date file of this extended version is downloadable as:
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In this note, we write down several observations concerning generic supercompactness and Laver-generic supercompactness.

In the following, we write down several observations concerning generic supercompactness and Laver-generic supercompactness. Most of the assertions presented here are either trivial, simple application of well-known ideals, or folklore. Their details are written just to clarify the situation.

This article is still in a state of a work in progress, and there may be some additional topics in new sections, as well as improvements and extension of the material presented here in the most recent version of this note uploaded at the URL mentioned in the footnote of the previous page.

1 First-order definability of generic supercompactness

After the original version of the present paper was submitted, we improved the following Theorem 1.3 and Corollary 1.5 tremendously. We can now prove the first-order definability of generic and Laver-generic supercompactness as well as the first-order definability of other generic versions of large cardinals for any (appropriate, e.g iterable) class \mathcal{P} of posets.

first-order

This result is written in

[8] [S. Fuchino, H. Sakai, The first-order definability of generic large cardinals, pre-preprint.](#)

For a class \mathcal{P} of posets, a cardinal κ is said to be *generically supercompact by \mathcal{P}* , if, for any $\lambda \geq \kappa$, there is a poset $\mathbb{P} \in \mathcal{P}$ with (\mathbf{V}, \mathbb{P}) -generic \mathbb{G} , and classes j , $M \subseteq \mathbf{V}[\mathbb{G}]$ such that

$$(1.1) \quad j : \mathbf{V} \xrightarrow{\kappa} M \subseteq \mathbf{V}[\mathbb{G}];^1$$

x-gen-sc-0

$$(1.2) \quad \text{crit}(j) = \kappa, j(\kappa) > \lambda; \text{ and}$$

x-gen-sc-1

$$(1.3) \quad j''\lambda \in M.$$

x-gen-sc-2

We call the class mapping j as above a *λ -generically supercompact embedding for κ* .

This section should be updated according to definability-of-glc-sakai-corr

If M is obtained as an inner model of \mathbf{V} by ultraproduct construction with a $< \omega_1$ -complete ultrafilter in \mathbf{V} , the condition (1.3) implies ${}^\lambda M \subseteq M$ (see Proposition 22.4 in [5]).

¹ When we write $j : \mathbf{V} \xrightarrow{\kappa} M \subseteq \mathbf{V}[\mathbb{G}]$, we always assume that M is transitive in $\mathbf{V}[\mathbb{G}]$.

In the context of generic supercompactness, the condition (1.3) still implies a certain kind of closedness of M . This can be seen in the following Lemma:

Lemma 1.1 (Lemma 2.5 in [2]) *Suppose that \mathbb{G} is a (\mathbb{V}, \mathbb{P}) -generic filter for a poset $\mathbb{P} \in \mathbb{V}$, and $j : \mathbb{V} \xrightarrow{\simeq} M \subseteq \mathbb{V}[\mathbb{G}]$ is such that, for cardinals κ, λ in \mathbb{V} with $\kappa \leq \lambda$, $\text{crit}(j) = \kappa$ and $j''\lambda \in M$. Then, we have the following:*

- (1) For any set $A \in \mathbb{V}$ with $\mathbb{V} \models |A| \leq \lambda$, we have $j''A \in M$.
- (2) $j \upharpoonright \lambda, j \upharpoonright \lambda^2 \in M$.
- (3) For any $A \in \mathbb{V}$ with $A \subseteq \lambda$ or $A \subseteq \lambda^2$ we have $A \in M$.
- (4) $(\lambda^+)^M \geq (\lambda^+)^{\mathbb{V}}$, Thus, if $(\lambda^+)^{\mathbb{V}} = (\lambda^+)^{\mathbb{V}[\mathbb{G}]}$, then $(\lambda^+)^M = (\lambda^+)^{\mathbb{V}}$.
- (5) $\mathcal{H}(\lambda^+)^{\mathbb{V}} \subseteq M$.
- (6) $j \upharpoonright A \in M$ for all $A \in \mathcal{H}(\lambda^+)^{\mathbb{V}}$. □

It is consistent (modulo a supercompact cardinal) that a successor cardinal of a regular uncountable cardinal is generically supercompact. In the following, we use Kanamori's notation of collapsing posets (see §10 of [5]).

Fact 1.2 *Suppose that κ is a (really) supercompact cardinal, $\mu < \kappa$ a regular uncountable cardinal, and $\mathbb{P}_0 = \text{Col}(\mu, \kappa)$. Then, for a $(\mathbb{V}, \mathbb{P}_0)$ -generic \mathbb{G}_0 ,*

$$\mathbb{V}[\mathbb{G}_0] \models \text{“}\mu^+ \text{ is a generically supercompact cardinal by } < \mu\text{-closed posets”}.$$

Proof. Note that $\mathbb{V}[\mathbb{G}_0] \models \text{“}\mu^+ = \kappa\text{”}$.

For $\lambda \geq \kappa$, let $j : \mathbb{V} \xrightarrow{\simeq} M$ be a λ -supercompact embedding for κ . Then we have

$$j(\mathbb{P}_0) \underset{\text{by elementarity}}{=} \text{Col}(\underbrace{j(\mu)}_{=\mu}, j(\kappa))^M \overset{\text{by closedness of } M}{=} \text{Col}(\mu, j(\kappa))^{\mathbb{V}}.$$

For a $(\mathbb{V}[\mathbb{G}_0], \text{Col}(\mu, j(\kappa) \setminus \kappa))$ -generic filter \mathbb{G} , the lifting

$$\tilde{j} : \mathbb{V}[\mathbb{G}_0] \xrightarrow{\simeq} \underbrace{M[\mathbb{G}_0][\mathbb{G}]}_{\subseteq \mathbb{V}[\mathbb{G}_0][\mathbb{G}]}; \tilde{a}^{\mathbb{G}_0} \mapsto j(\tilde{a})^{\mathbb{G}_0 * \mathbb{G}}$$

witnesses the generic λ -supercompactness of $\underbrace{\kappa}_{=(\mu^+)^{\mathbb{V}[\mathbb{G}_0]}}$ by μ -closed posets in $\mathbb{V}[\mathbb{G}_0]$. □ (Fact 1.2)

For a class \mathcal{P} of posets such that no $\mathbb{P} \in \mathcal{P}$ adds any new ω -sequence of ground model sets, the generic supercompactness by \mathcal{P} is first-order definable. This is seen in the following Proposition. The Proposition is proved by imitating the proof of the characterization of supercompactness by Solovay and Reinhardt in terms of the existence of normal ultrafilters (see e.g. Theorem 22.7 in [5]).

Theorem 1.3 *Suppose that \mathcal{P} is a class of posets such that no $\mathbb{P} \in \mathcal{P}$ adds any new ω -sequence of ground model sets, and \mathcal{P} is closed with respect to restriction (i.e., if $\mathbb{P} \in \mathcal{P}$ and $\mathbb{p} \in \mathbb{P}$, then $\mathbb{P} \restriction \mathbb{p} \in \mathcal{P}$).* P-gen-sc-1

An uncountable cardinal κ is generically supercompact by \mathcal{P} if and only if, for any $\lambda \geq \kappa$, there is a $\mathbb{P} \in \mathcal{P}$ such that

$$\Vdash_{\mathbb{P}} \text{“there is a } \mathbf{V}\text{-normal ultrafilter on } \mathcal{P}^{\mathbf{V}}(\mathcal{P}_{\kappa}(\lambda)^{\mathbf{V}})\text{”}.$$

Here, the notion of \mathbf{V} -normal ultrafilter is defined as follows: Suppose that we are living in a universe \mathbf{W} and \mathbf{V} is an inner model. Let λ be an ordinal in \mathbf{V} , $\mathcal{I} \in \mathbf{V}$, $\mathcal{I} \subseteq \mathcal{P}^{\mathbf{V}}(\lambda)$ a σ -ideal with $\{\xi\} \in \mathcal{I}$ for all $\xi < \lambda$, and $\mathcal{B} \in \mathbf{V}$ the sub-Boolean algebra $\mathcal{B} = \mathcal{P}^{\mathbf{V}}(\mathcal{I})$ of $\mathcal{P}^{\mathbf{W}}(\mathcal{I})$.

In \mathbf{W} , $U \subseteq \mathcal{B}$ is a \mathbf{V} -normal ultrafilter if

(1.4) U is a ultrafilter on the Boolean algebra \mathcal{B} . I.e., x-gen-sc-3

- (i) $\emptyset \notin U$;
- (ii) $A \cap A' \in U$ for any $A, A' \in U$;
- (iii) if $A \in U$, $A \subseteq A' \in \mathcal{B}$, then $A' \in U$; and
- (iv) for any $A \in \mathcal{B}$, either $A \in U$ or $\mathcal{I} \setminus A \in U$;

(1.5) For any $x_0 \in \mathcal{I}$, we have $\{x \in \mathcal{I} : x_0 \subseteq x\} \in U$; x-gen-sc-4

(1.6) For any $\langle A_{\xi} : \xi \in \lambda \rangle \in \mathbf{V}$, if $\{A_{\xi} : \xi < \lambda\} \subseteq U$, we have x-gen-sc-5

$\Delta_{\xi \in \lambda} A_{\xi} \in U$. Here, $\Delta_{\xi \in \lambda} A_{\xi}$ is the diagonal intersection of A_{ξ} 's defined by

$$(1.7) \quad \Delta_{\xi \in \lambda} A_{\xi} := \{x \in \mathcal{I} : x \in A_{\xi} \text{ for all } \xi \in x\}. \quad \text{x-gen-sc-5-a}$$

Lemma 1.4 *Suppose that $U \subseteq \mathcal{B}$ is a \mathbf{V} -normal ultrafilter.* P-gen-sc-2

(1) For $\delta < \lambda$ such that $\delta \in \mathcal{I}$, and $\langle A_{\xi} : \xi \in \delta \rangle \in \mathbf{V}$ with $A_{\xi} \in U$ for all $\xi \in \delta$, we have $\bigcap_{\xi \in \delta} A_{\xi} \in U$.

(2) (Pressing Down Lemma) For any $f \in \mathbf{V}$ with $f : \mathcal{I} \rightarrow \mathbf{V}$, if $\{x \in \mathcal{I} : f(x) \in x\} \in U$, then there is $\xi < \lambda$ such that $\{x \in \mathcal{I} : f(x) = \xi\} \in U$.

Proof. (1): Let $A_{\xi} := \mathcal{I}$ for all $\xi \in \lambda \setminus \delta$. Then

$$U \ni \underbrace{\Delta_{\xi \in \lambda} A_{\xi}}_{\in U \text{ by (1.6)}} \cap \underbrace{\{x \in \mathcal{I} : \delta \subseteq x\}}_{\in U \text{ by (1.5)}} \subseteq \bigcap_{\xi \in \delta} A_{\xi}.$$

by (1.4), (ii)

Hence, $\bigcap_{\xi \in \delta} A_{\xi} \in U$ by (1.4), (iii).

(2): Suppose that f is a counter-example to the assertion. That is,

$$(1.8) \quad A := \{x \in \mathcal{I} : f(x) \in x\} \in U, \text{ but}$$

x-gen-sc-5-0

$$(1.9) \quad A_\xi := \{x \in \mathcal{I} : f(x) \neq \xi\} \in U \text{ for all } \xi \in \lambda.$$

x-gen-sc-5-1

Then $\Delta_{\xi < \lambda} A_\xi \cap A \in U$ by (1.6) and (1.4), (ii). By (1.4), (i), there is an element x^* of this set. $f(x^*) \in x^*$ by (1.8) but $f(x^*) \neq \xi$ for all $\xi \in x^*$ by (1.9) and the definition (1.7) of diagonal intersection. This is a contradiction. \square (Lemma 1.4)

Proof of Theorem 1.3: (\Rightarrow): Let $\lambda \geq \kappa$ and let \mathbb{P} be a $< \mu$ -closed poset with (\mathbb{V}, \mathbb{P}) -generic \mathbb{G} and classes $j, M \subseteq \mathbb{V}[\mathbb{G}]$ such that $j : \mathbb{V} \xrightarrow{\simeq} M$ is a λ -generically supercompact embedding for κ . In particular, we have $j''\lambda \in M$. Note that

$$(1.10) \quad M \models j''\lambda \in \mathcal{P}_{j(\kappa)}(j(\lambda)) = j(\mathcal{P}_\kappa(\lambda)^\mathbb{V}).$$

x-gen-sc-5-2

In $\mathbb{V}[\mathbb{G}]$, let

$$U_j := \{A \in \mathbb{V} : A \subseteq \mathcal{P}_\kappa(\lambda)^\mathbb{V}, j''\lambda \in j(A)\}.$$

Claim 1.4.1 U_j is a \mathbb{V} -normal ultrafilter on $\mathcal{P}^\mathbb{V}(\mathcal{P}_\kappa(\lambda)^\mathbb{V})$.

Cl-gen-sc-0

$\vdash U_j \models (1.4)$, (i): $j(\emptyset) = \emptyset$ by elementarity (and transitivity of M). Thus $\emptyset \notin U$ by definition.

(ii): Suppose $A, A' \in U_j$. By definition this means that $j''\lambda \in j(A)$ and $j''\lambda \in j(A')$. It follows that $j''\lambda \in j(A) \cap j(A') = \underbrace{j(A \cap A')}_{\text{by elementarity}}$. This shows that

$$A \cap A' \in U.$$

(iii): Suppose that $A \in U_j$ and $A' \in \mathbb{V}$ is such that $A \subseteq A' \subseteq \mathcal{P}_\kappa(\lambda)^\mathbb{V}$. Then by elementarity we have $M \models j(A) \subseteq j(A')$. Hence $j''\lambda \in j(A) \subseteq j(A')$, and $A' \in U_j$.

(iv): If $A \in \mathcal{P}^\mathbb{V}(\mathcal{P}_\kappa(\lambda)^\mathbb{V}) \setminus U_j$, then by (1.10), $j''\lambda \in j(\mathcal{P}_\kappa(\lambda)^\mathbb{V}) \setminus j(A) = j(\mathcal{P}_\kappa(\lambda)^\mathbb{V} \setminus A)$. Thus $\mathcal{P}_\kappa(\lambda)^\mathbb{V} \setminus A \in U_j$.

$U_j \models (1.5)$: Suppose $x_0 \in \mathcal{P}_\kappa(\lambda)^\mathbb{V}$ and let $A := \{x \in \mathcal{P}_\kappa(\lambda)^\mathbb{V} : x_0 \subseteq x\}$. Clearly $A \in \mathcal{P}^\mathbb{V}(\mathcal{P}_\kappa(\lambda)^\mathbb{V})$. By elementarity, and noting that $j(x_0) = j''x_0$ since $|x_0| < \kappa$, we have

$$M \models j(A) = \{x \in \mathcal{P}_{j(\kappa)}(j(\lambda)) : \underbrace{j(x_0)}_{= j''x_0} \subseteq x\}.$$

Thus $M \models j''\lambda \in j(A)$. Hence $A \in U_j$.

$U_j \models (1.6)$: Suppose that $\vec{A} := \langle A_\xi : \xi \in \lambda \rangle \in \mathbb{V}$ is such that $A_\xi \in U$, i.e.

$$(1.11) \quad j''\lambda \in j(A_\xi)$$

x-gen-sc-5-2-

for all $\xi < \lambda$.

By elementarity, we have

0

$$(1.12) \quad j(\Delta_{\xi \in \lambda} A_\xi) = \{x \in \mathcal{P}_{j(\kappa)}(j(\lambda))^M : \forall \eta \in x (x \in j(\vec{A}(\eta)))\} \quad \text{x-gen-sc-5-3}$$

For $\eta \in j''\lambda$, there is $\eta_0 \in \lambda$ such that $\eta = j(\eta_0)$. Thus

$$(1.13) \quad j(\vec{A})(\eta) = j(\vec{A})(j(\eta_0)) \stackrel{\text{by elementarity}}{=} \underbrace{j(\vec{A}(\eta_0))}_{= j(A_{\eta_0})} \stackrel{(1.11)}{\ni} j''\lambda. \quad \text{x-gen-sc-5-4}$$

By (1.12) and (1.13), it follows that $j''\lambda \in j(\Delta_{\xi \in \lambda} A_\xi)$, and thus $\Delta_{\xi \in \lambda} A_\xi \in U$.

⊢ (Claim 1.4.1)

(\Leftarrow): Let $\lambda \geq \kappa$ and let \mathbb{P} be a $< \mu$ -closed poset with (\mathbb{V}, \mathbb{P}) -generic \mathbb{G} and \mathbb{V} -normal ultrafilter $U \in \mathbb{V}[\mathbb{G}]$ on $\mathcal{P}^{\mathbb{V}}(\mathcal{P}_\kappa(\lambda)^{\mathbb{V}})$.

Let

$$(1.14) \quad \mathcal{W} := \{f \in \mathbb{V} : f : \mathcal{P}^{\mathbb{V}}(\mathcal{P}_\kappa(\lambda)^{\mathbb{V}}) \rightarrow \mathbb{V}\}$$

$$(1.15) \quad \text{For } f, g \in \mathcal{W}, f \sim_U g :\Leftrightarrow \{x \in \mathcal{P}_\kappa(\lambda)^{\mathbb{V}} : f(x) = g(x)\} \in U;$$

$$f \in_U g :\Leftrightarrow \{x \in \mathcal{P}_\kappa(\lambda)^{\mathbb{V}} : f(x) \in g(x)\} \in U.$$

\sim_U is a congruence relation to \in_U .

We write $f/\sim_U \in_U g/\sim_U :\Leftrightarrow f \in_U g$.²⁾

Let $i_U : \mathbb{V} \rightarrow \mathcal{W}/\sim_U$ be defined by

$$(1.16) \quad i_U(a) = \text{const}_a/\sim_U \quad \text{x-gen-sc-6}$$

for $a \in \mathbb{V}$ where const_a denote the function on $\mathcal{P}^{\mathbb{V}}(\mathcal{P}_\kappa(\lambda)^{\mathbb{V}})$ whose value is constantly a . Łoś's Theorem holds:

Claim 1.4.2 *For any formula $\varphi = \varphi(x_0, \dots, x_{n-1})$ in \mathcal{L}_\in (the language of ZF), and $f_0, \dots, f_{n-1} \in \mathcal{W}$, we have $\langle \mathcal{W}/\sim_U, \in_U \rangle \models \varphi(f_0/\sim_U, \dots, f_{n-1}/\sim_U)$, if and only if $\{x \in \mathcal{P}^{\mathbb{V}}(\mathcal{P}_\kappa(\lambda)^{\mathbb{V}}) : \mathbb{V} \models \varphi(f_0(x), \dots, f_{n-1}(x))\} \in U$.* Cl-gen-sc-0-0

⊢ By induction on φ . ⊢ (Claim 1.4.2)

By Claim 1.4.2, the class mapping i_U above is an elementary embedding of \mathbb{V} into $\langle \mathcal{W}/\sim_U, \in_U \rangle$.

Claim 1.4.3 \in_U is (i) an extensional, (ii) well-founded and (iii) set-like relation on \mathcal{W}/\sim_U . Cl-gen-sc-0-1

²⁾ We apply here “Scott’s trick” and define the equivalence class f/\sim_U by

$$f/\sim_U := \{g \in \mathcal{W} : g \sim_U f \text{ and } g \text{ is of minimal } \in\text{-rank} \\ \text{among elements of } \mathcal{W} \text{ with this property}\}$$

to make the equivalence class f/\sim_U a set.

⊢ (i): The extensionality of \in_U follows from the elementarity of i_U .

(ii): Assume, toward a contradiction, that there is a sequence $\langle f_n : n \in \omega \rangle$ in \mathcal{W} such that $f_{n+1} \in_U f_n$ for all $n \in \omega$. By the definition of \in_U , this means that $A_n = \{x \in \mathcal{P}^V(\mathcal{P}_\kappa(\lambda)^V) : f_{n+1}(x) \in f_n(x)\} \in U$ for all $n \in \omega$. Since \mathbb{P} does not add any new ω -sequence, $\langle f_n : n \in \omega \rangle \in V$. Thus, we also have $\langle A_n : n \in \omega \rangle \in V$. By Lemma 1.4, (1), it follows that $\bigcap_{n \in \omega} A_n \in U$. For an element x of this intersection, we have

$$f_0(x) \ni f_1(x) \ni f_2(x) \ni f_3(x) \ni \dots$$

by definition of A_n 's. This is a contradiction.

(iii): Let $f \in \mathcal{W}$ be arbitrary, and let $S = \bigcup_{x \in \mathcal{P}^V(\mathcal{P}_\kappa(\lambda)^V)} f(x)$. Then, by Łoś's Theorem, we have

$$\{g/\sim_U : g/\sim_U \in_U f/\sim_U\} \subseteq \{g/\sim_U : g : \mathcal{P}^V(\mathcal{P}_\kappa(\lambda)^V) \rightarrow S\}$$

The right side of the inclusion is clearly a set.

⊢ (Claim 1.4.3)

Let $\mu_U : \langle \mathcal{W}/\sim_U, \in_U \rangle \rightarrow \langle M, \in \rangle$ be the Mostowski-collapse, and let $[\cdot]_U : \mathcal{W} \rightarrow M$; $f \mapsto [f]_U := \mu_U(f/\sim_U)$.

Łoś's Theorem (Claim 1.4.2) translates to the following:

Claim 1.4.4 *For any formula $\varphi = \varphi(x_0, \dots, x_{n-1})$ in \mathcal{L}_\in (the language of ZF), and $f_0, \dots, f_{n-1} \in \mathcal{W}$, we have $M \models \varphi([f_0]_U, \dots, [f_{n-1}]_U)$, if and only if $\{x \in \mathcal{P}^V(\mathcal{P}_\kappa(\lambda)^V) : V \models \varphi(f_0(x), \dots, f_{n-1}(x))\} \in U$.*

Cl-gen-sc-1

⊢

Let

$$j_U : V \xrightarrow{\cong} M; a \mapsto [a]_U := \mu_U(i_U(a)) = [const_a]_U.$$

We show that $j_U : V \xrightarrow{\cong} M$ is a λ -generically supercompact embedding for κ .

Claim 1.4.5 (1) $j_U(\xi) = \xi$ for all $\xi \in \kappa$.

Cl-gen-sc-2

(2) $j_U(\kappa) > \kappa$.

(3) $j_U''\lambda \in M$.

⊢ (1): Note that $j_U(\xi) = \mu_U(i_U(\xi)) = [const_\xi]_U$. Thus, for $\xi < \kappa$ and $f \in \mathcal{W}$,

$$\begin{aligned} [f]_U \in j_U(\xi) &\Leftrightarrow [f]_U \in [const_\xi]_U \\ &\Leftrightarrow \underbrace{\{x \in \mathcal{P}^V(\mathcal{P}_\kappa(\lambda)^V) : f(x) \in \xi\}}_{\text{Claim 1.4.4}} \in U \\ &\Leftrightarrow \underbrace{\{x \in \mathcal{P}^V(\mathcal{P}_\kappa(\lambda)^V) : f(x) = \eta^*\}}_{\text{by Lemma 1.4, (2) and (1.5)}} \in U \text{ for some } \eta^* \in \xi \\ &\Leftrightarrow \underbrace{\{x \in \mathcal{P}^V(\mathcal{P}_\kappa(\lambda)^V) : f(x) = \eta^*\}}_{= const_{\eta^*}(x)} \in U \end{aligned}$$

$$\underbrace{\Leftrightarrow [f]_U = j_U(\eta^*) \text{ for some } \eta^* \in \xi.}_{\text{Claim 1.4.4}}$$

Thus, by induction on $\xi < \kappa$, we obtain $j_U(\xi) = \xi$ for all $\xi < \kappa$.

(2): Let $\iota : \mathcal{P}^V(\mathcal{P}_\kappa(\lambda)^V) \rightarrow \mathbf{V}$; $x \mapsto \sup(x \cap \kappa)$

For all $\xi < \kappa$, we have

$$\xi = \underbrace{j_U(\xi)}_{(1)} = [const_\xi]_U < \underbrace{[\iota]_U}_{\text{Claim 1.4.4 and (1.5)}} < \underbrace{[const_\kappa]_U}_{\text{Claim 1.4.4 and (1.5)}} = j_U(\kappa).$$

Thus $\kappa \leq [\iota]_U < j(\kappa)$.

(3): We show that $[id_{\mathcal{P}_\kappa(\lambda)^V}]_U = j_U''\lambda$.

For an arbitrary $f \in \mathcal{W}$

$$\begin{aligned} [f]_U \in [id_{\mathcal{P}_\kappa(\lambda)^V}]_U &\Leftrightarrow \{x \in \mathcal{P}_\kappa(\lambda)^V : f(x) \in \underbrace{x}_{= id_{\mathcal{P}_\kappa(\lambda)^V}(x)}\} \in U \\ &\quad \text{by Claim 1.4.4} \\ &\Leftrightarrow \{x \in \mathcal{P}_\kappa(\lambda)^V : f(x) = \underbrace{\xi^*}_{= const_{\xi^*}(x)}\} \in U \text{ for some } \xi^* < \lambda \\ &\quad \text{by Lemma 1.4, (2)} \\ &\Leftrightarrow [f]_U = j_U(\xi^*) \text{ for some } \xi^* < \lambda. \\ &\quad \text{by Claim 1.4.4} \end{aligned} \quad \dashv \text{ (Claim 1.4.5)}$$

It follows that there is $\mathbb{p} \in \mathbb{G}$ such that

$$\mathbb{p} \Vdash_{\mathbb{P}} \text{“there is a } \mathbf{V}\text{-normal ultrafilter on } \mathcal{P}^V(\mathcal{P}_\kappa(\lambda)^V)\text{”}.$$

Since $\mathbb{P} \upharpoonright \mathbb{p} \in \mathcal{P}$ by the assumption on \mathcal{P} , we obtain the desired condition for λ by replacing \mathbb{P} with $\mathbb{P} \upharpoonright \mathbb{p}$. □ (Theorem 1.3)

Note that the proof of Claim 1.4.3 relies on the property of \mathcal{P} that no \mathbb{P} adds any new ω -sequence ground model sets. Note also that the argument using the fact that the well-foundedness of a relation is Δ_1 is irrelevant here since the relation \in_U is not in the ground model.

Thus, the proof of Theorem 1.3 cannot simply be applied to the generic supercompactness by a class of posets \mathcal{P} whose elements might add new ω -sequences of ground model sets.

By Theorem 1.3 we obtain another characterization of generic supercompactness by a \mathcal{P} as in Theorem 1.3:

Corollary 1.5 *Suppose that \mathcal{P} is a class of posets such that no $\mathbb{P} \in \mathcal{P}$ adds any new ω -sequence of ground model sets, and \mathcal{P} is closed with respect to restriction. Then, the following are equivalent:* P-gen-sc-2-0

- (a) κ is generically supercompact by \mathcal{P} .
(b) For any $\lambda \geq \kappa$, there is a $\mathbb{P} \in \mathcal{P}$ such that

$\Vdash_{\mathbb{P}}$ “there is a \mathbf{V} -normal ultrafilter on $\mathcal{P}^{\mathbf{V}}(\mathcal{P}_{\kappa}(\lambda)^{\mathbf{V}})$ ”.

- (c) For any $\lambda \geq \kappa$, there is a $\mathbb{P} \in \mathcal{P}$ such that for any (\mathbf{V}, \mathbb{P}) -generic \mathbb{G} , there are classes $j, M \subseteq \mathbf{V}[\mathbb{G}]$ such that $j : \mathbf{V} \xrightarrow{\cong} M$; $\text{crit}(j) = \kappa$; $j(\kappa) > \lambda$ and $j''\lambda \in M$.

□

2 Rado Conjectures of height $> \omega_1$

For an infinite cardinal μ , a tree $T = \langle T, \leq_T \rangle$ is said to be μ -special if T is the union of μ -many antichains (i.e. subsets whose elements are pairwise incomparable). Note that higher-rado

- (2.1) Any tree of height $< \mu^+$ is μ -special, and any tree of height $> \mu^+$ is not μ -special. x-gen-sc-7

For cardinals μ, κ with $\kappa > \mu^+$, the *Rado Conjecture of height μ^+ with reflection point $< \kappa$* is the principle:

- $\text{RC}(\mu, < \kappa)$: For any tree T , if T is not μ -special, then there is $T' \in [T]^{< \kappa}$ such that T' is not μ -special.

The following is a straight-forward generalization of Lemma 12 in [6]:

Lemma 2.1 *If a tree T is μ -special and \mathbb{P} a $< \mu^+$ -closed poset, then we have $\Vdash_{\mathbb{P}}$ “ T is not μ -special”.* P-gen-sc-3

Proof. By (2.1), we may assume that $ht(T) = \mu^+$. Suppose that $\Vdash_{\mathbb{P}}$ “ T is μ -special”, and let \tilde{f} be a \mathbb{P} -name such that

- (2.2) $\Vdash_{\mathbb{P}}$ “ $\tilde{f} : \check{T} \rightarrow \check{\mu}$ and $\tilde{f}^{-1}''\{\xi\}$ is an antichain in \check{T} for all $\xi < \check{\mu}$ ”.
- x-gen-sc-7-0

We want to prove that T is μ -special (in \mathbf{V}).

By induction on $\alpha < \mu$, we can take $\mathbb{P}_t \in \mathbb{P}$ and $\xi_t \in \mu$ for $t \in T_\alpha$ such that

- (2.3) if $t' \leq_T t$ then $\mathbb{P}_t \leq_{\mathbb{P}} \mathbb{P}_{t'}$; and x-gen-sc-8

- (2.4) $\mathbb{P}_t \Vdash_{\mathbb{P}}$ “ $\tilde{f}(\check{t}) = \check{\xi}_t$ ”.
- x-gen-sc-9

Note that, for each $t \in T$, if $\mathbb{P}_{t'}$, for all $t' \leq_T t$ have been defined according to (2.3) and (2.4), there is $\mathbb{P} \in \mathbb{P}$ with $\mathbb{P} \leq_T \mathbb{P}_{t'}$ for all $t' \leq_T t$ by $< \mu^+$ -closedness of \mathbb{P} . Thus we can choose $\mathbb{P}_t \leq_{\mathbb{P}} \mathbb{P}$ such that it satisfies (2.4).

For $\xi < \mu$, let

$$A_\xi := \{t \in T : \xi_t = \xi\}.$$

Then $T = \bigcup_{\xi < \mu} A_\xi$, and each A_ξ for $\xi < \mu$ is an antichain by (2.2), (2.3), and (2.4).

□ (Lemma 2.1)

Proposition 2.2 *Suppose that $\mu^+ < \kappa$ and κ is a generically supercompact cardinal by $< \mu^+$ -closed posets. Then $\text{RC}(\mu', < \kappa)$ holds for all $\omega \leq \mu' \leq \mu$.* P-gen-sc-4

Proof. Suppose that $\omega \leq \mu' \leq \mu$ and T is not μ' -special. Let $|T| = \lambda$. We want to show that there is a subtree T' of T of cardinality $< \kappa$ which is not μ' -special.

Without loss of generality, we may assume that the underlying set of T is λ . That is, we assume that $T = \langle \lambda, \leq_T \rangle$.

Let \mathbb{P} be a $< \mu^+$ -closed poset, and \mathbb{G} a (\mathbb{V}, \mathbb{P}) -generic set with j , $M \subseteq \mathbb{V}[\mathbb{G}]$ such that

$$(1.1) \quad j : \mathbb{V} \xrightarrow{\cong} M \subseteq \mathbb{V}[\mathbb{G}]; \tag{x-gen-sc-0}$$

$$(1.2) \quad \text{crit}(j) = \kappa, j(\kappa) > \lambda; \text{ and} \tag{x-gen-sc-1}$$

$$(1.3) \quad j''\lambda \in M. \tag{x-gen-sc-2}$$

By $< \mu^+$ -closedness of \mathbb{P} ³⁾ and Lemma 2.1, we have

$$(2.5) \quad \mathbb{V}[\mathbb{G}] \models \text{“}T \text{ is not } \mu'\text{-special”}. \tag{P-gen-sc-5}$$

The tree $j''\lambda = \langle j''\lambda, j''\leq_T \rangle$ is isomorphic to T . Thus we have

$$\mathbb{V}[\mathbb{G}] \models \text{“}j''T \text{ is not } \mu'\text{-special”}.$$

Since the tree $j''T$ is an element of M by Lemma 1.1, it follows that $M \models \text{“}j''T \text{ is not } \mu'\text{-special”}$. Thus, we have

$$M \models \text{“there is a subtree } T' \text{ of } j(T) \text{ of size } < j(\kappa) \text{ which is not } \underbrace{\mu'\text{-special}}_{= j(\mu')}\text{”}.$$

By elementarity, it follows that

$$\mathbb{V} \models \text{“there is a subtree } T' \text{ of } T \text{ of size } < \kappa \text{ which is not } \mu'\text{-special”}.$$

□ (Proposition 2.2)

³⁾Note that $< \mu^+$ -closedness of \mathbb{P} follows from this.

3 Laver-generically supercompact cardinals

The notion of Laver-generically large cardinal was introduced in [2]. The Laver-generically for a class \mathcal{P} of posets, as we define here, is stronger than the one given in [2], and it corresponds to the definition of Laver-genericity for $(\mathcal{P}, \mathcal{P})$ in [3].

Laver-g

A class \mathcal{P} of posets is *iterable* if

- (3.1) \mathcal{P} is closed with respect to forcing equivalence. That is, if $\mathbb{P} \in \mathcal{P}$ and \mathbb{P}' is forcing equivalent to \mathbb{P} , then $\mathbb{P}' \in \mathcal{P}$; x-gen-sc-10
- (3.2) $\mathbb{P} \restriction \mathbb{p} \in \mathcal{P}$ for any $\mathbb{P} \in \mathcal{P}$ and $\mathbb{p} \in \mathbb{P}$; and x-gen-sc-11
- (3.3) if $\mathbb{P} \in \mathcal{P}$ and $\Vdash_{\mathbb{P}} \text{“}\mathbb{Q} \varepsilon \mathcal{P}\text{”}$, then $\mathbb{P} * \mathbb{Q} \in \mathcal{P}$. x-gen-sc-12

For a cardinal κ and an iterable class \mathcal{P} of posets, we call κ a *Laver-generically supercompact for \mathcal{P}* (or *L-g supercompact*, for short) if, for any $\lambda \geq \kappa$ and any $\mathbb{P} \in \mathcal{P}$, there is a \mathbb{P} -name of a poset \mathbb{Q} with $\Vdash_{\mathbb{P}} \text{“}\mathbb{Q} \varepsilon \mathcal{P}\text{”}$ such that, for any $(\mathbb{V}, \mathbb{P} * \mathbb{Q})$ -generic filter \mathbb{H} , there are $M, j \subseteq \mathbb{V}[\mathbb{H}]$ such that

- (3.4) $j : \mathbb{V} \xrightarrow{\simeq} M$, x-gen-sc-13
- (3.5) $\text{crit}(j) = \kappa, j(\kappa) > \lambda$, x-gen-sc-14
- (3.6) $\mathbb{P}, \mathbb{H} \in M$ and x-gen-sc-15
- (3.7) $j''\lambda \in M$. x-gen-sc-16

We shall call j as above a λ *L-g supercompact embedding (with the critical point κ , associated with \mathbb{H} over \mathbb{V})*.

Even in the case that the class of \mathbb{P} of posets consists of $< \mu$ -closed posets, the first-order formulizability of the notion of Laver-generic supercompactness is unknown: An argument like that of Proposition 1.3 cannot help because it apparently cannot create the situation with (3.6).

Thus, at least at the moment, we have to treat a Laver-generic large cardinal merely as a scheme. In each of the concrete instances we encounter, this is no problem since we know exactly how the elementary embeddings j , and inner models M are constructed.

The situation depicted in the following theorem is archetypal for this:

Theorem 3.1 μ^+ is L-g supercompact in the model given in Fact 1.2. More precisely, if κ is a (really) supercompact cardinal, $\mu < \kappa$ a regular uncountable cardinal, and $\mathbb{P}_0 = \text{Col}(\mu, \kappa)$, then, for a $(\mathbb{V}, \mathbb{P}_0)$ -generic \mathbb{G}_0 ,

P-gen-sc-5-0

$\mathbb{V}[\mathbb{G}_0] \models \text{“}\mu^+ \text{ is a L-g supercompact cardinal for } < \mu\text{-closed posets”}$.

The theorem above follows from the corollary (Corollary 3.4) of the next theorem which is a generalization of Proposition 10.20 in Kanamori [5]:

Theorem 3.2 (see Theorem 1.5 in [2]) *Suppose that μ , and λ are regular with $\mu < \lambda$. If \mathbb{P} is a separative poset such that $|\mathbb{P}| = \lambda$, \mathbb{P} is μ -closed, and* *P-gen-sc-5-1*

$$(3.8) \quad \Vdash_{\mathbb{P}} \text{“there is a surjection } \check{\mu} \rightarrow \check{\lambda}\text{”}, \quad \text{col-1}$$

then $\text{ro}(\mathbb{P}) \cong \text{ro}(\text{Col}(\mu, \{\lambda\}))$. □

The following are well-known and easy to prove:

Lemma 3.3 *Let μ be an uncountable regular cardinal. Then* *P-gen-sc-5-1-*

- (1) *For disjoint sets S_0, S_1 , we have $\text{Col}(\mu, S_0 \dot{\cup} S_1) \sim \text{Col}(\mu, S_1) \times \text{Col}(\mu, S_1)$.* *0*
- (2) *If \mathbb{P}_0 and \mathbb{P}_1 are $< \mu$ -closed, then $\mathbb{P}_0 \times \mathbb{P}_1$ is $< \mu$ -closed.*
- (3) *If \mathbb{P}_0 is $< \mu$ -closed and $\Vdash_{\mathbb{P}_0}$ “ \mathbb{P}_1 is $< \mu$ -closed”, then $\mathbb{P}_0 * \mathbb{P}_1$ is $< \mu$ -closed.* □

Corollary 3.4 (Corollary 1.6, (2) in [2]) *For any $< \mu$ -closed poset \mathbb{P} and cardinals ν, λ_0, λ with $|\mathbb{P}| \leq \lambda_0 = (\lambda_0)^{< \mu} < \lambda$, and $\nu \leq \lambda_0$, we have* *P-gen-sc-5-2*

$$\text{Col}(\mu, \lambda \setminus \nu) \underset{\textcircled{1}}{\sim} \text{Col}(\mu, \lambda) \underset{\textcircled{2}}{\sim} \mathbb{P} \times \text{Col}(\mu, \lambda) \underset{\textcircled{3}}{\sim} \mathbb{P} * \text{Col}(\mu, \lambda)^{\mathbb{V}^{\mathbb{P}}}.$$

Proof. ① : Since $|\text{Col}(\mu, \lambda_0 + 2 \setminus \nu)| = |\text{Col}(\mu, \lambda_0 + 2)| = \lambda_0$ and both of the posets add a surjection from μ to λ_0 , we have

$$(3.9) \quad \text{Col}(\mu, \lambda_0 + 2 \setminus \nu) \sim \text{Col}(\mu, \{\lambda_0\}) \sim \text{Col}(\mu, \lambda_0 + 2) \quad \text{x-gen-sc-16-0}$$

by Theorem 3.2. Thus

$$\begin{aligned} \text{Col}(\nu, \lambda \setminus \nu) &\underset{\text{by Lemma 3.3, (1)}}{\sim} \text{Col}(\mu, \lambda_0 + 2 \setminus \nu) \times \text{Col}(\nu, \lambda \setminus \lambda_0 + 2) \\ &\underset{\text{by (3.9)}}{\sim} \text{Col}(\mu, \lambda_0 + 2) \times \text{Col}(\nu, \lambda \setminus \lambda_0 + 2) \underset{\text{by Lemma 3.3, (1)}}{\sim} \text{Col}(\mu, \lambda) \end{aligned}$$

② : By Lemma 3.3, (2) and Theorem 3.2, we have

$$(3.10) \quad \mathbb{P} \times \text{Col}(\mu, \lambda_0 + 1) \sim \text{Col}(\mu, \{\lambda_0\}) \sim \text{Col}(\mu, \lambda_0 + 1). \quad \text{x-gen-sc-16-1}$$

Thus

$$\begin{aligned}
\mathbb{P} \times \text{Col}(\mu, \lambda) &\sim \mathbb{P} \times \underbrace{\text{Col}(\mu, \lambda_0 + 1)}_{\text{by Lemma 3.3, (1)}} \times \text{Col}(\mu, \lambda \setminus \lambda_0 + 1) \\
&\sim \underbrace{\text{Col}(\mu, \lambda_0 + 1)}_{\text{by (3.10)}} \times \text{Col}(\mu, \lambda \setminus \lambda_0 + 1) \sim \underbrace{\text{Col}(\mu, \lambda)}_{\text{by Lemma 3.3, (1)}}.
\end{aligned}$$

③ : follows from the $< \mu$ -closedness of \mathbb{P} .

□ (Corollary 3.4)

Proof of Theorem 3.1: Suppose that $\mathbb{V}[\mathbb{G}_0] \models \mathbb{P}$ is $< \mu$ -closed.

Let \mathbb{P} be a \mathbb{P}_0 -name of \mathbb{P} , and let $\lambda \geq \kappa$ be arbitrary. Let λ_0 be such that $|\mathbb{P}_0 * \mathbb{P}|, \lambda \leq \lambda_0$ and $(\lambda_0)^{< \mu} = \lambda_0$. Without loss of generality, we may assume that the underlying set of $\mathbb{P}_0 * \mathbb{P}$ is a cardinal $\leq \lambda_0$.

Let $j : \mathbb{V} \xrightarrow{\cong} M \subseteq \mathbb{V}$ be a λ_0 -supercompact embedding for κ . Note that $\lambda_0 < j(\kappa) \leq j(\lambda_0)$.⁴⁾

For an arbitrary $(\mathbb{V}[\mathbb{G}_0], \mathbb{P})$ -generic set \mathbb{G} , let \mathbb{H}_0 be a $(\mathbb{V}[\mathbb{G}_0][\mathbb{G}], \underbrace{\text{Col}(\mu, j(\lambda_0))^{\mathbb{V}[\mathbb{G}_0][\mathbb{G}]}}_{(=\text{Col}(\mu, j(\lambda_0))^{\mathbb{V}})})$ -generic set. In $\mathbb{V}[\mathbb{G}_0]$, Let $\mathbb{Q} = \mathbb{P} * \text{Col}(\mu, j(\lambda_0))^{\mathbb{V}[\mathbb{G}_0]^{\mathbb{P}}}$. $\mathbb{G} * \mathbb{H}_0$ is then a $(\mathbb{V}[\mathbb{G}_0], \mathbb{Q})$ -generic set.

By Corollary 3.4, there is a $(\mathbb{V}, \text{Col}(\mu, j(\lambda_0)))$ -generic set \mathbb{H} such that $j''\mathbb{G}_0 = \mathbb{G}_0 \subseteq \mathbb{H}$ and $\mathbb{V}[\mathbb{H}] = \mathbb{V}[\mathbb{G}_0][\mathbb{G}][\mathbb{H}_0]$.

Let

$$(3.11) \quad \tilde{j} : \mathbb{V}[\mathbb{G}_0] \xrightarrow{\cong} M[\mathbb{H}] \subseteq \mathbb{V}[\mathbb{H}]; \quad \underset{\sim}{a}^{\mathbb{G}_0} \mapsto j(\underset{\sim}{a})^{\mathbb{H}}.$$

x-gen-sc-16-3

Since $\mathbb{P}_0 \in M$ by the closedness of M (as a target model of λ_0 -supercompact embedding for κ) and Lemma 1.1, we have $\mathbb{P}_0 \in M[\mathbb{G}_0]$. Hence we also have $\mathbb{G}_0 \in M[\mathbb{G}_0]$. By the closedness of $M[\mathbb{G}_0]$ ($\tilde{j}''\lambda_0 = j''\lambda_0 \in M \subseteq M[\mathbb{G}_0]$), we have $\mathbb{P} \in M[\mathbb{G}_0]$ and $\text{Col}(\mu, j(\lambda_0))^{\mathbb{V}[\mathbb{G}_0][\mathbb{G}]} = \text{Col}(\mu, j(\lambda_0))^{\mathbb{V}} \in M \subseteq M[\mathbb{H}]$.

Thus we have $\mathbb{G}, \mathbb{H}_0 \in M[\mathbb{H}]$ and $M[\mathbb{H}] = M[\mathbb{G}_0][\mathbb{G}][\mathbb{H}_0]$. It follows that \tilde{j} is a λ -L-g supercompact embedding with the critical point κ , associated with $\mathbb{G} * \mathbb{H}_0$ over $\mathbb{V}[\mathbb{G}_0]$. □ (Theorem 3.1)

In [2], it is proved that a/the L-g supercompact cardinal for $< \aleph_1$ -closed poset is \aleph_2 (if it exists). The proof can be generalized to show that a L-g supercompact cardinal for $< \aleph_n$ -closed poset is \aleph_{n+1} for each $n \in \omega$.

In general we have the following. Let us first see the situation with an arbitrary class \mathcal{P} of posets:

Lemma 3.5 *If κ is generically supercompact by a class \mathcal{P} of posets, and κ is a limit cardinal, then κ is a Mahlo cardinal.*⁵⁾ P-gen-sc-5-3

⁴⁾ $j(\lambda_0)$ is going to play the role of λ in Corollary 3.4.

⁵⁾ Actually, for the following proof, it is enough to assume that κ is generically measurable.

Proof. We prove first that κ is a regular cardinal. Suppose not. Then there is a strictly increasing sequence $\langle \alpha_\xi : \xi < \delta \rangle$ of ordinals such that $\delta < \kappa$ and $\lim_{\xi < \delta} \alpha_\xi = \kappa$.

Let $\mathbb{P} \in \mathcal{P}$ be such that, for a (\mathbb{V}, \mathbb{P}) -generic \mathbb{G} , and $j, H \subseteq \mathbb{V}[\mathbb{G}]$,

$$(3.12) \quad j : \mathbb{V} \xrightarrow{\cong} M, \text{ and}$$

$$(3.13) \quad \text{crit}(j) = \kappa.$$

x-gen-sc-16-4

x-gen-sc-16-5

By the elementarity (3.12) and (3.13), we have $j(\langle \alpha_\xi : \xi < \delta \rangle) = \langle \alpha_\xi : \xi < \delta \rangle$. Hence, again by elementarity, $\mathbb{V}[\mathbb{G}] \models j(\kappa) = \lim_{\xi < \delta} \alpha_\xi = \kappa$. This is a contradiction to (3.13).

Suppose now that $C \subseteq \kappa$ is a club. Then, for $\mathbb{P}, \mathbb{G}, j, M$ as above, we have $M \models$ “ $j(C)$ is a club in $j(\kappa)$ ” and $M \ni j(c) \cap \kappa = C$. It follows that $M \models \kappa \in j(C)$. Since $M \models$ “ κ is regular”, we have $M \models$ “there is a regular cardinal $\in j(C)$ ”. By elementarity, it follows that $\mathbb{V} \models$ “there is a regular cardinal $\in C$ ”. \square (Lemma 3.5)

Lemma 3.6 (1) *Suppose that κ is a generically measurable cardinal by a $< \mu$ -closed poset. If κ is a successor cardinal then $\mu < \kappa$.*

P-gen-sc-5-4

(2) *Suppose that κ is a L -g supercompact cardinal for a class \mathcal{P} of posets with $\text{Col}(\mu, \{\mu^+\}) \in \mathcal{P}$ for $\mu < \kappa$. Then we have $\kappa = \mu^+$.*

(3) *Suppose that κ is a L -g supercompact cardinal for $< \mu$ -closed posets. If κ is a successor cardinal, then $\kappa = \mu^+$.*

Proof. (1): Suppose that $\kappa = (\kappa_0)^+$. Toward a contradiction, assume $\mu \geq \kappa$. Let poset \mathbb{P} be a $< \mu$ -closed poset such that, for (\mathbb{V}, \mathbb{P}) -generic \mathbb{G} and $j, M \subseteq \mathbb{V}[\mathbb{G}]$, we have $j : \mathbb{V} \xrightarrow{\cong} M \subseteq \mathbb{V}[\mathbb{G}]$ and $\text{crit}(j) = \kappa$.

Then

$$(3.14) \quad M \models \underbrace{(j(\kappa_0))^+}_{= \kappa_0} = j(\kappa)$$

x-gen-sc-16-6

by elementarity. On the other hand, $\mathbb{V}[\mathbb{G}] \models$ “ κ is a cardinal” by the $< \mu$ -closedness of \mathbb{P} . Hence $M \models$ “ κ is a cardinal” and $M \models \kappa_0 < \kappa < j(\kappa)$. This is a contradiction to (3.14).

(2): Suppose that $\kappa > \mu^+$. Let $\mathbb{P} = \text{Col}(\mu, \{\mu^+\})$ and let \mathbb{Q} be such that $\mathbb{P} \leq \mathbb{Q}$, \mathbb{Q} is $< \mu$ -closed, and, for (\mathbb{V}, \mathbb{Q}) -generic \mathbb{H} there are $j, M \subseteq \mathbb{V}[\mathbb{H}]$ with $j : \mathbb{V} \xrightarrow{\cong} M$, $\kappa = \text{crit}(j)$, and $\mathbb{P}, \mathbb{H} \in M$.

By elementarity, we have

Here, a cardinal κ is said to be *generically mesearable* by \mathcal{P} , if there is a $\mathbb{P} \in \mathcal{P}$ with (\mathbb{V}, \mathbb{P}) -generic $\mathbb{G}, j, M \in \mathbb{V}[\mathbb{G}]$ such that $j : \mathbb{V} \xrightarrow{\cong} M \subseteq \mathbb{V}[\mathbb{G}]$; and $\text{crit}(j) = \kappa$.

$$M \models \underbrace{j((\mu^+)^V)}_{=(\mu^+)^V} \text{ is the successor cardinal of } \underbrace{j(\mu)}_{=\mu}.$$

However, $\mathbb{H} \cap \mathbb{P} (\in M)$ collapses $(\mu^+)^V$ to an ordinal of cardinality μ . This is a contradiction.

(3): follows from (1) and (2).

□ (Lemma 3.6)

Problem 3.7 *Is it consistent that for some regular uncountable μ , there is a limit cardinal κ which is L - g supercompact for $< \mu$ -closed posets?*

4 Takeuti's Axioms of Reflection

In the article [10] published in “数学セミナー”, late Professor Gaishi Takeuti mentioned the following strengthening of an axiom he once formulated in [9]:

takeuti

(4.1) **Reflection Axiom** ([9]): For (any?) ordinal $\alpha_0 > \omega_1$ and $A \subseteq \mathcal{P}(\omega)$, there is a transitive set M^* such that (1) $\alpha_0 \in M^*$, (2) $\mathcal{P}(\omega) \notin M^*$, and (3) $\langle M^*, A \cap M^*, \alpha_0, \in, \alpha \rangle_{\alpha \in \omega_1} \equiv \langle V, A, \alpha_0, \in, \alpha \rangle_{\alpha \in \omega_1}$.

x-gen-sc-21

It is not clear in which axiom system of set theory this axiom should be formalized. Surely it cannot be formalized in ZFC since if there were a formalization in ZFC then we would obtain a contradiction because of Tarski's Theorem of Undefinability of Truth. It may be formulated as a statement in BG as a statement about the existence of certain closed unbounded classes of ordinals. Takeuti neither discusses about the consistency of this axiom. In [7], Takeuti's former student Paul E. Cohen discusses about the consistency and characterization of some related axioms which however seem to be much weaker than the Reflection Axiom (4.1).

In spite of this inaccuracy, the axiom expresses an interesting condition about the continuum: We can interpret the axiom as saying that the continuum cannot be captured by all transitive models of an arbitrary height however strong absoluteness they should possess. The axiom thus can be considered as a statement about certain aspects of richness of the continuum.

In the following, we try to extract a first-order formalizable part of the axiom and show that an extremely “large” Laver-generically large cardinal implies this part of the axiom.

The idea of the extract is that we replace the elementary equivalence mentioned in (4.1) with a collection of statements similar to Lévy-Montague Reflection Theorem. While this makes our axiom possibly much weaker than Takeuti's original version, we add a detail which says that there are many generic reals over M .

(4.2) **(First-order (ZFC) variant of Takeuti's Reflection Axiom)** For x-gen-sc-22
each formula $\varphi = \varphi(x_0, \dots, x_{\ell-1})$ in the language $\mathcal{L}_{\varepsilon, \underline{A}} = \{\underline{A}, \varepsilon\}$, we have
that for any $A \subseteq \mathcal{P}(\omega)$, $\delta < 2^{\aleph_0}$, $\alpha \in \text{On} \setminus 2^{\aleph_0}$ and any c.c.c. poset \mathbb{P} of
size $\leq 2^{\aleph_0}$, there are $\alpha^* \in \text{On} \setminus \alpha$ and a transitive set M^* such that

- (1) $\alpha^* \in M^*$;
- (2) there are poset $\mathbb{P}' \in M^*$ with $\mathbb{P} \cong \mathbb{P}'$, and (M^*, \mathbb{P}') -generic $\mathbb{G} (\in \mathbf{V})$;
and
- (3) for all $\alpha_0, \dots, \alpha_{\ell-1} \in \delta \cup \{2^{\aleph_0}, \alpha^*\}$, we have

$$\langle M^*, A \cap M^*, \varepsilon \rangle \models \varphi[\alpha_0, \dots, \alpha_{\ell-1}] \Leftrightarrow \langle \mathbf{V}, A, \varepsilon \rangle \models \varphi[\alpha_0, \dots, \alpha_{\ell-1}].$$

The axiom can be shown to be consistent under a large cardinal property of extreme consistency strength.

Theorem 4.1 *If there is a L -g super $I2$ -cardinal then the Reflection Axiom (4.2) holds.* P-gen-sc-7 □

At the moment, we do not know if this extremely strong variant of Laver-genericity is really necessary to show the consistency of the Reflection Axiom (4.2).

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