Workshop on Galois point and related topics (2017/7/16)

On birational transformations belonging to Galois points, II

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References

- Dolgachev, Iskovskikh, " Finite Subgroups of the Plane Cremona Group" Algebra, Arithmetic and Geometry, Vol I, Progress in Math. 269.
- Bayle, Beauville, " Birational involutions of $\mathbb{P}^2$ " Asian J. Math. 4 (2000) 11-17.
- Beauville, Blanc, " On Cremona transformations of prime order" C.R.Acad.Sci. Paris, Ser I 339 (2004) 257-259.

"This paper completes the classic and modern results on classification of conjugacy classes of finite subgroups of the group of birational automorphisms of the complex projective plane."

Introduction

- $k = \mathbb{C}$
- $C \subset \mathbb{P}^2$: irred. plane curve of degree d (≥ 4)
- $P \in \mathbb{P}^2$
- $\pi_P : C \dashrightarrow \mathbb{P}^1$: the projection with the center P.
- $\pi_P^* : k(C) \hookleftarrow \pi_P^*(k(\mathbb{P}^1))$ i.e., $\pi_P^* : k(C)/K_P$

$$\pi_P^*(k(\mathbb{P}^1)) = K_P$$

Def. 1.

P : Galois point for C $\overset{\text{def}}{\Longleftrightarrow}$ $\pi_P^* : k(C)/K_P$ is a Galois extension

Suppose that P is a Galois pt for C.

- $G_P := \mathrm{Gal}(k(C)/K_P)$: Galois group at P
- $G_P \ni \sigma$

  $\Rightarrow$ σ induces a birational transformation of C.

Def. 2 We call σ a birational transformation belonging to Galois point P.

Problem Study the properties of σ !!

In particular, we want to know when this birational transformation will become a restriction of a projective (or Cremona) transformation of the ambient space $\mathbb{P}^2$.

i.e., $\sigma \in G_P$

$\Rightarrow$ (i) $\sigma \in PGL(3, k)$?

(ii) $\sigma \in \mathrm{Bir}(\mathbb{P}^2)$?

(iii) neither (i) nor (ii) ?

In general, this problem seems to be a kind of difficult.

Today

$$\boxed{\sigma \in \mathrm{Bir}(\mathbb{P}^2)}$$

Example 1

C : quartic with simple cusp of multiplicity 3.

$\Rightarrow$ C is projectively equivalent to

(a) $X^4 - X^3Y + Y^3Z = 0$

(b) $X^4 - Y^3Z = 0$

Case (b) $P = (0:1:0)$ is a Galois point, $G_P \cong \mathbb{Z}/3\mathbb{Z}$

$G_P = \langle \sigma \rangle$

$\Rightarrow$ σ is a restriction of a projective transformation

$$\begin{pmatrix} \omega & & \\ & 1 & \\ & & \omega \end{pmatrix}.$$

Case (a) $P_1 = (1:1:0)$, $P_2 = (8:-16:3)$ are Galois points $G_{P_i} \cong \mathbb{Z}/3\mathbb{Z}$ $(i=1,2)$

$G_{P_1} = \langle \sigma_1 \rangle$

$\Rightarrow$ σ_1 is a restriction of a Cremona transformation

$$\boxed{\sigma_1(X:Y:Z) = (XY : Y((\omega-1)X+\omega Y) : Z((\omega-1)X+\omega Y))}$$

$G_{P_2} = \langle \sigma_2 \rangle$

$\Rightarrow$ $\sigma_2 = ??$

In fact,

$$A := \begin{pmatrix} 16 & -8 & 0 \\ 0 & -16 & 0 \\ 4 & -1 & -16 \end{pmatrix}$$

$$\Rightarrow \quad A(C) = C \quad \text{and} \quad A(P_1) = P_2$$

Example 2

$$C : g_2(X,Y)^m + f_m(X,Y)Z^m + Z^{2m} = 0$$

$P = (0:0:1)$ is an outer Galois point

$G_P \cong D_{2m}$

$D_{2m} = \langle \sigma, \tau \rangle$

$$\Rightarrow \boxed{\begin{array}{l} \sigma(X:Y:Z) = (XZ : YZ : g_2(X,Y)) \\ \tau(X:Y:Z) = (X : Y : \zeta Z) \quad (\zeta^m = 1) \\ \qquad \Leftrightarrow \begin{pmatrix} 1 & & \\ & 1 & \\ & & \zeta \end{pmatrix} \end{array}}$$

de Jonquières subgroup $dJ(2)$

$$(x, y) \longmapsto \left(\frac{ax+b}{cx+d}, \frac{r_1(x)y + r_2(x)}{r_3(x)y + r_4(x)} \right)$$

$a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$

$r_i(x) \in \mathbb{C}(x)$

$r_1(x)r_4(x) - r_2(x)r_3(x) \neq 0.$

these transformations form a subgroup of $\mathrm{Bir}(\mathbb{P}^2)$ called a de Jonquières subgroup and denoteed by $dJ(2)$.

$$dJ(2) \cong PGL(2, \mathbb{C}(x)) \rtimes PGL(2, \mathbb{C})$$

A de Jonquières map is a plane Cremona map $\mathcal{J}$ of degree $d \geq 2$ satisfying any one of the following equivalent conditions:

- $\mathcal{J}$ has homaloidal type $(d; d-1, 1^{2d-2})$
- $\exists$ a point $O \in \mathbb{P}^2$ s.t. the restriction of $\mathcal{J}$ to a general line passing through O maps it birationally to a line passing through O.
- Up to projective coordinate change $\mathcal{J}$ is defined by d-forms $\{ g x_0, g x_1, f \}$ s.t. $f, g \in k[x_0, x_1, x_2]$ are relatively prime x_2-monoids one of which at least has degree 1 in x_2

Theorem (?)

P : Galois pt for C

$G_P \ni \sigma$

if $\sigma \in \mathrm{Bir}(\mathbb{P}^2)$, then σ is a de Jonquières map.

Classification of conjugacy classes of elements of prime order p in $\mathrm{Bir}(\mathbb{P}^2)$

$p=2$
- linear transformation
- de Jonquières involution
- Bertini involution
- Geiser involution

$p=3$
- linear transformation
- automorphism of special Del Pezzo surface of degree 3
- 〃 degree 1

Example 1 (again)

(a) $X^4 - X^3Y + Y^3Z = 0$, $P = (1:1:0)$, $G_P \cong \mathbb{Z}/3\mathbb{Z}$.

$$\sigma(X:Y:Z) = (XY : Y((\omega-1)X+\omega Y) : Z((\omega-1)X+\omega Y))$$

Theorem σ is conjugate to a linear transformation.

proof $C : (Y+X)^3 Z - X^3 Y = 0$, $P = (1:0:0)$, $\pi_P(X:Y:Z) = (Y:Z)$

put $x = X/Z$, $y = Y/Z$

$\Rightarrow (x+y)^3 - x^3 y = 0$

$$\begin{array}{c} k(C) = k(x,y) \\ | \\ k(y) \end{array}$$

$\Rightarrow \sigma(x) = \dfrac{yx}{(\omega-1)x + \omega y}$ $\left(\Leftrightarrow \underset{A}{\underset{\shortparallel}{\begin{pmatrix} y & 0 \\ \omega-1 & \omega y \end{pmatrix}}} \in PGL(2, k(y)) \right)$

$\Rightarrow$ put $P = \begin{pmatrix} -y & 0 \\ 1 & 1 \end{pmatrix}$, we have $P^{-1} A P = \begin{pmatrix} y & 0 \\ 0 & \omega y \end{pmatrix}$

$\Rightarrow \tilde{\sigma}(x) = \dfrac{yx}{\omega y} = \omega^2 x$ $\therefore$ $\tilde{\sigma}$ is given by $\begin{pmatrix} \omega^2 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ //