EGOROFF'S THEOREM FOR FAMILIES OF FUNCTIONS WITH CONTINUOUS PARAMETER

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1. Introduction

We recall Egoroff's theorem as follows.

Theorem 1.1. *Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set such that $\mu(E) < \infty$, where μ denotes the Lebesgue measure. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on E. Suppose that $\{f_n(x)\}_{n=1}^{\infty}$ is convergent for all $x \in E$. Let $\epsilon > 0$. Then there exists a subset $F \subset E$ such that $\mu(E \setminus F) < \epsilon$ and $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent on F.*

Next, we state a continuous parameter version of Theorem 1.1.

Theorem 1.2. *Let E be a measurable subset of $\mathbb{R}^n$. Let $\mu(E) < \infty$. Let f_h, $0 < h \leq 1$, be continuous on E. Suppose that $\lim_{h\to 0} f_h(x)$ exists for all $x \in E$. Let $\epsilon > 0$. Then there exist a subset F of E and a continuous function f on F such that $\mu(E \setminus F) < \epsilon$ and $\lim_{h\to 0} f_h(x) = f(x)$ uniformly on F.*

See [1, p. 60] and [2, p. 7, p. 93] for Theorem 1.1 and Theorem 1.2, respectively.

2. Proof of Theorem 1.1

Define $f(x) = \lim_{n\to\infty} f_n(x)$, $x \in E$. For $n, k, m \in \mathbb{Z} \cap [1, \infty)$ let

$$E_{n,k} = \left\{ x \in E : |f_n(x) - f(x)| < \frac{1}{k} \right\}, \qquad F_{m,k} = \bigcap_{n \geq m} E_{n,k}.$$

Then $F_{m,k} \subset F_{m+1,k}$ and $F_{m,k} \to E$ as $m \to \infty$. So we can find a positive integer m_k such that $\mu(E \setminus F_{m_k,k}) < 2^{-k}\epsilon$. Let $F = \cap_{k=1}^{\infty} F_{m_k,k}$. Then

$$\mu(E \setminus F) \leq \sum_{k=1}^{\infty} \mu(E \setminus F_{m_k,k}) < \sum_{k=1}^{\infty} 2^{-k}\epsilon = \epsilon.$$

Now we prove that $f_n \to f$ uniformly on F. For any $\tau > 0$, take a positive integer k_0 with $1/k_0 < \tau$. Let $x \in F$. Then $x \in F_{m_{k_0},k_0}$. So $x \in \cap_{n \geq m_{k_0}} E_{n,k_0}$ and hence if $n \geq m_{k_0}$, we have $|f(x) - f_n(x)| < 1/k_0 < \tau$. This implies the uniform convergence, completing the proof.

2020 *Mathematics Subject Classification.* 28A20.

Key Words and Phrases. Egoroff's theorem, uniform convergence.

The author is partly supported by Grant-in-Aid for Scientific Research (C) No. 20K03651, Japan Society for the Promotion of Science.

3. Proof of Theorem 1.2

Let $\{h_n\}$ be a sequence in $(0,1]$ such that $h_n \to 0$. By Theorem 1.1 there exists a measurable set F_0 in E such that $\mu(E \setminus F_0) < \epsilon$ and $f_{h_n} \to f$ uniformly on F_0. Since each f_{h_n} is continuous on F_0, f is continuous on F_0 as a limit function of uniformly convergent continuous functions. Let

$$E_{h,k} = \left\{ x \in F_0 : |f_h(x) - f(x)| \le \frac{1}{k} \right\}.$$

Then $E_{h,k}$ is a closed set in F_0. For positive integers k, m, define

$$F_{m,k} = \bigcap_{h \in (0,1/m)} E_{h,k}.$$

Then $F_{m,k}$ is closed in F_0 and $F_{m,k} \subset F_{m+1,k}$, $F_{m,k} \to F_0$ as $m \to \infty$ for every fixed k. For each k we chose m_k so that $\mu(F_0 \setminus F_{m_k,k}) < 2^{-k}\epsilon$. Let

$$F = \bigcap_{k=1}^{\infty} F_{m_k,k}.$$

Then F is closed in F_0 and

$$\mu(F_0 \setminus F) \le \sum_{k=1}^{\infty} \mu(F_0 \setminus F_{m_k,k}) < \sum_{k=1}^{\infty} 2^{-k}\epsilon = \epsilon.$$

Thus $\mu(E \setminus F) \le \mu(E \setminus F_0) + \mu(F_0 \setminus F) < 2\epsilon$.

To complete the proof of Theorem 1.2, we show that $f_h \to f$ uniformly on F. Given $\tau > 0$, we take $k_0 \in \mathbb{Z} \cap [1, \infty)$ such that $k_0^{-1} < \tau$. Let $h \in (0, m_{k_0}^{-1})$. Then, if $x \in F$, we have $x \in F_{m_{k_0},k_0}$, which implies that $x \in E_{h,k_0}$ and so $|f_h(x) - f(x)| \le 1/k_0 < \tau$. Thus we have the uniform convergence claimed.

References

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