ON DECOMPOSITION OF NEIGHBORHOOD OF A CIRCULAR CONE RELATED TO PRINCIPAL CURVATURES

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Abstract. We give an alternative proof of a result on the uniform overlap of the algebraic sums of the sets arising from a decomposition of a neighborhood of a circular cone in $\mathbb{R}^3$. It is known that the uniform overlap result can be applied to make a unified approach for the proofs of a theorem on the maximal Bochner-Riesz operator on $\mathbb{R}^2$ and a theorem on the maximal spherical means on $\mathbb{R}^2$.

1. Introduction

Let

$$T_R^\lambda f(x) = \int_{|\xi|<R} \hat{f}(\xi)(1 - |R^{-1}\xi|^2)_+^\lambda e^{2\pi i\langle x,\xi\rangle}\, d\xi$$

be the Bochner-Riesz operator of order λ on $\mathbb{R}^2$, where

$$\hat{f}(\xi) = \int_{\mathbb{R}^2} f(x)e^{-2\pi i\langle x,\xi\rangle}\, dx$$

is the Fourier transform and $\langle x, \xi\rangle = x_1\xi_1 + x_2\xi_2$, $x = (x_1, x_2)$, $\xi = (\xi_1, \xi_2)$, denotes the inner product. Let

$$T_*^\lambda f(x) = \sup_{R>0} |T_R^\lambda f(x)|$$

be the maximal Bochner-Riesz operator.

The following is known ([2]).

Theorem A. *If $\lambda > 0$, T_*^λ is bounded on $L^4(\mathbb{R}^2)$:*

$$\|T_*^\lambda f\|_4 \le C_\lambda \|f\|_4.$$

The L^4 boundedness for T_1^λ is shown in [4]. See also [3] for related results.

Let

$$S_t f(x) = \int_{S^1} f(x - t\theta)\, d\sigma(\theta)$$

be the spherical mean on $\mathbb{R}^2$, where $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ is the unit circle and σ denotes the Lebesgue arc length measure on S^1, and let

$$S_* f(x) = \sup_{t>0} |S_t f(x)|$$

be the maximal spherical mean.

The following result is known ([1]).

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Theorem B. *The maximal operator S_* is bounded on $L^p(\mathbb{R}^2)$ for $p > 2$:*

$$\|S_*f\|_p \leq C_p\|f\|_p.$$

We refer to [7] for a result analogous to Theorem B in $\mathbb{R}^n$, $n \geq 3$.

In [6, Chap. 2], a unified approach to the proofs of Theorems A and B are presented. In the arguments, a geometric overlap theorem concerning a circular cone in $\mathbb{R}^3$ plays a crucial role.

Let τ be a fixed large positive number. In this note we assume that $\tau > 10^6$. Set

$$\Gamma_\mu = \{\xi \in \mathbb{R}^2 \setminus \{0\} : \mu\tau^{-1/2} \leq \arg\xi < (\mu+1)\tau^{-1/2}\}, \tag{1.1}$$

where $\mu \in \mathbb{R}$ with $|\mu| \leq \tau^{1/2}(\pi/8) - 1$. Let

$$\mathcal{N} = \{\mu \in \mathbb{Z} : |\mu| \leq \tau^{1/2}(\pi/8) - 1\}, \tag{1.2}$$

where $\mathbb{Z}$ denotes the set of integers. Let $J = [\alpha, \beta] \subset [1, 2]$. We assume that $|J| = \beta - \alpha \leq \tau^{-1/2}$. For $\mu \in \mathcal{N}$, let

$$U_{\mu,J} = \{(\xi, |\xi|) \in \mathbb{R}^2 \times \mathbb{R} : \xi \in \Gamma_\mu, |\xi| \in J\} \tag{1.3}$$

and

$$\begin{aligned} u_{\mu,J} &= \{(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R} : |\eta - |\xi|| \leq \tau^{-1}, (\xi, |\xi|) \in U_{\mu,J}\} \\ &= \cup\{\{\xi\} \times [|\xi| - \tau^{-1}, |\xi| + \tau^{-1}] : (\xi, |\xi|) \in U_{\mu,J}\}. \end{aligned} \tag{1.4}$$

We note that $U_{\mu,J} \subset u_{\mu,J}$.

We have the following result.

Theorem 1.1. *Let $[\alpha_1, \beta_1], [\alpha_2, \beta_2] \subset [1, 2]$. Set $J_i = [\alpha_i, \beta_i]$ and suppose that $|J_i| \leq \tau^{-1/2}$, $i = 1, 2$. Let u_{μ,J_1}, u_{μ,J_2} be defined as in (1.4) with J_1, J_2 in place of J. Then there exists a constant C independent of τ and the intervals J_1, J_2 such that*

$$I = \sum_{(\mu,\nu)\in\mathcal{N}^2} \chi_{u_{\mu,J_1}+u_{\nu,J_2}} \leq C,$$

where $\mathcal{N}$ is as in (1.2), $u_{\mu,J_1} + u_{\nu,J_2} = \{(\xi, \eta) + (\xi', \eta') : (\xi, \eta) \in u_{\mu,J_1}, (\xi', \eta') \in u_{\nu,J_2}\}$ and χ_E denotes the characteristic function of a set E.

A more general result is stated in [6, (2.4.23), p. 87], and a variant of it is also given ([6, Lemma 2.4.5]). As in [6], in the arguments for the unified approach to the proofs of Theorems A and B applying half wave operators, Theorem 1.1 can be used to prove an orthogonality result through Fourier transform estimates, which is crucial in the arguments, since the orthogonality result leads to an effective application of sharp $L^2(\mathbb{R}^3)$-$L^2(\mathbb{R}^2)$ estimates for the Kakeya maximal functions defined by using rays on a light cone.

The result [6, (2.4.23)] includes Theorem 1.1 as a special case and the proof is given in [6, pp. 87-88]. In this note we shall give an alternative proof of Theorem 1.1. Also, we shall consider a variant of Theorem 1.1 (Theorem 4.2 below), which is a special case of [6, Lemma 2.4.5] and which is related to Theorem 1.1 as [6, Lemma 2.4.5] is related to [6, (2.4.23)]. We shall give the proof of Theorem 4.2 by applying Theorem 1.1.

A version of Theorem 1.1 on $\mathbb{R}^2$, where a cone is replaced with a circle, is given in [5] and it follows from Theorem 1.1 as a corollary.

Remark 1.2. If the condition on μ in the definition of $\mathcal{N}$ in (1.2) is replaced by $|\mu| \le \tau^{1/2}a - 1$ with some $\pi/2 < a < \pi$, then a result analogous to Theorem 1.1 will not exist. This can be seen by letting $J_1 = J_2 =: J$, $b \in J$ and considering (μ, ν) satisfying $(\xi, |\xi|) \in U_{\mu,J}$ and $(-\xi, |\xi|) \in U_{\nu,J}$ for some ξ such that $|\xi| = b$ and $\arg \xi$ is sufficiently close to $\pi/2$. For such (μ, ν) we have $(0, 2b) \in U_{\mu,J} + U_{\nu,J}$ and we note that the cardinality of the family of such (μ, ν) increases with τ unlimitedly.

Let

$$\mathcal{I} = \left\{ a : a = \frac{\mu + \nu}{2}, \mu, \nu \in \mathcal{N} \right\}. \tag{1.5}$$

Set $\mathbb{Z}^* = \{k/2 : k \in \mathbb{Z}\}$. Then we note that $\mathcal{I}$ is a subset of $\mathbb{Z}^*$ and if $a \in \mathcal{I}$, then $-(\pi/8)\tau^{1/2} + 1 \le a \le (\pi/8)\tau^{1/2} - 1$. To prove Theorem 1.1, we write

$$I = \sum_{(\mu,\nu)\in\mathcal{N}^2} \chi_{u_{\mu,J_1}+u_{\nu,J_2}} = \sum_{a\in\mathcal{I}} \sum_{(\mu,\nu)\in k_0^{-1}(\{a\})} \chi_{u_{\mu,J_1}+u_{\nu,J_2}},$$

where the surjection $k_0 : \mathcal{N}^2 \to \mathcal{I}$ is defined by

$$k_0(\mu, \nu) = \frac{\mu + \nu}{2}.$$

Let $\mathcal{I}' \subset \mathcal{I}$ be such that if $a, a' \in \mathcal{I}'$ and $a \ne a'$, then $|a - a'| \ge C_0$, where C_0 is sufficiently large. To prove $I \le C$, it suffices to show that

$$\sum_{a\in\mathcal{I}'} \sum_{(\mu,\nu)\in k_0^{-1}(\{a\})} \chi_{u_{\mu,J_1}+u_{\nu,J_2}} \le C', \tag{1.6}$$

by considering a suitable partition of $\mathcal{I}$.

In the proof of (1.6), one result we apply is the following.

Lemma 1.3. *Fix $a \in \mathcal{I}$. Let $\frac{\mu+\nu}{2} = a$, $\mu, \nu \in \mathcal{N}$ and*

$$E_a(\mu, \nu) = u_{\mu,J_1} + u_{\nu,J_2}.$$

Then $\{E_a(\mu,\nu)\}_{(\mu,\nu)\in k_0^{-1}(\{a\})}$ is finitely overlapping uniformly in $a \in \mathcal{I}$.

Let

$$\widetilde{J_1} = [\alpha_1 - \tau^{-1}, \beta_1 + \tau^{-1}], \quad \widetilde{J_2} = [\alpha_2 - \tau^{-1}, \beta_2 + \tau^{-1}].$$

Let $\eta \in \widetilde{J_1} + \widetilde{J_2} = [\alpha_1 + \alpha_2 - 2\tau^{-1}, \beta_1 + \beta_2 + 2\tau^{-1}]$ and define

$$E_a^\eta(\mu, \nu) = \left\{ \xi \in \mathbb{R}^2 : (\xi, \eta) \in E_a(\mu, \nu) \right\}, \quad (\mu, \nu) \in k_0^{-1}(\{a\}).$$

Then the following result implies Lemma 1.3.

Lemma 1.4. *Fix $a \in \mathcal{I}$. Let $\eta \in \widetilde{J_1} + \widetilde{J_2}$. Then $\{E_a^\eta(\mu,\nu)\}_{(\mu,\nu)\in k_0^{-1}(\{a\})}$ is finitely overlapping uniformly in a and η.*

To prove Lemma 1.4, we observe that

$$E_a^\eta(\mu, \nu) = \bigcup \left\{ u_{\mu,J_1}^{\eta'} + u_{\nu,J_2}^{\eta''} : \eta' + \eta'' = \eta, \eta' \in \widetilde{J_1}, \eta'' \in \widetilde{J_2} \right\},$$

where $u_{\mu,J_1}^{\eta'}$ and $u_{\nu,J_2}^{\eta''}$ are defined from u_{μ,J_1} and u_{ν,J_2} as

$$u_{\mu,J_1}^{\eta'} = \left\{ \xi \in \mathbb{R}^2 : (\xi, \eta') \in u_{\mu,J_1} \right\}, \quad u_{\nu,J_2}^{\eta''} = \left\{ \xi \in \mathbb{R}^2 : (\xi, \eta'') \in u_{\nu,J_2} \right\}.$$

Thus Lemma 1.4 is restated as follows.

Lemma 1.5. *Let $a \in \mathcal{I}$ and $\eta \in \widetilde{J_1} + \widetilde{J_2}$. The family of the sets*

$$\left\{ \bigcup_{\substack{\eta'+\eta''=\eta,\\ \eta'\in\widetilde{J_1},\eta''\in\widetilde{J_2}}} \left(u^{\eta'}_{\mu,J_1} + u^{\eta''}_{\nu,J_2}\right) \right\}_{(\mu,\nu)\in k_0^{-1}(\{a\})}$$

is finitely overlapping uniformly in $a \in \mathcal{I}$ and $\eta \in \widetilde{J_1} + \widetilde{J_2}$.

To prove (1.6) another result we need is the following.

Lemma 1.6. *For $a \in \mathcal{I}$, let*

$$E_a = \bigcup_{(\mu,\nu)\in k_0^{-1}(\{a\})} E_a(\mu,\nu).$$

Then there exists $C_0 > 0$ such that if $|a - a'| \geq C_0$, $a, a' \in \mathcal{I}$, then $E_a \cap E_{a'} = \emptyset$.

Lemma 1.6 follows from the following.

Lemma 1.7. *Let $\eta \in \widetilde{J_1} + \widetilde{J_2}$. Then there exists $C_0 > 0$ independent of η such that $E^\eta_a \cap E^\eta_{a'} = \emptyset$ if $|a - a'| \geq C_0$, $a, a' \in \mathcal{I}$, where*

$$E^\eta_a = \{\xi \in \mathbb{R}^2 : (\xi, \eta) \in E_a\}.$$

By Lemmas 1.3 and 1.6 we have (1.6), from which Theorem 1.1 will follow. As we have seen above, Lemmas 1.3 and 1.6 follow from Lemmas 1.5 and 1.7, respectively. So, to prove Theorem 1.1 it suffices to show Lemmas 1.5 and 1.7.

In Section 2, we shall prove Lemma 1.5 by applying arguments using principal curvatures of a circular cone. The proof of Lemma 1.7 will be given in Section 3. When $J_1 = J_2$, we can prove Lemma 1.7 by observing that E^η_a is contained in a $c\tau^{-1/2}$ neighborhood $\widetilde{\ell_a}$ of a line segment ℓ_a for some positive constant c, where

$$\ell_a = \{\xi \in \mathbb{R}^2 : \arg \xi = a\tau^{-1/2}, 1/2 \leq |\xi| \leq 9/2\}, \quad \widetilde{\ell_a} = \{\zeta \in \mathbb{R}^2 : d(\zeta, \ell_a) < c\tau^{-1/2}\},$$

with $d(\zeta, \ell_a) = \inf_{\xi\in\ell_a} |\zeta - \xi|$. The proof for the general case is slightly less straightforward. We shall provide a detailed proof. In Section 4, we shall state a variant of Theorem 1.1 (Theorem 4.2) and give the proof.

2. Proof of Lemma 1.5

We need the following.

Lemma 2.1. *Let $\eta' \in \widetilde{J_1}$, $\mu \in \mathcal{N}$.*

(1) *if $\xi \in u^{\eta'}_{\mu,J_1}$, then*

$$\xi = \eta'(\cos\theta, \sin\theta) + \zeta, \quad \zeta = \sigma(\cos\theta, \sin\theta)$$

for some $\theta, \sigma \in \mathbb{R}$ such that $\mu\tau^{-1/2} \leq \theta < (\mu + 1)\tau^{-1/2}$ and $|\sigma| \leq \tau^{-1}$;

(2) *if $\mu\tau^{-1/2} \leq \theta < (\mu + 1)\tau^{-1/2}$, there exists $\sigma \in \mathbb{R}$ such that $|\sigma| \leq \tau^{-1}$ and*

$$\eta'(\cos\theta, \sin\theta) + \sigma(\cos\theta, \sin\theta) \in u^{\eta'}_{\mu,J_1}.$$

Similar results hold for $u^{\eta''}_{\nu,J_2}$ with $\eta'' \in \widetilde{J_2}$, $\nu \in \mathcal{N}$.

Proof. If $\xi \in u^{\eta'}_{\mu,J_1}$, then $(\xi,\eta') \in u_{\mu,J_1}$, which implies that $(\xi,|\xi|) \in U_{\mu,J_1}$ and $|\eta'-|\xi|| \le \tau^{-1}$. Since $(\xi,|\xi|) \in U_{\mu,J_1}$, there exists $\theta \in \mathbb{R}$ such that $\mu\tau^{-1/2} \le \theta < (\mu+1)\tau^{-1/2}$ and $\xi = |\xi|(\cos\theta,\sin\theta)$. We write

$$\xi = \eta'(\cos\theta,\sin\theta) + (|\xi|-\eta')(\cos\theta,\sin\theta).$$

Putting $\sigma = |\xi|-\eta'$, we get the conclusion of part (1).

Proof of part (2). We take $\eta'_0 \in J_1$ such that $|\eta'_0-\eta'| \le \tau^{-1}$. Then $(\eta'_0(\cos\theta,\sin\theta),\eta'_0) \in U_{\mu,J_1}$. Thus $(\eta'_0(\cos\theta,\sin\theta),\eta') \in u_{\mu,J_1}$. It follows that $\eta'_0(\cos\theta,\sin\theta) \in u^{\eta'}_{\mu,J_1}$. Therefore, setting $\sigma = \eta'_0-\eta'$, we reach the conclusion. □

Proof of Lemma 1.5. We first assume that $a=0$. Let $(\mu,\nu) \in k_0^{-1}(\{0\})$, $\eta' \in \widetilde{J_1}$, $\eta'' \in \widetilde{J_2}$. Suppose that $\mu = \ell+m$, $\nu = -\ell-m$, with $\ell, m \ge 0$. By Lemma 2.1 (2), there exist $p \in u^{\eta'}_{\mu,J_1}$ and $q \in u^{\eta''}_{\nu,J_2}$ such that

$$|\eta'(\cos(\mu\tau^{-1/2}),\sin(\mu\tau^{-1/2})) - p| \le \tau^{-1}, \tag{2.1}$$

$$|\eta''(\cos(\nu\tau^{-1/2}),\sin(\nu\tau^{-1/2})) - q| \le \tau^{-1}. \tag{2.2}$$

Also, we have

(2.3)
$$\begin{aligned}\eta'(\cos(\ell+m)\tau^{-1/2},\sin(\ell+m)\tau^{-1/2}) &+ \eta''(\cos(\ell+m)\tau^{-1/2},-\sin(\ell+m)\tau^{-1/2})\\ &= (\eta\cos(\ell+m)\tau^{-1/2},(\eta'-\eta'')\sin(\ell+m)\tau^{-1/2}).\end{aligned}$$

We note that

$$\cos\ell\tau^{-1/2} - \cos(\ell+m)\tau^{-1/2} = 2\sin\left(\ell+\frac{m}{2}\right)\tau^{-1/2}\sin\frac{m}{2}\tau^{-1/2}.$$

By this it follows that

$$\left|\cos\ell\tau^{-1/2} - \cos(\ell+m)\tau^{-1/2}\right| \le 2\left(\ell+\frac{m}{2}\right)\frac{m}{2}\tau^{-1} \tag{2.4}$$

$$\left|\cos\ell\tau^{-1/2} - \cos(\ell+m)\tau^{-1/2}\right| \ge 2(2/\pi)^2\left(\ell+\frac{m}{2}\right)\frac{m}{2}\tau^{-1}, \tag{2.5}$$

where we have used well-known inequalities $\sin x \le x$, $x \ge 0$, and $\sin x \ge (2/\pi)x$, $0 \le x \le \pi/2$.

If $\xi \in u^{\eta'}_{\mu,J_1}$, $\xi = (\xi_1,\xi_2)$, $\mu = \ell+m$, by Lemma 2.1 (1) and the estimate $|\eta'| \le 3$ and by using (2.4) suitably, we have

$$\begin{aligned}|\eta'\cos(\ell+m)\tau^{-1/2} - \xi_1| &\le |\eta'\cos(\ell+m)\tau^{-1/2} - \eta'\cos\theta| + |\zeta_1|\\ &\le \eta'|\cos(\ell+m)\tau^{-1/2} - \cos(\ell+m+1)\tau^{-1/2}| + \tau^{-1}\\ &\le 3(\ell+m+1)\tau^{-1}.\end{aligned}$$

Also, if $\xi' \in u^{\eta''}_{\nu,J_2}$, $\xi' = (\xi'_1,\xi'_2)$, $\nu = -(\ell+m)$,

$$\xi' = \eta''(\cos\theta',\sin\theta') + \zeta', \quad \zeta' = \sigma'(\cos\theta',\sin\theta'),$$

with $\nu\tau^{-1/2} \le \theta' < (\nu+1)\tau^{-1/2}$, $|\sigma'| \le \tau^{-1}$, then

$$\begin{aligned}|\eta''\cos(\ell+m)\tau^{-1/2} - \xi'_1| &\le \eta''|\cos(\ell+m)\tau^{-1/2} - \cos(\ell+m-1)\tau^{-1/2}| + |\zeta'_1|\\ &\le \eta''|\ell+m-1/2|\tau^{-1} + \tau^{-1}\\ &\le 3(\ell+m+1)\tau^{-1}\end{aligned}$$

for $\ell, m \geq 0$. Thus we have $\operatorname{diam} P_1(u^{\eta'}_{\mu,J_1}) \leq 6(\ell+m+1)\tau^{-1}$, $\operatorname{diam} P_1(u^{\eta''}_{\nu,J_2}) \leq 6(\ell+m+1)\tau^{-1}$, where P_1 is the projection mapping defined by $P_1(\xi) = \xi_1$ when $\xi = (\xi_1, \xi_2)$.

Therefore

$$\operatorname{diam} P_1(u^{\eta'}_{\mu,J_1} + u^{\eta''}_{\nu,J_2}) \leq 12(\ell+m+1)\tau^{-1}. \tag{2.6}$$

Let $\widetilde{\eta}' + \widetilde{\eta}'' = \eta, \widetilde{\eta}' \in \widetilde{J_1}, \widetilde{\eta}'' \in \widetilde{J_2}$. By (2.1), (2.2) and (2.3), there exist $A \in u^{\eta'}_{\ell,J_1} + u^{\eta''}_{-\ell,J_2}$ and $B \in u^{\widetilde{\eta}'}_{\ell+m,J_1} + u^{\widetilde{\eta}''}_{-\ell-m,J_2}$ such that

$$|\eta\cos\ell\tau^{-1/2} - P_1(A)| \leq 2\tau^{-1}, \quad |\eta\cos(\ell+m)\tau^{-1/2} - P_1(B)| \leq 2\tau^{-1}.$$

Thus if

$$P_1(u^{\eta'}_{\ell,J_1} + u^{\eta''}_{-\ell,J_2}) \cap P_1(u^{\widetilde{\eta}'}_{\ell+m,J_1} + u^{\widetilde{\eta}''}_{-\ell-m,J_2}) \neq \emptyset,$$

then

$$\begin{aligned}12(\ell+1)\tau^{-1} + 12(\ell+m+1)\tau^{-1} &\geq |P_1(A) - P_1(B)| \\ &\geq \eta\cos\ell\tau^{-1/2} - \eta\cos(\ell+m)\tau^{-1/2} - 4\tau^{-1} \\ &\geq \eta 2\pi^{-2}(2\ell+m)m\tau^{-1} - 4\tau^{-1} \\ &\geq 2\pi^{-2}(2\ell+m)m\tau^{-1} - 4\tau^{-1},\end{aligned}$$

where the penultimate inequality follows by (2.5). This implies that

$$m^2 + 2(\ell - 3\pi^2)m - 2\pi^2(6\ell+7) \leq 0,$$

and hence we see that $m \leq C$ with a positive constant C. From this Lemma 1.5 for $a = 0$ can be deduced.

Let $\mathcal{R}_\sigma$ be a rotation around the origin such that $\mathcal{R}_\sigma((1,0)) = (\cos\sigma, \sin\sigma)$. To prove the general case, let $a \in \mathcal{I}$, $(\mu,\nu) \in k_0^{-1}(\{a\})$ and put $\alpha = \mu - a$, $\beta = \nu - a$. Then $\alpha + \beta = 0$. We note that $\alpha, \beta \in \mathbb{Z}^* \cap [-\tau^{1/2}\pi 8^{-1} + 1, \tau^{1/2}\pi 8^{-1} - 1]$, recalling $\mathbb{Z}^* = \{k/2 : k \in \mathbb{Z}\}$. Also, we observe that $\mathcal{R}_{-a\tau^{-1/2}}\Gamma_\mu = \Gamma_\alpha$ and $\mathcal{R}_{-a\tau^{-1/2}}\Gamma_\nu = \Gamma_\beta$. Thus, we can argue similarly to the case $a = 0$ to handle the family of the sets

$$\left\{ \bigcup_{\substack{\eta'+\eta''=\eta, \\ \eta' \in \widetilde{J_1}, \eta'' \in \widetilde{J_2}}} \mathcal{R}_{-a\tau^{-1/2}} \left(u^{\eta'}_{\mu,J_1} + u^{\eta''}_{\nu,J_2} \right) \right\}_{(\mu,\nu) \in k_0^{-1}(\{a\})}$$

to get a finitely overlapping property which can prove the desired result by applying the mapping $\mathcal{R}_{a\tau^{-1/2}}$. This completes the proof of Lemma 1.5. □

3. Proof of Lemma 1.7

In this section, we prove Lemma 1.7. Let $\delta = \tau^{-1}$. Let $\mathbb{N}_0$ be the set of non-negative integers and let $\mathbb{N}_0^* = \{k/2 : k \in \mathbb{N}_0\}$. Let $\eta' + \eta'' = \eta, \eta' \in \widetilde{J_1}, \eta'' \in \widetilde{J_2}$ and $\ell \in \mathbb{N}_0^*$, $\ell \leq \tau^{1/2}(\pi/8) - 1$. Put

$$p_{\ell,\delta}(\eta', \eta'') = (\eta\cos(\ell\delta^{1/2}), (\eta' - \eta'')\sin(\ell\delta^{1/2})).$$

Let $R_0(a,b) = [-a,a] \times [-b,b]$, $a, b > 0$ be the rectangle centered at 0. Let $\ell_* = \max(\ell, 1/2)$ and $c_1, c_2 > 0$. Define

$$R(p_{\ell,\delta}(\eta',\eta''); c_1\ell_*\delta, c_2\delta^{1/2}) = p_{\ell,\delta}(\eta',\eta'') + R_0(c_1\ell_*\delta, c_2\delta^{1/2}).$$

Let $a \in \mathcal{I}$, $\ell \in \mathbb{N}_0^*$. If $\mu = a + \ell$, $\nu = a - \ell$, then

$$\mathcal{R}_{-a\delta^{1/2}}(u^{\eta'}_{\mu,J_1} + u^{\eta''}_{\nu,J_2}) \subset R(p_{\ell,\delta}(\eta', \eta''); c_1\ell_*\delta, c_2\delta^{1/2}) \tag{3.1}$$

for some positive constants c_1, c_2, which can be seen since $u^{\eta'}_{\mu,J_1} + u^{\eta''}_{\nu,J_2}$ is contained in a ball of radius $c\delta^{1/2}$ and we have estimates similar to (2.6).

Let

$$\mathcal{E}^{(0)}(\eta', \eta'') = \left\{(\xi_1, \xi_2) : \frac{\xi_1^2}{\eta^2} + \frac{\xi_2^2}{(\eta' - \eta'')^2} = 1, \quad 1 \leq \xi_1 \leq \eta\right\}$$

when $\eta' \neq \eta''$; if $\eta' = \eta''$, let

$$\mathcal{E}^{(0)}(\eta', \eta'') = \{(\xi_1, 0) : 1 \leq \xi_1 \leq \eta\}.$$

Let

$$\mathcal{E}(\eta', \eta'') = \left\{(\xi_1, \xi_2) \in \mathcal{E}^{(0)}(\eta', \eta'') : \xi_2 \geq 0\right\}$$

if $\eta' \geq \eta''$; when $\eta' \leq \eta''$ let

$$\mathcal{E}(\eta', \eta'') = \left\{(\xi_1, \xi_2) \in \mathcal{E}^{(0)}(\eta', \eta'') : \xi_2 \leq 0\right\}.$$

We note that the point $p_{\ell,\delta}(\eta', \eta'')$ is on the curve $\mathcal{E}(\eta', \eta'')$. Let $\mathcal{E}_a(\eta', \eta'') = \mathcal{R}_{a\delta^{1/2}}\mathcal{E}(\eta', \eta'')$, $a \in \mathbb{R}$.

Also, for a technical reason, it is convenient to consider a slightly augmented version of $\mathcal{E}(\eta', \eta'')$:

$$\widetilde{\mathcal{E}}_a(\eta', \eta'') = \mathcal{R}_{a\delta^{1/2}}\widetilde{\mathcal{E}}(\eta', \eta''), \quad \widetilde{\mathcal{E}}(\eta', \eta'') = \mathcal{E}(\eta', \eta'') \cup \{(\xi_1, 0) : \eta \leq \xi_1 \leq 5\}.$$

Let $A(\alpha, \beta) = \{\xi \in \mathbb{R}^2 : \alpha \leq |\xi| \leq \beta\}$ be an annulus. To prove Lemma 1.7 we need the following.

Lemma 3.1. *Let $\ell \in \mathbb{N}_0^* \cap [0, \tau^{1/2}\pi 8^{-1} - 1]$. Let b_1 be a positive constant satisfying $|p_{\ell,\delta}| - b_1\ell_*\delta > |\eta' - \eta''|$, where $p_{\ell,\delta} = p_{\ell,\delta}(\eta', \eta'')$. Then there exist $b_2, b_3 > 0$ depending on b_1 such that*

$$A(|p_{\ell,\delta}| - b_1\ell_*\delta, |p_{\ell,\delta}| + b_1\ell_*\delta) \cap \widetilde{\mathcal{E}}(\eta', \eta'') \subset R(p_{\ell,\delta}; b_2\ell_*\delta, b_3\delta^{1/2}),$$

where b_2 and b_3 are independent of $\eta' \in \widetilde{J_1}$, $\eta'' \in \widetilde{J_2}$ and δ.

Proof. Let $\eta' \geq \eta''$. Let

$$\Phi(\xi_1) = (\eta' - \eta'')\sqrt{1 - \frac{\xi_1^2}{\eta^2}}, \quad \Psi(\xi_1) = \sqrt{(h - b_1\ell_*\delta)^2 - \xi_1^2},$$

where $h = |p_{\ell,\delta}|$. If $\Phi(\xi_1) = \Psi(\xi_1)$, $\xi_1 \geq 0$, then

$$\xi_1 = \frac{\eta}{\sqrt{\eta^2 - \beta^2}}\sqrt{(h - b_1\ell_*\delta)^2 - \beta^2},$$

where $\beta = \eta' - \eta''$. We note that

$$0 \leq \eta^2\cos^2(\ell\delta^{1/2}) - \frac{\eta^2}{\eta^2 - \beta^2}((h - b_1\ell_*\delta)^2 - \beta^2) \leq \frac{\eta^2}{\eta^2 - \beta^2}2b_1h\ell_*\delta, \tag{3.2}$$

as follows:

$$\begin{aligned}\eta^2\cos^2(\ell\delta^{1/2}) - \frac{\eta^2}{\eta^2 - \beta^2}((h - b_1\ell_*\delta)^2 - \beta^2) &= \frac{\eta^2}{\eta^2 - \beta^2}(h^2 - (h - b_1\ell_*\delta)^2) \\ &\leq \frac{\eta^2}{\eta^2 - \beta^2}2hb_1\ell_*\delta.\end{aligned}$$

Since $\eta\cos(\ell\delta^{1/2}) \geq 1$, by (3.2) we have

$$0 \leq \eta\cos(\ell\delta^{1/2}) - \frac{\eta}{\sqrt{\eta^2-\beta^2}}\sqrt{(h-b_1\ell_*\delta)^2-\beta^2} \tag{3.3}$$
$$\leq \eta^2\cos^2(\ell\delta^{1/2}) - \frac{\eta^2}{\eta^2-\beta^2}((h-b_1\ell_*\delta)^2-\beta^2)$$
$$\leq \frac{\eta^2}{\eta^2-\beta^2}2b_1h\ell_*\delta.$$

In the case $\ell = 0$, from (3.3) we can easily see that

$$A(\eta-(b_1/2)\delta, \eta+(b_1/2)\delta) \cap \widetilde{\mathcal{E}}(\eta', \eta'') \subset R((\eta,0); b_2\delta, b_3\delta^{1/2})$$

for some $b_2, b_3 > 0$, which is what we need. So, we assume that $\ell \geq 1/2$ and $\ell_* = \ell$ in what follows.

Let $\Phi(\xi_1)$ be as above and

$$\widetilde{\Psi}(\xi_1) = \sqrt{(h+b_1\ell\delta)^2-\xi_1^2}.$$

Solving the equation $\Phi(\xi_1) = \widetilde{\Psi}(\xi_1)$ for $\xi_1 \geq 0$ under the condition that $h+b_1\ell\delta \leq \eta$, we have

$$\xi_1 = \frac{\eta}{\sqrt{\eta^2-\beta^2}}\sqrt{(h+b_1\ell\delta)^2-\beta^2}.$$

We see that

$$0 \leq \frac{\eta^2}{\eta^2-\beta^2}((h+b_1\ell\delta)^2-\beta^2) - \eta^2\cos^2(\ell\delta^{1/2}) = \frac{\eta^2}{\eta^2-\beta^2}((h+b_1\ell\delta)^2-h^2)$$
$$= \frac{\eta^2}{\eta^2-\beta^2}(2hb_1+b_1^2\ell\delta)\ell\delta$$

and hence, arguing as in (3.3), we have

$$0 \leq \frac{\eta}{\sqrt{\eta^2-\beta^2}}\sqrt{(h+b_1\ell\delta)^2-\beta^2} - \eta\cos(\ell\delta^{1/2}) \tag{3.4}$$
$$\leq \frac{\eta^2}{\eta^2-\beta^2}((h+b_1\ell\delta)^2-\beta^2) - \eta^2\cos^2(\ell\delta^{1/2})$$
$$= \frac{\eta^2}{\eta^2-\beta^2}(2hb_1+b_1^2\ell\delta)\ell\delta,$$

assuming $h+b_1\ell\delta \leq \eta$.

Next, we estimate

$$I := \Phi(\eta\cos(\ell\delta^{1/2}) - \widetilde{b}_1\ell\delta) - \Phi(\eta\cos(\ell\delta^{1/2})),$$

where we assume that $\widetilde{b}_1\ell\delta \leq \eta\cos(\ell\delta^{1/2})$, $\widetilde{b}_1 \geq 0$, and

$$II := -\Phi(\eta\cos(\ell\delta^{1/2}) + \widetilde{b}_1\ell\delta) + \Phi(\eta\cos(\ell\delta^{1/2})),$$

when $\eta\cos(\ell\delta^{1/2})+\widetilde{b}_1\ell\delta \leq \eta$; when $\eta\cos(\ell\delta^{1/2})+\widetilde{b}_1\ell\delta > \eta$, let $II = \Phi(\eta\cos(\ell\delta^{1/2}))$. We note that if $\eta < \eta\cos(\ell\delta^{1/2}) + \widetilde{b}_1\ell\delta$, then

$$\widetilde{b}_1\ell\delta > \eta(1-\cos(\ell\delta^{1/2})) \geq \eta(2/\pi^2)\ell^2\delta,$$

and hence $\ell < \widetilde{b}_1\eta^{-1}\pi^2/2$.

We use

$$\Phi'(\xi_1) = \beta\left(1 - \frac{\xi_1^2}{\eta^2}\right)^{-1/2}\left(-\frac{\xi_1}{\eta^2}\right).$$

By the mean value theorem, it follows that

$$|I| \le \widetilde{b}_1\ell\delta\beta(1-\cos^2(\ell\delta^{1/2}))^{-1/2}\eta^{-1} \le \widetilde{b}_1\ell\delta\beta\sin(\ell\delta^{1/2})^{-1} \le (\pi/2)\widetilde{b}_1\beta\delta^{1/2}. \tag{3.5}$$

Obviously, we see that

$$|II| \le \Phi(\eta\cos(\ell\delta^{1/2})) = \beta\sin(\ell\delta^{1/2}) \le \beta\ell\delta^{1/2}. \tag{3.6}$$

If $\ell \ge \widetilde{b}_1\eta^{-1}\pi^2/2$ and so $\eta\cos(\ell\delta^{1/2}) + \widetilde{b}_1\ell\delta \le \eta$, then applying the mean value theorem, we see that

$$\begin{aligned}
|II| &\le \beta\widetilde{b}_1\ell\delta\left(1-\eta^{-2}(\eta\cos(\ell\delta^{1/2})+\widetilde{b}_1\ell\delta)^2\right)^{-1/2}\\
&= \beta\widetilde{b}_1\ell\delta\left(1-(\cos^2(\ell\delta^{1/2})+2\eta^{-1}\widetilde{b}_1\ell\delta\cos(\ell\delta^{1/2})+\eta^{-2}(\widetilde{b}_1)^2(\ell\delta)^2)\right)^{-1/2}\\
&= \beta\widetilde{b}_1\ell\delta\left(\sin^2(\ell\delta^{1/2})-2\eta^{-1}\widetilde{b}_1\ell\delta\cos(\ell\delta^{1/2})-\eta^{-2}(\widetilde{b}_1)^2(\ell\delta)^2\right)^{-1/2}\\
&\le \beta\widetilde{b}_1\ell\delta\left((2/\pi)^2\ell^2\delta-(2\eta^{-1}\widetilde{b}_1+\eta^{-2}(\widetilde{b}_1)^2)\ell\delta\right)^{-1/2} =: J,
\end{aligned}$$

where we assume that $\ell \ge 2(\pi/2)^2(2\eta^{-1}\widetilde{b}_1+\eta^{-2}(\widetilde{b}_1)^2) =: C_0$. Then, we see that

$$|II| \le J \le \beta\widetilde{b}_1\ell\delta\left(2^{-1}(2/\pi)^2\ell^2\delta\right)^{-1/2} = 2^{-1/2}\pi\beta\widetilde{b}_1\delta^{1/2}. \tag{3.7}$$

By (3.6) and (3.7), noting that $C_0 \ge \widetilde{b}_1\eta^{-1}\pi^2/2$, we have

$$|II| \le \beta(C_0 + 2^{-1/2}\pi\widetilde{b}_1)\delta^{1/2}. \tag{3.8}$$

By (3.3), (3.4), (3.5) and (3.8), we can prove Lemma 3.1 as follows. First, by (3.3), (3.4), we have

$$\begin{aligned}
&A(|p_{\ell,\delta}|-b_1\ell\delta, |p_{\ell,\delta}|+b_1\ell\delta)\cap\widetilde{\mathcal{E}}(\eta',\eta'')\\
&\qquad\subset [\eta\cos(\ell\delta^{1/2})-b_2\ell\delta, \eta\cos(\ell\delta^{1/2})+b_2\ell\delta]\times\mathbb{R}
\end{aligned} \tag{3.9}$$

for some $b_2 > 0$ under the condition $h + b_1\ell\delta \le \eta$. If $h + b_1\ell\delta > \eta$, we easily see that

$$2hb_1\ell\delta + b_1^2\ell^2\delta^2 > (\eta^2-\beta^2)\sin^2(\ell\delta^{1/2}),$$

which implies that $\ell \le C$ for some constant C. Using this, we have

$$\begin{aligned}
|h-\eta\cos(\ell\delta^{1/2})| &\le h^2-\eta^2\cos^2(\ell\delta^{1/2})\\
&= (\eta'-\eta'')^2\sin^2(\ell\delta^{1/2}) \le C_1\ell^2\delta \le C_1C\ell\delta.
\end{aligned} \tag{3.10}$$

Also, when $h + b_1\ell\delta > \eta$, by (3.3) we see that

$$A(|p_{\ell,\delta}|-b_1\ell\delta, |p_{\ell,\delta}|+b_1\ell\delta)\cap\widetilde{\mathcal{E}}(\eta',\eta'') \subset [\eta\cos(\ell\delta^{1/2})-b_2\ell\delta, h+b_1\ell\delta]\times\mathbb{R}. \tag{3.11}$$

By (3.10) and (3.11), we also have (3.9) for some b_2 when $h + b_1\ell\delta > \eta$.

Next, by (3.5) and (3.8) with $\widetilde{b}_1 = b_2$ and (3.9) we have

$$
\begin{aligned}
&A(|p_{\ell,\delta}| - b_1\ell\delta, |p_{\ell,\delta}| + b_1\ell\delta) \cap \widetilde{\mathcal{E}}(\eta', \eta'') \\
&\subset \left([\eta\cos(\ell\delta^{1/2}) - b_2\ell\delta, \eta\cos(\ell\delta^{1/2}) + b_2\ell\delta] \times \mathbb{R}\right) \cap \widetilde{\mathcal{E}}(\eta', \eta'') \\
&\subset [\eta\cos(\ell\delta^{1/2}) - b_2\ell\delta, \eta\cos(\ell\delta^{1/2}) + b_2\ell\delta] \\
&\quad \times [\beta\sin(\ell\delta^{1/2}) - b_3\delta^{1/2}, \beta\sin(\ell\delta^{1/2}) + b_3\delta^{1/2}]
\end{aligned}
$$

for some positive constant b_3. This proves Lemma 3.1 when $\eta' \geq \eta''$.

The case $\eta' \leq \eta''$ can be handled similarly. This completes the proof of Lemma 3.1. □

We also need the following lemmas (Lemmas 3.2, 3.3 and 3.4) in proving Lemma 1.7.

Lemma 3.2. *Let $\ell \in \mathbb{N}_0^* \cap [0, \delta^{-1/2}(\pi/8) - 1]$. Let c_1, c_2 be positive constants. Then, there exist constants $c_3, c_4 > 0$ depending on c_1, c_2 such that*

$$R(p_{\ell,\delta}; c_1\ell_*\delta, c_2\delta^{1/2}) \subset A(|p_{\ell,\delta}| - c_3\ell_*\delta, |p_{\ell,\delta}| + c_4\ell_*\delta),$$

where $p_{\ell,\delta} = p_{\ell,\delta}(\eta', \eta'')$ and $\ell_* = \max(\ell, 1/2)$.

Proof. We write (α, β) for $p_{\ell,\delta}$. Let $(\alpha + \epsilon_1, \beta + \epsilon_2) \in R(p_{\ell,\delta}; c_1\ell_*\delta, c_2\delta^{1/2})$. Then $|\epsilon_1| \leq c_1\ell_*\delta$, $|\epsilon_2| \leq c_2\delta^{1/2}$. To prove the lemma, it suffices to show that

$$\left|\sqrt{\alpha^2 + \beta^2} - \sqrt{(\alpha + \epsilon_1)^2 + (\beta + \epsilon_2)^2}\right| \leq c_0\ell_*\delta$$

for some $c_0 > 0$. Since $\alpha \geq c > 0$, this follows from the estimate

$$\left|(\alpha^2 + \beta^2) - \left((\alpha + \epsilon_1)^2 + (\beta + \epsilon_2)^2\right)\right| \leq c_0'\ell_*\delta. \tag{3.12}$$

Now we see that $|\alpha| \leq 5$, $|\beta| \leq |\eta' - \eta''|\ell\delta^{1/2}$, $|\eta' - \eta''| \leq 3/2$ and

$$
\begin{aligned}
&\left|(\alpha^2 + \beta^2) - \left((\alpha + \epsilon_1)^2 + (\beta + \epsilon_2)^2\right)\right| = \left|2\alpha\epsilon_1 + \epsilon_1^2 + 2\beta\epsilon_2 + \epsilon_2^2\right| \\
&\leq 10c_1\ell_*\delta + (c_1\ell_*\delta)^2 + 2|\eta' - \eta''|\ell\delta^{1/2}c_2\delta^{1/2} + c_2^2\delta \\
&\leq 10c_1\ell_*\delta + (c_1\ell_*\delta)^2 + 3c_2\ell_*\delta + c_2^2\delta \\
&\leq (10c_1 + c_1^2 + 3c_2 + 2c_2^2)\ell_*\delta.
\end{aligned}
$$

This proves (3.12) and hence completes the proof of Lemma 3.2. □

Let $\eta' \in \widetilde{J_1}$, $\eta'' \in \widetilde{J_2}$, $\eta' + \eta'' = \eta$. Let $\ell \in \mathbb{N}_0^*$ and

$$p_{\ell,\delta}(\eta', \eta'') = (\eta\cos(\ell\delta^{1/2}), (\eta' - \eta'')\sin(\ell\delta^{1/2})).$$

Let

$$R_{\ell,\delta}(\eta', \eta'') = p_{\ell,\delta}(\eta', \eta'') + R_0(c_1\ell_*\delta, (c_2 + 4)\delta^{1/2}),$$

where c_1, c_2 are as in (3.1) and we recall that $\ell_* = \max(\ell, 1/2)$.

Lemma 3.3. *If $\eta', \eta_0' \in \widetilde{J_1}$, $\eta'', \eta_0'' \in \widetilde{J_2}$, $\eta' + \eta'' = \eta$, $\eta_0' + \eta_0'' = \eta$, then*

$$\widetilde{\mathcal{E}}_a(\eta', \eta'') \cap R_{\ell,\delta}(\eta_0', \eta_0'') \subset \mathcal{R}_{a\delta^{1/2}} R(p_{\ell,\delta}(\eta', \eta''); c_1'\ell_*\delta, c_2'\delta^{1/2})$$

for some positive constants c_1', c_2' independent of a, δ and ℓ, where $\widetilde{\mathcal{E}}_a(\eta', \eta'') = \mathcal{R}_{a\delta^{1/2}}\widetilde{\mathcal{E}}(\eta', \eta'')$ for $a \in \mathbb{R}$.

We have

$$R_{\ell,\delta}(\eta',\eta'') \cup R(p_{\ell,\delta}(\eta',\eta''); c_1'\ell_*\delta, c_2'\delta^{1/2}) \subset B(\eta,\ell,\delta,c_3) \tag{3.13}$$

for all $\eta' \in \widetilde{J_1}$, $\eta'' \in \widetilde{J_2}$ satisfying $\eta' + \eta'' = \eta$ with some positive number c_3, where

$$B(\eta,\ell,\delta,c_3) = \bigcup_{\substack{\eta'+\eta''=\eta,\\ \eta'\in\widetilde{J_1},\eta''\in\widetilde{J_2}}} B(p_{\ell,\delta}(\eta',\eta''), c_3\delta^{1/2}).$$

Here $B(x,r)$ denotes a ball with radius r centered at x. We may assume that $c_3\delta^{1/2}$ is small enough so that

(1) $$\mathcal{R}_\sigma B(\eta,\ell,\delta,c_3) \subset D = \{\xi \in \mathbb{R}^2 : 3/2 \le |\xi| \le 9/2, \xi_1 \ge 0\}$$
for $|\sigma| \le \pi/4$;

(2) there exists $a_0 > 0$ independent of η, ℓ, δ such that if $a_0 \le |a| \le (\pi/4)\delta^{-1/2}$, then

$$B(\eta,\ell,\delta,c_3) \cap \mathcal{R}_{a\delta^{1/2}} B(\eta,\ell,\delta,c_3) = \emptyset. \tag{3.14}$$

Proof of Lemma 3.3. We note that $|\eta' - \eta_0'| < 2\delta^{1/2}$, $|\eta'' - \eta_0''| < 2\delta^{1/2}$. Thus

$$R_{\ell,\delta}(\eta_0',\eta_0'') \subset R(p_{\ell,\delta}(\eta',\eta''); c_1\ell_*\delta, (c_2+8)\delta^{1/2}).$$

So, by Lemma 3.2 we have

$$R_{\ell,\delta}(\eta_0',\eta_0'') \subset R(p_{\ell,\delta}; c_1\ell_*\delta, (c_2+8)\delta^{1/2}) \subset A(|p_{\ell,\delta}| - c\ell_*\delta, |p_{\ell,\delta}| + c\ell_*\delta)$$

for some $c > 0$, where $p_{\ell,\delta} = p_{\ell,\delta}(\eta',\eta'')$. Thus

$$\widetilde{\mathcal{E}}_a(\eta',\eta'') \cap R_{\ell,\delta}(\eta_0',\eta_0'') \subset \widetilde{\mathcal{E}}_a(\eta',\eta'') \cap A(|p_{\ell,\delta}| - c\ell_*\delta, |p_{\ell,\delta}| + c\ell_*\delta).$$

By Lemma 3.1 and the rotation invariance of annulus, we see that

$$\widetilde{\mathcal{E}}_a(\eta',\eta'') \cap A(|p_{\ell,\delta}| - c\ell_*\delta, |p_{\ell,\delta}| + c\ell_*\delta) \subset \mathcal{R}_{a\delta^{1/2}} R(p_{\ell,\delta}; c'\ell_*\delta, c'\delta^{1/2})$$

for some $c' > 0$. Combining results, we have

$$\widetilde{\mathcal{E}}_a(\eta',\eta'') \cap R_{\ell,\delta}(\eta_0',\eta_0'') \subset \mathcal{R}_{a\delta^{1/2}} R(p_{\ell,\delta}; c'\ell_*\delta, c'\delta^{1/2}).$$

This completes the proof of Lemma 3.3. □

Let

$$\widetilde{E}_{00}^{\eta} = \bigcup_{\substack{\ell,\eta'+\eta''=\eta,\\ \eta'\in\widetilde{J_1},\eta''\in\widetilde{J_2}}} R(p_{\ell,\delta}(\eta',\eta''); c_1\ell_*\delta, (c_2+4)\delta^{1/2}) = \bigcup_{\substack{\ell,\eta'+\eta''=\eta,\\ \eta'\in\widetilde{J_1},\eta''\in\widetilde{J_2}}} R_{\ell,\delta}(\eta,\eta''),$$

where ℓ ranges over a subset of $\mathbb{N}_0^*$ such that $0 \le \ell \le \delta^{-1/2}(\pi/8) - 1$ and c_1, c_2 are as in (3.1).

Lemma 3.4. *Fix $\eta' \in \widetilde{J_1}$ and $\eta'' \in \widetilde{J_2}$ with $\eta' + \eta'' = \eta$. There exists $a_0 > 0$ independent of δ such that if $a_0 \le a \le (\pi/8)\delta^{-1/2}$, then*

$$\widetilde{\mathcal{E}}_a(\eta',\eta'') \cap \widetilde{E}_{00}^{\eta} = \emptyset, \quad \mathcal{R}_{a\delta^{1/2}}(\widetilde{E}_{00}^{\eta}) \cap \widetilde{\mathcal{E}}(\eta',\eta'') = \emptyset.$$

Proof. Let $\eta_0' \in \widetilde{J_1}$ and $\eta_0'' \in \widetilde{J_2}$ with $\eta_0' + \eta_0'' = \eta$. We first show that

$$\widetilde{\mathcal{E}}_a(\eta', \eta'') \cap R_{\ell,\delta}(\eta_0', \eta_0'') = \emptyset, \tag{3.15}$$

if $a_0 \le a \le (\pi/8)\delta^{-1/2}$ and a_0 is sufficiently large, where $R_{\ell,\delta}(\eta_0', \eta_0'')$ is as in Lemma 3.3. By Lemma 3.3 and (3.13), we have

$$\widetilde{\mathcal{E}}_a(\eta', \eta'') \cap R_{\ell,\delta}(\eta_0', \eta_0'') \subset \mathcal{R}_{a\delta^{1/2}} B(\eta, \ell, \delta, c_3).$$

Since $R_{\ell,\delta}(\eta_0', \eta_0'') \subset B(\eta, \ell, \delta, c_3)$, by (3.14) we have (3.15) if $a_0 \le a \le (\pi/8)\delta^{-1/2}$ and a_0 is as in (3.14), from which it can be deduced that $\widetilde{\mathcal{E}}_a(\eta', \eta'') \cap \widetilde{E}_{00}^{\eta} = \emptyset$ as claimed.

Next, we prove that $\mathcal{R}_{a\delta^{1/2}}(\widetilde{E}_{00}^{\eta}) \cap \widetilde{\mathcal{E}}(\eta', \eta'') = \emptyset$ if $a_0 \le a \le (\pi/8)\delta^{-1/2}$ and a_0 is sufficiently large. This follows from $\widetilde{E}_{00}^{\eta} \cap \widetilde{\mathcal{E}}_{-a}(\eta', \eta'') = \emptyset$, which can be shown as above by using Lemma 3.3, (3.13) and (3.14). This completes the proof of Lemma 3.4. □

Proof of Lemma 1.7. Let

$$\widetilde{E}_0^{\eta} = \bigcup_{\substack{\ell, \eta'+\eta''=\eta, \\ \eta' \in \widetilde{J_1}, \eta'' \in \widetilde{J_2}}} R(p_{\ell,\delta}(\eta', \eta''); c_1\ell_*\delta, c_2\delta^{1/2}),$$

where the constants c_1, c_2 are as in (3.1) and ℓ ranges over a subset of $\mathbb{N}_0^*$ such that $0 \le \ell \le \delta^{-1/2}(\pi/8) - 1$. We note that

$$E_\alpha^{\eta} \subset \mathcal{R}_{\alpha\delta^{1/2}} \widetilde{E}_0^{\eta} = \bigcup_{\substack{\ell, \eta'+\eta''=\eta, \\ \eta' \in \widetilde{J_1}, \eta'' \in \widetilde{J_2}}} \mathcal{R}_{\alpha\delta^{1/2}} R(p_{\ell,\delta}(\eta', \eta''); c_1\ell_*\delta, c_2\delta^{1/2}) \tag{3.16}$$

(see (3.1)). Recall that $D = \{\xi \in \mathbb{R}^2 : 3/2 \le |\xi| \le 9/2, \xi_1 \ge 0\}$. We may assume that $\mathcal{R}_{\alpha\delta^{1/2}} \widetilde{E}_0^{\eta} \subset D$ for $|\alpha| \le (\pi/4)\delta^{-1/2}$, $\eta = \eta' + \eta''$, $\eta' \in \widetilde{J_1}, \eta'' \in \widetilde{J_2}$.

Let $\ell, \ell' \in \mathbb{N}_0^*$ with $0 \le \ell, \ell' \le \delta^{-1/2}(\pi/8) - 1$, $\eta_0', \eta_1' \in \widetilde{J_1}$, $\eta_0'', \eta_1'' \in \widetilde{J_2}$ with $\eta_0' + \eta_0'' = \eta$, $\eta_1' + \eta_1'' = \eta$. Let $a, a' \in \mathcal{I}$. To prove Lemma 1.7, by (3.16) it suffices to show that

$$R(p_{\ell,\delta}(\eta_0', \eta_0''), c_1\ell_*\delta, c_2\delta^{1/2}) \cap \mathcal{R}_{(a-a')\delta^{1/2}} R(p_{\ell',\delta}(\eta_1', \eta_1''), c_1\ell'_*\delta, c_2\delta^{1/2}) = \emptyset, \tag{3.17}$$

if $a - a'$ is sufficiently large with $a - a' \le (\pi/4)\delta^{-1/2} - 2$. Let $b = a - a'$. We observe that for $\eta' \in \widetilde{J_1}$ and $\eta'' \in \widetilde{J_2}$ with $\eta' + \eta'' = \eta$,

$$p_{\ell,\delta}(\eta', \eta'') \in R(p_{\ell,\delta}(\eta_0', \eta_0''), c_1\ell_*\delta, (c_2+4)\delta^{1/2}) = R_{\ell,\delta}(\eta_0', \eta_0''), \tag{3.18}$$

$$\mathcal{R}_{b\delta^{1/2}} p_{\ell',\delta}(\eta', \eta'') \in \mathcal{R}_{b\delta^{1/2}} R(p_{\ell',\delta}(\eta_1', \eta_1''), c_1\ell'_*\delta, (c_2+4)\delta^{1/2}) = \mathcal{R}_{b\delta^{1/2}} R_{\ell',\delta}(\eta_1', \eta_1'').$$

Obviously (3.17) follows from

$$R_{\ell,\delta}(\eta_0', \eta_0'') \cap \mathcal{R}_{b\delta^{1/2}} R_{\ell',\delta}(\eta_1', \eta_1'') = \emptyset. \tag{3.19}$$

Applying Lemma 3.4, we see that

$$\widetilde{\mathcal{E}}_{b/2}(\eta', \eta'') \cap R_{\ell,\delta}(\eta_0', \eta_0'') = \emptyset, \tag{3.20}$$

$$\mathcal{R}_{b\delta^{1/2}} R_{\ell',\delta}(\eta_1', \eta_1'') \cap \widetilde{\mathcal{E}}_{b/2}(\eta', \eta'') = \emptyset$$

for a sufficiently large $b > 0$, $0 < b \le (\pi/4)\delta^{-1/2} - 2$.

We can divide D as $D \setminus \widetilde{\mathcal{E}}_{b/2}(\eta', \eta'') = D_1 \cup D_2$ with $D_1 \cap D_2 = \emptyset$. Since $\mathcal{E}(\eta', \eta''), \mathcal{E}_b(\eta', \eta'') \subset D$ and, obviously, $\widetilde{\mathcal{E}}_{b/2}(\eta', \eta'') \cap \mathcal{E}(\eta', \eta'') = \emptyset$, $\widetilde{\mathcal{E}}_{b/2}(\eta', \eta'') \cap \mathcal{E}_b(\eta', \eta'') = \emptyset$, we may assume that $\mathcal{E}(\eta', \eta'') \subset D_1$ and $\mathcal{E}_b(\eta', \eta'') \subset D_2$. Since

$p_{\ell,\delta}(\eta',\eta'') \in \mathcal{E}(\eta',\eta'')$, $\mathcal{R}_{b\delta^{1/2}}p_{\ell',\delta}(\eta',\eta'') \in \mathcal{E}_b(\eta',\eta'')$, we have $p_{\ell,\delta}(\eta',\eta'') \in D_1$ and $\mathcal{R}_{b\delta^{1/2}}p_{\ell',\delta}(\eta',\eta'') \in D_2$. Thus by (3.18) and (3.20) we have $R_{\ell,\delta}(\eta_0',\eta_0'') \subset D_1$ and $\mathcal{R}_{b\delta^{1/2}}R_{\ell',\delta}(\eta_1',\eta_1'') \subset D_2$, which implies (3.19). This completes the proof of Lemma 1.7. □

4. Applications of Theorem 1.1

Recall that

$$\begin{aligned} u_{\mu,J_i} = u^{(\delta)}_{\mu,J_i} &= \{(\xi,\eta) \in \mathbb{R}^2 \times \mathbb{R} : |\eta - |\xi|| \le \delta, (\xi,|\xi|) \in U_{\mu,J_i}\} \\ &= \cup\{\{\xi\} \times [|\xi| - \delta, |\xi| + \delta] : (\xi,|\xi|) \in U_{\mu,J_i}\} \quad i = 1,2, \end{aligned}$$

where $\delta = \tau^{-1}$ is a small positive number,

$$\begin{aligned} U_{\mu,J_i} &= U^{(\delta)}_{\mu,J_i} = \{(\xi,|\xi|) \in \mathbb{R}^2 \times \mathbb{R} : \xi \in \Gamma_\mu, |\xi| \in J_i\}, \\ \Gamma_\mu &= \Gamma^{(\delta)}_\mu = \{\xi \in \mathbb{R}^2 : \mu\delta^{1/2} \le \arg\xi < (\mu+1)\delta^{1/2}\}, \quad \mu \in \mathcal{N}, \\ \mathcal{N} &= \mathcal{N}^{(\delta)} = \{\mu \in \mathbb{Z} : |\mu| \le \frac{\pi}{8}\delta^{-1/2} - 1\}, \\ J_i &= J^{(\delta)}_i = [\alpha_i,\beta_i] \subset [1,2], \quad |J_i| \le \delta^{1/2}, \quad i = 1,2. \end{aligned}$$

We also consider

$$\mathcal{N}^{(\delta)}_* = \{\mu \in \mathbb{Z} : |\mu| \le \frac{\pi}{7}\delta^{-1/2} - 1\}.$$

For $\mu \in \mathcal{N}^{(\delta)}_*$, we define an enlargement $(u^{(\delta)}_{\mu,J_i})^*$ of $u^{(\delta)}_{\mu,J_i}$, i=1, 2, by

$$\begin{aligned} (u^{(\delta)}_{\mu,J_i})^* &= \{(\xi,\eta) \in \mathbb{R}^2 \times \mathbb{R} : |\eta - |\xi|| \le 6\delta, (\xi,|\xi|) \in (U^{(\delta)}_{\mu,J_i})^*\} \\ &= \cup\{\{\xi\} \times [|\xi| - 6\delta, |\xi| + 6\delta] : (\xi,|\xi|) \in (U^{(\delta)}_{\mu,J_i})^*\}, \end{aligned}$$

where

$$\begin{aligned} (U^{(\delta)}_{\mu,J_i})^* &= \{(\xi,|\xi|) \in \mathbb{R}^2 \times \mathbb{R} : \xi \in (\Gamma^{(\delta)}_\mu)^*, |\xi| \in (J^{(\delta)}_i)^*\}, \\ (\Gamma^{(\delta)}_\mu)^* &= \{\xi \in \mathbb{R}^2 : (\mu-1)\delta^{1/2} \le \arg\xi < (\mu+2)\delta^{1/2}\}, \\ (J^{(\delta)}_i)^* &= [\alpha_i - 2\delta, \beta_i + 2\delta]. \end{aligned}$$

We have the following result by examining the proof of Theorem 1.1.

Theorem 4.1. *There exists a positive constant C independent of J_1, J_2 and δ such that*

$$\sum_{(\mu,\nu)\in(\mathcal{N}^{(\delta)}_*)^2} \chi_{(u^{(\delta)}_{\mu,J_1})^* + (u^{(\delta)}_{\nu,J_2})^*} \le C.$$

Let $0 < \epsilon < 1/2$ and set

$$\widetilde{u}_{\mu,J_i} = \{(\xi,\eta) \in \mathbb{R}^2 \times \mathbb{R} : d(u_{\mu,J_i},(\xi,\eta)) < \delta^{1-\epsilon}\}, \quad i = 1,2,$$

where

$$d(u_{\mu,J_i},(\xi,\eta)) = \inf_{(\xi',\eta')\in u_{\mu,J_i}} (|\xi_1 - \xi_1'|^2 + |\xi_2 - \xi_2'|^2 + |\eta - \eta'|^2)^{1/2}.$$

Then we have the following.

Theorem 4.2. *We can find a positive constant C independent of J_1, J_2 and δ such that*

$$\sum_{(\mu,\nu)\in\mathcal{N}^2} \chi_{\widetilde{u}_{\mu,J_1} + \widetilde{u}_{\nu,J_2}} \le C\delta^{-\epsilon}.$$

To prove Theorem 4.2 by applying Theorem 4.1, we need the following.

Lemma 4.3. *Let* $N = [\delta^{-\epsilon/2}]$, $\delta_\epsilon = \delta[\delta^{-\epsilon/2}]^2 \sim \delta^{1-\epsilon}$, *where* $[\alpha] = \max\{m \in \mathbb{Z} : m \le \alpha\}$ *for* $\alpha \in \mathbb{R}$. *Suppose that* $\mu \in \mathcal{N}^{(\delta)}$ *and* $\mu = \ell N + k$ *for some* $\ell \in \mathbb{Z}$ *and* $k \in [0, N-1] \cap \mathbb{Z}$. *Then*

$$\Gamma^{(\delta)}_{\ell N+k} \subset \{\ell\delta_\epsilon^{1/2} \le \arg\xi < (\ell+1)\delta_\epsilon^{1/2}\}.$$

Proof. We have

$$\begin{aligned}\Gamma^{(\delta)}_{\ell N+k} &= \{(\ell N+k)\delta^{1/2} \le \arg\xi < (\ell N+k+1)\delta^{1/2}\}\\ &\subset \{\ell N\delta^{1/2} \le \arg\xi < (\ell N+N)\delta^{1/2}\}\\ &= \{\ell N\delta^{1/2} \le \arg\xi < (\ell+1)N\delta^{1/2}\}\\ &= \{\ell\delta_\epsilon^{1/2} \le \arg\xi < (\ell+1)\delta_\epsilon^{1/2}\}.\end{aligned}$$

This completes the proof. □

We also need the following.

Lemma 4.4. *Let* μ, ℓ, N, k *be as in Lemma* 4.3 *with* $\ell \in \mathcal{N}_*^{(\delta_\epsilon)}$. *Then, if* δ *is small enough, we have*

$$\widetilde{u}_{\mu,J_i} = \widetilde{u}_{\ell N+k,J_i} \subset (u^{(\delta_\epsilon)}_{\ell,J_i})^*, \quad i = 1, 2.$$

Proof. We show the result by using Lemma 4.3 and the definitions of $\widetilde{u}_{\mu,J_i}$, $u^{(\delta_\epsilon)}_{\ell,J_i}$ and $(u^{(\delta_\epsilon)}_{\ell,J_i})^*$ as follows.

Fix i and let $(\xi,\eta) \in \widetilde{u}_{\ell N+k,J_i}$. Then there exists $(\xi_0,\eta_0) \in u_{\ell N+k,J_i}$ such that

$$|(\xi,\eta) - (\xi_0,\eta_0)| < \delta^{1-\epsilon}. \tag{4.1}$$

Since $(\xi_0,\eta_0) \in u_{\ell N+k,J_i}$, we have $(\xi_0,|\xi_0|) \in U_{\ell N+k,J_i}$ and

$$|\eta_0 - |\xi_0|| < \delta. \tag{4.2}$$

The fact $(\xi_0,|\xi_0|) \in U_{\ell N+k,J_i}$ implies that $\xi_0 \in \Gamma^{(\delta)}_{\ell N+k}$ and $|\xi_0| \in J_i$. By Lemma 4.3 it follows that

$$\ell\delta_\epsilon^{1/2} \le \arg\xi_0 < (\ell+1)\delta_\epsilon^{1/2}, \tag{4.3}$$

where $|\ell| \le (\pi/7)\delta_\epsilon^{-1/2} - 1$ by the assumption that $\ell \in \mathcal{N}_*^{(\delta_\epsilon)}$.

To prove $(\xi,\eta) \in (u^{(\delta_\epsilon)}_{\ell,J_i})^*$, we need the estimate

$$|\arg\xi - \arg\xi_0| \le \delta_\epsilon^{1/2}, \tag{4.4}$$

if $0 < \epsilon < 1/2$ and δ is small enough. By (4.1), (4.3) and the fact that $|\xi_0| \in J_i$, we have $1/2 < |\xi| < 5/2$ and $|\arg\xi| < \pi/4$. Let $\xi_0 = (\zeta_1,\zeta_2)$. Using (4.1), if δ is small enough, we see that

$$\begin{aligned}|\arg\xi - \arg\xi_0| &= |\arctan(\xi_2/\xi_1) - \arctan(\zeta_2/\zeta_1)|\\ &\le |\xi_2/\xi_1 - \zeta_2/\zeta_1| = |\xi_2\zeta_1 - \xi_1\zeta_2|/|\xi_1\zeta_1|\\ &\le c|\xi_1-\zeta_1| + c|\xi_2-\zeta_2| \le 2c\delta^{1-\epsilon}\\ &\le 4c\delta_\epsilon \le \delta_\epsilon^{1/2},\end{aligned}$$

which proves (4.4), where we have also used the estimates $\delta^{1-\epsilon}/2 \le \delta_\epsilon \le \delta^{1-\epsilon}$, which are valid when δ is small enough.

By (4.3) and (4.4) we have

$$(\ell-1)\delta_\epsilon^{1/2} \le \arg\xi < (\ell+2)\delta_\epsilon^{1/2}. \tag{4.5}$$

Since $|\xi_0| \in J_i = [\alpha_i, \beta_i]$, by (4.1) it follows that

$$\alpha_i - 2\delta_\epsilon \le |\xi| \le \beta_i + 2\delta_\epsilon. \tag{4.6}$$

Also (4.1) and (4.2) imply that

$$|\eta - |\xi|| \le |\eta - \eta_0| + |\eta_0 - |\xi_0|| + ||\xi_0| - |\xi|| < \delta^{1-\epsilon} + \delta + \delta^{1-\epsilon} < 3\delta^{1-\epsilon} < 6\delta_\epsilon. \tag{4.7}$$

By (4.5) with $\ell \in \mathcal{N}_*^{(\delta_\epsilon)}$ and (4.6) we see that $(\xi, |\xi|) \in (U_{\ell,J_i}^{(\delta_\epsilon)})^*$, which combined with (4.7) will imply that $(\xi,\eta) \in (u_{\ell,J_i}^{(\delta_\epsilon)})^*$. This completes the proof of Lemma 4.4. ☐

The assumption that $\ell \in \mathcal{N}_*^{(\delta_\epsilon)}$ in Lemma 4.4 is always satisfied when δ is small.

Lemma 4.5. *If $\mu \in \mathcal{N}^{(\delta)}$, then there exist $\ell \in \mathcal{N}_*^{(\delta_\epsilon)}$ and $k \in \mathbb{Z}$ with $0 \le k \le N-1$ such that $\mu = \ell N + k$, where N is as in Lemma* 4.3.

Proof. We write $\mu = mN + k$, $m, k \in \mathbb{Z}$ with $0 \le k \le N-1$. Then

$$\begin{aligned}|m| &= N^{-1}|\mu - k| \le N^{-1}(\frac{\pi}{8}\delta^{-1/2} - 1 + k) \le N^{-1}(\frac{\pi}{8}\delta^{-1/2} + N - 2)\\ &= \frac{\pi}{8}\delta_\epsilon^{-1/2} + 1 - 2/N \le \frac{\pi}{7}\delta_\epsilon^{-1/2} - 1,\end{aligned}$$

if δ is small enough, which implies that $m \in \mathcal{N}_*^{(\delta_\epsilon)}$. ☐

Proof of Theorem 4.2. We may assume that δ is small enough. Let $\ell, \ell' \in \mathcal{N}_*^{(\delta_\epsilon)}$, $0 \le k, k' \le N-1$ and $\ell N + k, \ell' N + k' \in \mathcal{N}^{(\delta)}$. Then by Lemma 4.4 we have

$$\chi_{\widetilde{u}_{\ell N+k,J_1} + \widetilde{u}_{\ell' N+k',J_2}} \le \chi_{(u_{\ell,J_1}^{(\delta_\epsilon)})^* + (u_{\ell',J_2}^{(\delta_\epsilon)})^*}.$$

Therefore applying Lemma 4.5 and Theorem 4.1 with δ_ϵ in place of δ, we have

$$\begin{aligned}\sum_{(\mu,\nu)\in\mathcal{N}^2} \chi_{\widetilde{u}_{\mu,J_1}+\widetilde{u}_{\nu,J_2}} &\le \sum_{\ell,\ell'\in\mathcal{N}_*^{(\delta_\epsilon)}} \sum_{k,k'\in[0,N-1]\cap\mathbb{Z}} \chi_{\widetilde{u}_{\ell N+k,J_1}+\widetilde{u}_{\ell' N+k',J_2}}\\ &\le N^2 \sum_{(\ell,\ell')\in(\mathcal{N}_*^{(\delta_\epsilon)})^2} \chi_{(u_{\ell,J_1}^{(\delta_\epsilon)})^*+(u_{\ell',J_2}^{(\delta_\epsilon)})^*}\\ &\le CN^2\\ &\le C\delta^{-\epsilon}.\end{aligned}$$

This completes the proof of Theorem 4.2. ☐

References

[1] J. Bourgain, *Averages in the plane over convex curves and maximal operators*, J. Analyse Math. **47** (1986), 69–85.

[2] A. Carbery, *The boundedness of the maximal Bochner-Riesz operators on $L^4(\mathbb{R}^2)$*, Duke math. J. **50** (1983), 409–416.

[3] A. Carbery, *A weighted inequality for the maximal Bochner-Riesz operator on $\mathbb{R}^2$*, Trans. Amer. Math. Soc. **287** (1985), 673–680.

[4] L. Carleson and P. Sjölin, *Oscillatory integrals and a multiplier problem for the disc*, Studia Math. **44** (1972), 287–299.

[5] C. Fefferman, *A note on spherical summation multipliers*, Israel J. Math. **15** (1973), 44-52.

[6] C. D. Sogge, *Fourier Integrals in Classical Analysis*, Cambridge University Press, 1993.

[7] E. M. Stein and S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. **84** (1978), 1239–1295.

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