Nonisotropic dilations and the Calderón-Zygmund method of rotations with weight

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§1. Maximal functions and singular integrals along curves. Let $\{A_t\}_{t>0}$ be a dilation group on $\mathbb{R}^n$ defined by

$$\boxed{A_t = t^P = \exp((\log t)P),}$$

where P is an $n \times n$ real matrix whose eigenvalues have positive real parts. We assume $n \geq 2$. We can define **a norm function r** on $\mathbb{R}^n$ from $\{A_t\}_{t>0}$ such that

(1) $r(x) \geq 0, r(x) = r(-x)$ for all $x \in \mathbb{R}^n$, $r(x) = 0$ if and only if $x = 0$;

(2) r is continuous on $\mathbb{R}^n$ and infinitely differentiable in $\mathbb{R}^n \setminus \{0\}$;

(3) $r(A_t x) = tr(x)$ for all $t > 0$ and $x \in \mathbb{R}^n$;

(4) $r(x + y) \leq C(r(x) + r(y))$ for some $C > 0$;

(5) $\Sigma = \{\theta \in \mathbb{R}^n : \langle B\theta, \theta\rangle = 1\}$ for a positive symmetric matrix B, where $\Sigma = \{x \in \mathbb{R}^n : r(x) = 1\}$ and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$;

(6) $dx = t^{\gamma-1}\, d\mu\, dt$, that is,

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_0^\infty \int_\Sigma f(A_t\theta) t^{\gamma-1}\, d\mu(\theta)\, dt, \quad d\mu = \omega\, d\mu_0,$$

where ω is a strictly positive C^∞ function on Σ, $d\mu_0$ is the Lebesgue surface measure on Σ and $\gamma = \mathrm{trace}\, P$;

(7) there are positive constants $c_1, c_2, c_3, c_4, \alpha_1, \alpha_2, \beta_1$ and β_2 such that

$$c_1|x|^{\alpha_1} < r(x) < c_2|x|^{\alpha_2} \quad \text{if } r(x) \geq 1,$$
$$c_3|x|^{\beta_1} < r(x) < c_4|x|^{\beta_2} \quad \text{if } 0 < r(x) \leq 1.$$

For $t < 0$, define A_t by $A_t = (\mathrm{sgn}\, t)A_{|t|} = -A_{|t|}$.

Let S^{n-1} denote the unit sphere of $\mathbb{R}^n$. For $(x, \theta) \in \mathbb{R}^n \times S^{n-1}$, we define

$$Mf(x,\theta) = \sup_{h>0} h^{-1} \left| \int_0^h f(x - A_t\theta)\, dt \right|,$$

$$Hf(x,\theta) = \mathrm{p.v.} \int_{-\infty}^{\infty} f(x - A_t\theta)\, dt/t,$$

$$H_*f(x,\theta) = \sup_{0<\epsilon<R} \left| \int_{\epsilon<|t|<R} f(x - A_t\theta)\, dt/t \right|.$$

Let w be a weight function. We recall that

$$\|F\|_{L^p_w(L^q)} = \left(\int_{\mathbb{R}^n} \left(\int_{S^{n-1}} |F(x,\theta)|^q \, d\sigma(\theta)\right)^{p/q} w(x)\, dx\right)^{1/p}$$

for functions $F \in L^p_w(L^q(S^{n-1}))$, with usual modifications when $q = \infty$ or $p = \infty$, where $d\sigma$ denotes the Lebesgue surface measure on S^{n-1}.

Also, we write

$$\|f\|_{L^p_w} = \|f w^{1/p}\|_{L^p} = \|f w^{1/p}\|_p$$

for $f \in L^p_w(\mathbb{R}^n)$.

For $1 \le p, q \le \infty$, let $\Delta_{(p,q)} \subset [0,1] \times [0,1]$ be the interior of the convex hull of the points $(0,0)$, $(1,1)$, $(0,1)$, $(1/p, 1/q)$. Put

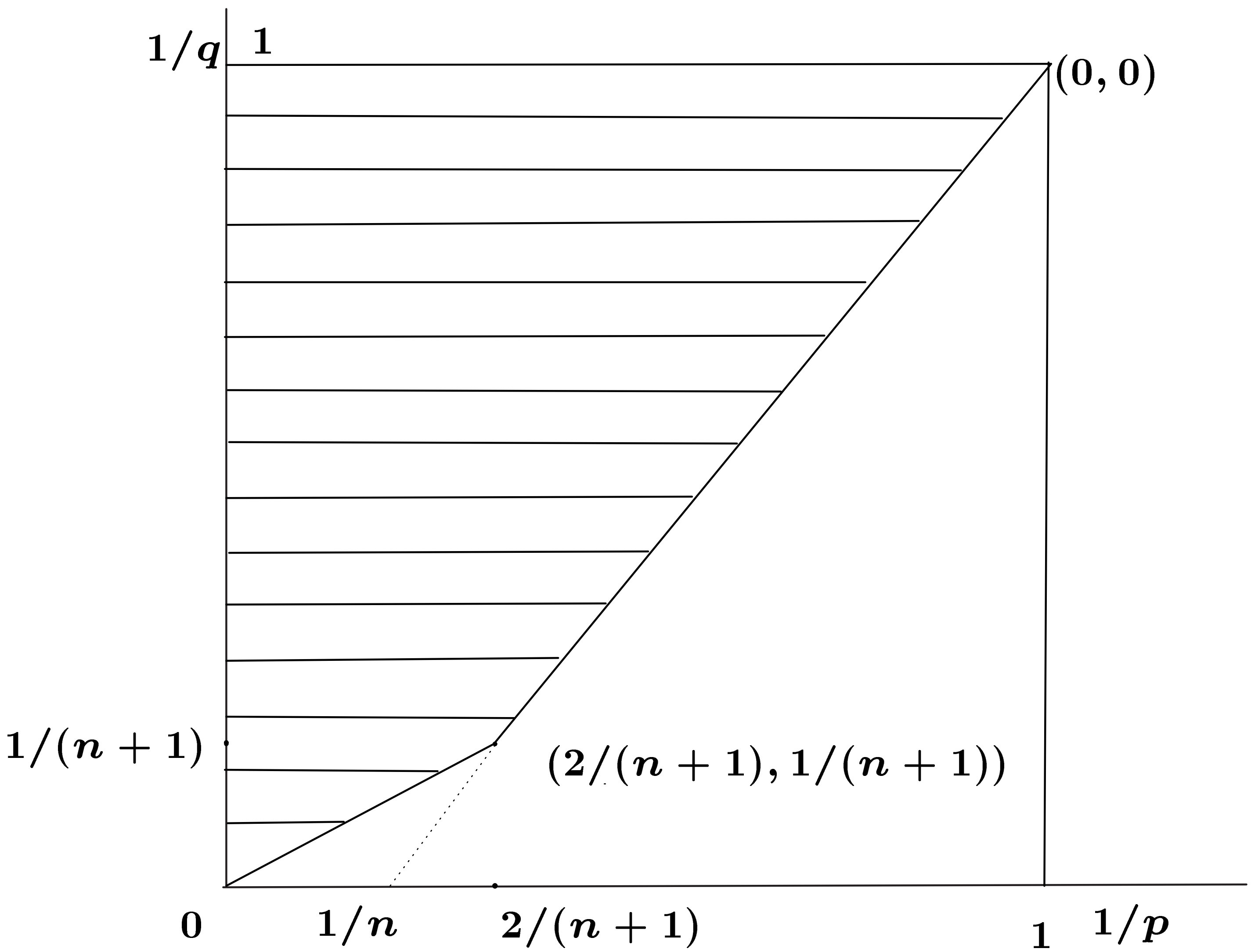
$$\boxed{q_n(p) = p(n-1)/(n-p), \qquad p_n = \max(2, (n+1)/2),}$$

$$\boxed{q_n(p) = \infty, \qquad p \ge n.}$$

Then, the following result was proved by M. Christ, J. Duoandikoetxea and J. L. Rubio de Francia (1986).

Theorem A. Suppose $P = E$ (the identity matrix). Let $(1/p, 1/q) \in \Delta_{(p_n, q_n(p_n))}$. Then,

$$M, H, H_* : L^p(\mathbb{R}^n) \to L^p(L^q).$$


$1/q$
1
$(0,0)$
$1/(n+1)$
$(2/(n+1), 1/(n+1))$
0
$1/n$
$2/(n+1)$
1
$1/p$

Theorem A for M, H was extended to the case of nonisotropic dilations by Bez (2008) as follows.

Theorem B.

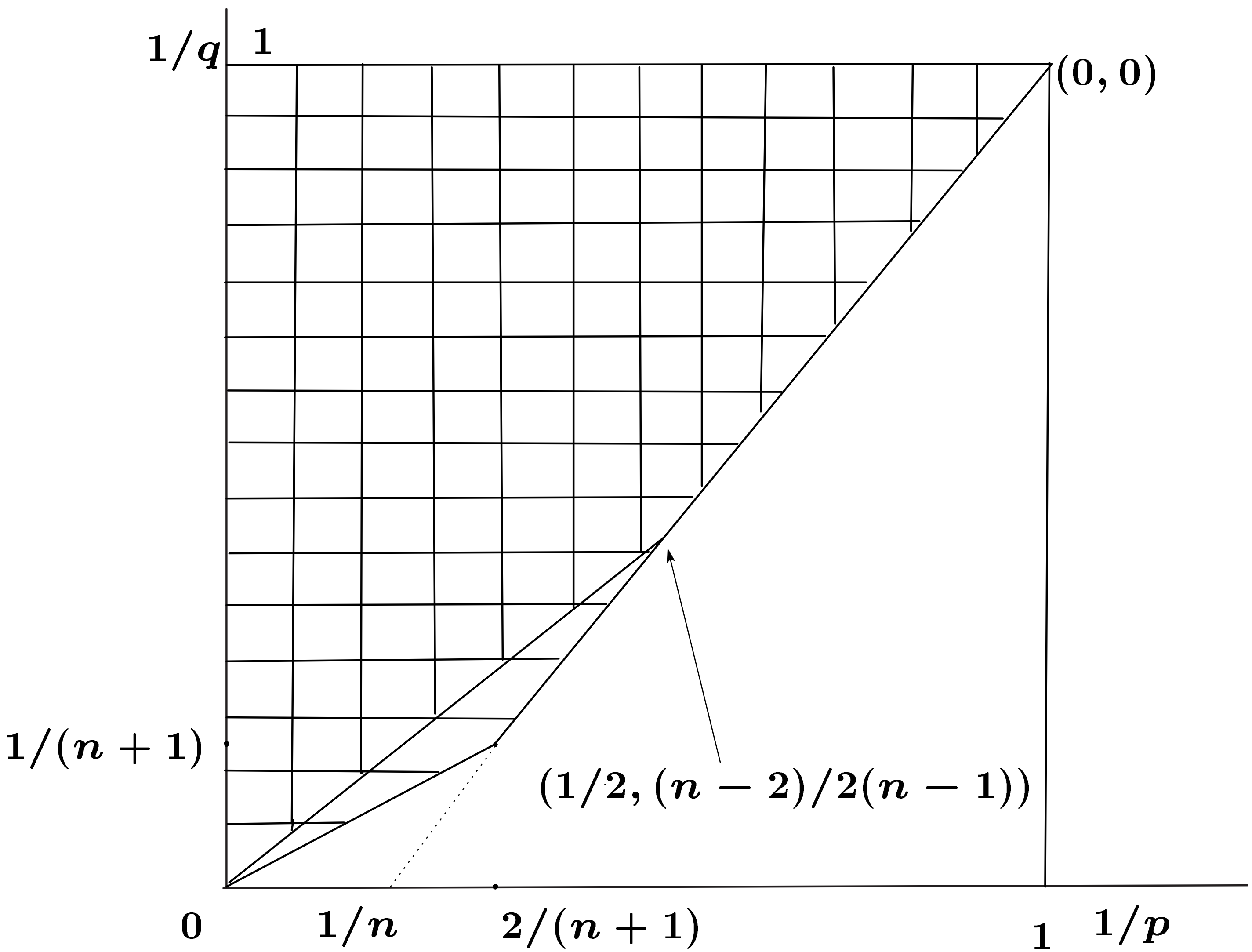
(1) **If** $(1/p, 1/q) \in \Delta_{(p_n, q_n(p_n))}$,

$$M : L^p(\mathbb{R}^n) \to L^p(L^q);$$

(2)

$$H : L^p(\mathbb{R}^n) \to L^p(L^q)$$

whenever $(1/p, 1/q) \in \Delta_{(2, q_n(2))}$.


$1/q$
1
$(0,0)$
$1/(n+1)$
$(1/2,(n-2)/2(n-1))$
0
$1/n$
$2/(n+1)$
1
$1/p$

We assume that $\Sigma = S^{n-1}$, $d\mu = \omega \, d\sigma$ and ω is even.

Theorem 1. Suppose that $(1/p, 1/q) \in \Delta_{(2,q_n(2))}$. Then,

$$H_* : L^p(\mathbb{R}^n) \to L^p(L^q).$$

We recall the following result for H_* shown by Lung-Kee Chen (1988).

Theorem C. Suppose $n = 2$ and $P = \mathrm{diag}(\alpha_1, \alpha_2)$ with $1 < \alpha_2/\alpha_1 < 4/3$. Then, H_* is bounded from $L^p(\mathbb{R}^2)$ to $L^p(L^q(S^1))$ whenever $(1/p, 1/q) \in \Delta_{(2,4)}$.

Theorem 1 improves on Theorem C, when $n = 2$. It is known that if H is bounded from L^p to $L^p(L^q)$, $p \in (1, \infty)$, then $q \leq q_n(p)$. This implies the same result for H_*. Thus, in particular, we can see that Theorem 1 is a sharp result when $n = 2$ (we note that $\Delta_{(2,q_2(2))} = \Delta_{(2,\infty)}$).

If B is a subset of $\mathbb{R}^n$ such that $B = \{x \in \mathbb{R}^n : r(x-a) < t\}$ for some $a \in \mathbb{R}^n$ and $t > 0$, then we call B an r-ball. Let w be a weight function on $\mathbb{R}^n$. For $1 \leq p < \infty$, we recall the Muckenhoupt class $\mathcal{A}_p$. We say $w \in \mathcal{A}_p$, $1 < p < \infty$, if

$$\boxed{\sup_B \left(|B|^{-1} \int_B w(x)\, dx\right) \left(|B|^{-1} \int_B w(x)^{-1/(p-1)} dx\right)^{p-1} < \infty,}$$

where the supremum is taken over all r-balls B. The class $\mathcal{A}_1$ is defined to be the set of weight functions w satisfying

$$\boxed{M_{HL} w \leq C w \text{ a.e.}}$$

where M_{HL} is the Hardy-Littlewood maximal operator defined by

$$M_{HL} g(x) = \sup_{t>0} t^{-\gamma} \int_{r(x-y)<t} |g(y)|\, dy.$$

We note that $\mathcal{A}_p \subset \mathcal{A}_u$ if $p \le u$. We shall prove the following weighted estimates.

Theorem 2. Let $2 \le q < q_n(2)$. Then,

$$M, H, H_* : L^2_w(\mathbb{R}^n) \to L^2_w(L^q)$$

for $w \in \mathcal{A}_1^\tau$, $\tau = 2(n-1)/q - n + 2$, where $\mathcal{A}_1^\tau$ is a subclass of $\mathcal{A}_1$ defined by $\mathcal{A}_1^\tau = \{v^\tau : v \in \mathcal{A}_1\}$.

By Stein-Wainger (1978) we know that the operator H_* is bounded from $L^p(\mathbb{R}^n)$ to $L^p(L^p)$, $1 < p < \infty$, so Theorem 1 follows by interpolation from the part of Theorem 2 concerning H_* with $w = 1$.

Theorem 2 implies in particular

$$\int_{S^{n-1}} \int_{\mathbb{R}^n} |Mf(x,\theta)|^2 |x|^\alpha \, dx \, d\sigma(\theta) \le C \int_{\mathbb{R}^n} |f(x)|^2 |x|^\alpha \, dx$$

for

$$-n < \alpha \le 0,$$

where

$$Mf(x,\theta) = \sup_{h>0} h^{-1} \left| \int_0^h f(x - t\theta) \, dt \right| .$$

Difference between the cases of isotropic dilation and nonisotropic dilation.

- In the case of nonisotropic dilations, to get necessary estimates for the maximal operator M near $(1/p_n, 1/q_n(p_n))$ we cannot apply a result for the X-ray transform X of M. Christ (1984) in the same way as in the case of nonisotropic dilation, where

$$X f(x,\theta) = \int_{-\infty}^{\infty} f(x, t\theta)\, dt, \qquad (x,\theta) \in S,$$

$$S = \{(x,\theta) \in \mathbb{R}^n \times S^{n-1} : \langle x, \theta \rangle = 0\};$$

then

$$X : L^{(n+1)/2}(\mathbb{R}^n) \to L^{n+1}(S, d\nu), \qquad d\nu(x,\theta) = d\lambda_\theta(x)\, d\sigma(\theta),$$

with $d\lambda_\theta(x)$ denoting $n-1$ dimensional Lebesgue measure on the hyperplane $\{x : \langle x, \theta \rangle = 0\}$.

- In the case of nonisotropic dilations, the maximal operator M cannot be used to control H, H_* as in the case of nonisotropic dilation;
a reason for this is that certain weighted inequalities which will be required in the arguments are not yet available in the case of nonisotropic dilations.

- In the case of nonisotropic dilations, to estimate the maximal operator M near $(1/p_n, 1/q_n(p_n))$, Bez (2008) applied a result of

P. Gressman, L^p-improving properties of X-ray like transforms, Math. Res. Lett. 13 (2006), 10001–10017.

- Bez (2008) proved certain estimates for trigonometric integrals by using the decay estimates for the Fourier transform of $d\sigma$.

- Consequently, Bez (2008) can prove boundedness of the maximal operator M in the same range of $(1/p, 1/q)$ as in the case of isotropic dilations.

§2. Applications.

Let $K(x, y)$ be a kernel on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$K(x, A_t y) = t^{-\gamma} K(x, y) \quad \text{for all } t > 0 \text{ and } (x, y) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}).$$

We assume that $K(x, y)$ is locally integrable with respect to y in $\mathbb{R}^n \setminus \{0\}$ and

$$\int_{a \le r(y) \le b} K(x, y)\, dy = 0 \quad \text{for all } a, b,\ 0 < a < b,$$

for every $x \in \mathbb{R}^n$. We consider the singular integral

$$Tf(x) = \mathrm{p.v.} \int K(x, y) f(x - y)\, dy = \lim_{\epsilon \to 0} \int_{r(y) \ge \epsilon} K(x, y) f(x - y)\, dy,$$

and the maximal singular integral

$$T_* f(x) = \sup_{\epsilon, R > 0} \left| \int_{\epsilon \le r(y) \le R} K(x, y) f(x - y)\, dy \right|.$$

We can apply Theorems 1 and 2 to study mapping properties of T and T_*.

Theorem 3. Let $(1/p, 1/q') \in \Delta_{(2,q_n(2))}$, $q' = q/(q-1)$. Suppose that $K(x,y)$ is odd in y, that is, $K(x,-y) = -K(x,y)$ for all $(x,y) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ and suppose that

$$\sup_{x\in\mathbb{R}^n} \left(\int_{S^{n-1}} |K(x,\theta)|^q \, d\sigma(\theta) \right)^{1/q} = \|K\|_{L^\infty(L^q)} < \infty.$$

Then, T_* is bounded on $L^p(\mathbb{R}^n)$.

This is an analogue for T_* of Theorem 12 of Bez (2008) concerning T. Theorem 3 is an extension to the case of nonisotropic dilations of a result due to M. Cowling and G. Mauceri (1985) for isotropic dilation.

Theorem 4. Let $2(n-1)/n < q \le 2$, $w \in \mathcal{A}_1$. Suppose that $\|K\|_{L^\infty(L^q)} < \infty$. Then, T and T_* are bounded on L^2_w, $w \in \mathcal{A}_1^\tau$, $\tau = n - 2(n-1)/q$.

Since $w^b \in \mathcal{A}_1$ for some $b > 1$ when $b \in \mathcal{A}_1$, from Theorem 4 we readily obtain the following result.

Corollary. Suppose that $\|K\|_{L^\infty(L^q)} < \infty$ for all $q < 2$. Then, T and T_* are bounded on L^2_w for all $w \in \mathcal{A}_1$.

Using this result and the extrapolation theorem of Rubio de Francia, we can obtain the L^p_w boundedness of T and T_* for $w \in \mathcal{A}_{p/2}$, $p \ge 2$.

Proof of Theorem 3.

The method of rotations of Calderón-Zygmund and Hölder's inequality imply

$$\begin{aligned} T_* f(x) &\le C \int_{S^{n-1}} |K(x,\theta)| H_* f(x,\theta)\, d\sigma(\theta) \\ &\le C \|K\|_{L^\infty(L^q(S^{n-1}))} \|H_* f(x,\cdot)\|_{L^{q'}}. \end{aligned}$$

Thus, the conclusion follows from Theorem 1.

Similarly, Theorem 4 follows from Theorem 2.

Remark 1.

Introducing nonisotropic Riesz transforms, we expect that Theorems $3, 4$ extend to the case where kernels $K(x, y)$ are even in y.

§3. $L^2_w(L^q)$ estimates for maximal functions.

§4. $L^2_w(L^q)$ estimates for H.

§5. $L^2_w(L^q)$ estimates for H_*.

Idea of proof.

Theory of Duoandikoetxea and Rubio de Francia (1986):

- **Orthogonality arguments with L^2 estimates via Fourier transform estimates and Plancherel's theorem for vector valued functions**
- **Sobolev embedding theorem**
- **Littlewood-Paley theory**
- **Interpolation arguments**

§3. $L^2_w(L^q)$ estimates for maximal functions.

In this section we prove

$$\|Mf\|_{L^2_w(L^q)} \le C\|f\|_{L^2_w}, \quad 2 \le q < q_n(2), \quad f \in \mathcal{S}(\mathbb{R}^n),$$

where $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class, and q and w are related as in Theorem 2.

We denote by $\hat{f}$ the Fourier transform of f:

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle}\, dx.$$

Let $\{D_k\}_{-\infty}^{\infty}$ be a sequence of non-negative functions in $C^\infty((0,\infty))$ such that

$$\text{supp}(D_k) \subset [2^{-k-1}, 2^{-k+1}], \quad \sum_k D_k(t)^2 = 1,$$

$$|(d/dt)^m D_k(t)| \le c_m/t^m \quad (m = 1, 2, \dots).$$

To apply the Littlewood-Paley theory, we define S_k by

$$\boxed{(S_k(f))\,\hat{}\,(\xi) = D_k(s(\xi))\hat{f}(\xi), \quad k \in \mathbb{Z},}$$

where $\mathbb{Z}$ denotes the set of integers, and the norm function $s(\xi)$ is coming from A_t^* (the adjoint). For $k \in \mathbb{Z}$, let

$$N_k f(x, \theta) = \int_{-\infty}^{\infty} f(x - A_t\theta)\varphi_k(t)\,dt - \int_{\mathbb{R}^n} f(x - y)\Phi_{2^k}(y)\,dy,$$

where $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi \geq 0$, $\mathrm{supp}(\varphi) \subset (1/2, 2)$, $\int \varphi\,dt = 1$, $\varphi_k(t) = 2^{-k}\varphi(2^{-k}t)$, and $\Phi \in C_0^\infty(\mathbb{R}^n)$, $\int \Phi\,dx = 1$. We define $f_t(x) = t^{-\gamma}f(A_t^{-1}x)$, $t > 0$.

Put $\tilde{S}_k = S_k^2$. Then, $\sum_k \tilde{S}_k f = f$. We may assume $f \geq 0$. We note that

$$
\begin{aligned}
Mf(x,\theta) &\leq C \sup_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} f(x - A_t\theta)\varphi_k(t)\,dt \\
&\leq C \sup_k |N_k f(x,\theta)| + C M_{HL} f(x) \\
&\leq C \sum_{j=-\infty}^{\infty} \left(\sum_k \left| N_k \tilde{S}_{j+k} f(x,\theta) \right|^q \right)^{1/q} + C M_{HL} f(x).
\end{aligned}
$$

Let $2 \leq q < q_n(2)$. Since $q \geq 2$, this and the Hardy-Littlewood maximal theorem imply

$$\begin{aligned}
\|Mf\|_{L^2_w(L^q)} &\leq C \sum_j \left\| \left(\sum_k \|N_k \tilde{S}_{j+k} f\|^q_{L^q(S^{n-1})} \right)^{1/q} \right\|_{L^2_w} + C\|f\|_{L^2_w} \\
&\leq C \sum_j \left\| \left(\sum_k \|N_k \tilde{S}_{j+k} f\|^2_{L^q(S^{n-1})} \right)^{1/2} \right\|_{L^2_w} + C\|f\|_{L^2_w},
\end{aligned}$$

for $w \in \mathcal{A}_2$.

We prove the following result.

Lemma 1. If $0 \leq \alpha < 1/2$, then

$$\left\| \left(\sum_k \| N_k \tilde{S}_{j+k} f \|^2_{L^2_\alpha(S^{n-1})} \right)^{1/2} \right\|_2 \leq C 2^{-\epsilon|j|} \| f \|_2 \quad \text{for some } \epsilon > 0.$$

Proof. Let $0 < a < 1/2 - \alpha$. It suffices to prove

$$\int_{\mathbb{R}^n} \|N_k \tilde{S}_{j+k} f(x, \cdot)\|_{L^2_\alpha}^2 \, dx \\ \leq C \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 |D_{k+j}(s(\xi))|^4 \min(|A^*_{2^k}\xi|, |A^*_{2^k}\xi|^{-1})^{2a} \, d\xi.$$

Note that $(N_k f(\cdot, \theta))\hat{}\,(\xi) = \hat{f}(\xi)\Psi(A^*_{2^k}\xi, \theta)$, where

$$\Psi(\xi, \theta) = \Psi_0(\xi, \theta) - \hat{\Phi}(\xi), \quad \Psi_0(\xi, \theta) = \int_{-\infty}^{\infty} \exp(-2\pi i\langle A_t\theta, \xi\rangle)\varphi(t) \, dt.$$

Therefore, it suffices to show the pointwise inequality

$$\|\Psi(\xi, \cdot)\|_{L^2_\alpha}^2 \leq C \min(|\xi|, |\xi|^{-1})^{2a}.$$

If $|\xi| \leq 1$, this is easily obtained, since $\Psi(\xi, \theta)$ is C^∞ and vanishes when $\xi = 0$. The estimate for $|\xi| > 1$ follows from the following result of Bez (2008).

Lemma 2. Let $0 < c_1 < c_2$ and $\xi \in \mathbb{R}^n$, $|\xi| > 1$. Then

$$\int_{S^{n-1}} \left| \int_{c_1}^{c_2} \exp\left(i\langle A_t\theta, \xi\rangle\right)\, dt \right|^2 d\sigma(\theta) \le C_\delta |\xi|^{-1+\delta}$$

for all $\delta > 0$.

On the other hand, we can easily see that

$$\|N_k f\|_{L^2_w(L^2)} \le C\|f\|_{L^2_{M_{HL}(w)}}.$$

Therefore, if $w \in \mathcal{A}_1$, by the Littlewood-Paley inequality we have

$$\left\| \left(\sum_k \|N_k \tilde{S}_{j+k} f\|^2_{L^2(S^{n-1})} \right)^{1/2} \right\|^2_{L^2_w} \le C \sum_k \|\tilde{S}_{j+k} f\|^2_{L^2_w} \le C\|f\|^2_{L^2_w}.$$

If $2 \leq q < q_n(2)$, then by the Sobolev embedding theorem we have

$$\boxed{L^2_\alpha(S^{n-1}) \subset L^q(S^{n-1}) \text{ for some } \alpha = \alpha(q) \in [0, 1/2).}$$

Thus, Lemma 1 implies

$$\left\| \left(\sum_k \| N_k \tilde{S}_{j+k} f \|^2_{L^q(S^{n-1})} \right)^{1/2} \right\|_2 \leq C 2^{-\epsilon|j|} \| f \|_2.$$

for some $\epsilon > 0$. By interpolation with change of measure, we get

$$\left\| \left(\sum_k \| N_k \tilde{S}_{j+k} f \|^2_{L^q(S^{n-1})} \right)^{1/2} \right\|_{L^2(w^\tau)} \leq C 2^{-\epsilon|j|} \| f \|_{L^2(w^\tau)}$$

for some $\epsilon > 0$, where q and τ are related as in Theorem 2. This implies the desired result.

Remark 2. Let $0 < c_1 < c_2$ and $\eta, \zeta \in \mathbb{R}^n \setminus \{0\}$. Then, we have

$$\left| \int_{c_1}^{c_2} \exp\left(i \langle A_t \eta, \zeta \rangle \right) \, dt \right| \le C \left| \langle P\eta, \zeta \rangle \right|^{-1/d}$$

for some positive constant C independent of η and ζ, where d is the degree of the minimal polynomial of P. We note that this result implies Lemma 2 when $d = 1, 2$.

§4. $L^2_w(L^q)$ estimates for H.

Let $2 \le q < q_n(2)$. In this section we prove

$$\|Hf\|_{L^2_w(L^q)} \le C\|f\|_{L^2_w}, \quad f \in \mathcal{S}(\mathbb{R}^n),$$

where q and w are related as in Theorem 2.

Decompose

$$Hf(x,\theta) = \sum_{k=-\infty}^{\infty} H_k f(x,\theta), \quad H_k f(x,\theta) = \int_{-\infty}^{\infty} f(x - A_t\theta)\psi_k(t)\,dt,$$

where $\psi_k(t) = 2^{-k}\psi(2^{-k}t)$, $\psi \in C_0^\infty(\mathbb{R})$, $\mathrm{supp}(\psi) \subset \{1/2 \le |t| \le 2\}$, $\int \psi(t)\,dt = 0$. We write

$$Hf = \sum_k H_k f = \sum_j U_j f, \quad U_j f = \sum_k H_k S^2_{j+k} f.$$

Let $0 \leq \alpha < 1/2$. We prove

$$\boxed{\|U_j f\|_{L^2(L^2_\alpha)} \leq C 2^{-\epsilon|j|}\|f\|_2,}$$

for some $\epsilon > 0$. Then, arguing as in the case of M, from this and the Sobolev embedding theorem we can get

$$\|U_j f\|_{L^2(L^q)} \leq C 2^{-\epsilon|j|}\|f\|_2, \quad 2 \leq q < q_n(2).$$

Let

$$\tilde{\Psi}(\xi, \theta) = \int_{-\infty}^{\infty} \exp(-2\pi i \langle A_t \theta, \xi \rangle) \psi(t)\, dt.$$

Then

$$(H_k f(\cdot, \theta))\hat{}\,(\xi) = \hat{f}(\xi) \tilde{\Psi}(A^*_{2^k} \xi, \theta).$$

If $0 < a < 1/2 - \alpha$, we have the estimate

$$\|\tilde{\Psi}(\xi, \cdot)\|^2_{L^2_\alpha} \leq C \min(|\xi|, |\xi|^{-1})^{2a}.$$

Therefore, by the Littlewood-Paley theory for vector valued functions,

$$\|U_j f\|_{L^2(L^2_\alpha)}^2 \le C \sum_k \|H_k S_{j+k} f\|_{L^2(L^2_\alpha)}^2$$
$$\le C \sum_k \int_{\mathbb{R}^n} |D_{j+k}(s(\xi))\hat{f}(\xi)|^2 \min(|A^*_{2^k}\xi|, |A^*_{2^k}\xi|^{-1})^{2a}\, d\xi,$$

where $0 < a < 1/2 - \alpha$. This implies

$$\|U_j f\|_{L^2(L^2_\alpha)}^2 \le C2^{-\epsilon|j|} \sum_k \int_{\mathbb{R}^n} |D_{j+k}(s(\xi))\hat{f}(\xi)|^2\, d\xi \le C2^{-\epsilon|j|}\|f\|_2^2$$

for some $\epsilon > 0$, as claimed.

If $w \in \mathcal{A}_1$, we can show that

$$\|H_k S_{j+k} f\|_{L^2_w(L^2)} \le C\|S_{j+k} f\|_{L^2_w}.$$

Thus, by the Littlewood-Paley inequality, we have

$$\begin{aligned}\|U_j f\|_{L^2_w(L^2)} &\le C\left(\sum_k \|H_k S_{j+k} f\|^2_{L^2_w(L^2)}\right)^{1/2} \\ &\le C\left(\sum_k \|S_{j+k} f\|^2_{L^2_w}\right)^{1/2} \le C\|f\|_{L^2_w}.\end{aligned}$$

Interpolation between the unweighted and weighted estimates implies

$$\|U_j f\|_{L^2_{w^\tau}(L^q)} \le C2^{-\epsilon|j|}\|f\|_{L^2_{w^\tau}}$$

for some $\epsilon > 0$, where q and τ are related as in Theorem 2. Using this and the

triangle inequality, we can see that

$$\|Hf\|_{L^2_w(L^q)} \le \sum_j \|U_j f\|_{L^2_w(L^q)} \le C \sum_j 2^{-\epsilon|j|}\|f\|_{L^2_w} \le C\|f\|_{L^2_w}.$$

§5. $L^2_w(L^q)$ estimates for H_*.

Let q, w be as in Theorem 2. In this section we prove

$$\|H_* f\|_{L^2_w(L^q)} \le C\|f\|_{L^2_w}, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

Lemma 3. Let

$$H_{**}f(x,\theta) = \sup_{N\in\mathbb{Z}} \left| \sum_{k=N}^{\infty} H_k f(x,\theta) \right|.$$

Then

$$\|H_{**} f\|_{L^2_w(L^q)} \le C\|f\|_{L^2_w}.$$

We need the following result, for $p \le q$, to show Lemma 3.

Lemma 4. Let $1 < p < \infty$, $1 < q \leq \infty$, $w \in \mathcal{A}_p$. For functions $F(x, \theta)$ on $\mathbb{R}^n \times S^{n-1}$, define $(M_{HL}F)(x, \theta) = (M_{HL}F(\cdot, \theta))(x)$. Then

$$\|M_{HL}F\|_{L^p_w(L^q)} \leq C\|F\|_{L^p_w(L^q)}.$$

Proof of Lemma 3. Let $\hat{Q} \in C_0^\infty(\mathbb{R}^n)$, $\mathrm{supp}(\hat{Q}) \subset \{s(\xi) < 2\}$, $\hat{Q}(\xi) = 1$ if $s(\xi) < 1$. Decompose

$$\sum_{k=N}^{\infty} H_k f = Q_{2^N} * Hf - Q_{2^N} * \sum_{k=-\infty}^{N-1} H_k f + (\delta - Q_{2^N}) * \sum_{k=N}^{\infty} H_k f,$$

where δ denotes the delta function and the convolution is taken with respect to the x variable.

The first term on the right hand side can be handled by Theorem 2 for H and

Lemma 4 as follows:

$$\left\| \sup_N \left| Q_{2N} * Hf \right| \right\|_{L^2_w(L^q)} \le C \left\| M_{HL} Hf \right\|_{L^2_w(L^q)} \le C \|Hf\|_{L^2_w(L^q)} \le C \|f\|_{L^2_w}.$$

Also, by inspection we see that

$$\sup_N \left| Q_{2N} * \sum_{k=-\infty}^{N-1} H_k f(x,\theta) \right| \le C M_{HL} f(x)$$

with the constant C independent of θ. Therefore, the second term on the right hand side can be handled by the weighted norm inequality for the Hardy-Littlewood maximal operator.

It remains to estimate

$$I(f) = \sup_N \left| (\delta - Q_{2N}) * \sum_{k=N}^{\infty} H_k f \right|.$$

We note that

$$I(f) \le \sum_{j=0}^{\infty} I_j(f), \quad I_j(f) = \sup_{N\in\mathbb{Z}} \left|(\delta - Q_{2^N}) * H_{N+j} f\right|.$$

Let $0 \le \alpha < 1/2$ and $0 < a < 1/2 - \alpha$. Then, we have

$$\left\|(\delta - Q_{2^N}) * H_{N+j} f\right\|^2_{L^2(L^2_\alpha)} \le C \int_{\mathbb{R}^n} |(1 - \hat{Q}(A^*_{2^N}\xi))\hat{f}(\xi)|^2 |A^*_{2^{N+j}}\xi|^{-2a}\, d\xi.$$

Therefore,

$$\sum_N \left\|(\delta - Q_{2^N}) * H_{N+j} f\right\|^2_{L^2(L^2_\alpha)} \le C 2^{-j\epsilon} \|f\|_2^2$$

and hence, if $2 \le q < q_n(2)$, the Sobolev embedding theorem implies

$$\sum_N \left\|(\delta - Q_{2^N}) * H_{N+j} f\right\|^2_{L^2(L^q)} \le C 2^{-j\epsilon} \|f\|_2^2.$$

We write

$$\delta - Q_{2N} = \sum_{m \le N} \Delta_m, \quad \hat{\Delta}_m(\xi) = \Gamma(A^*_{2^m}\xi),$$

where $\Gamma \in C_0^\infty$, $\mathrm{supp}(\Gamma) \subset \{c_1 < s(\xi) < c_2\}$ for some $c_1, c_2 > 0$.
Then, by Plancherel's theorem we have

$$\|\Delta_m * H_{N+j} f\|^2_{L^2(L^2)} \le C 2^{-\epsilon(N-m+j)} \|f\|_2^2.$$

On the other hand, if $w \in \mathcal{A}_1$,

$$\|\Delta_m * H_{N+j} f\|^2_{L^2_w(L^2)} \le C \|f\|^2_{L^2_w}.$$

For $w \in \mathcal{A}_1$, choose $b > 1$ such that $w^b \in \mathcal{A}_1$. Then, interpolating between these estimates with w^b in place of w, we get

$$\|\Delta_m * H_{N+j} f\|^2_{L^2_w(L^2)} \le C 2^{-\epsilon(N-m+j)} \|f\|^2_{L^2_w}$$

for some $\epsilon > 0$.

Choose $G_m \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{G}_m(\xi) = F(A^*_{2^m}\xi)$, $F \in C_0^\infty$, $\mathrm{supp}(F) \subset \{d_1 < s(\xi) < d_2\}$ for some $d_1, d_2 > 0$, and

$$\Delta_m * G_m * f = \Delta_m * f.$$

Then, by Littlewood-Paley inequality

$$\begin{aligned}
\sum_{N\in\mathbb{Z}} \left\|(\delta - Q_{2^N}) * H_{N+j} f\right\|^2_{L^2_w(L^2)} &\leq C \sum_{N\in\mathbb{Z}} \sum_{m\leq N} \|\Delta_m * H_{N+j} f\|^2_{L^2_w(L^2)} \\
&\leq C \sum_{N\in\mathbb{Z}} \sum_{m\leq N} 2^{-\epsilon(N-m+j)} \|G_m * f\|^2_{L^2_w} \\
&\leq C \sum_{m\in\mathbb{Z}} 2^{-j\epsilon} \|G_m * f\|^2_{L^2_w} \\
&\leq C 2^{-j\epsilon} \|f\|^2_{L^2_w}.
\end{aligned}$$

Interpolation between this and $L^2(L^q)$ estimate implies

$$\sum_N \left\|(\delta - Q_{2^N}) * H_{N+j} f\right\|^2_{L^2_w(L^q)} \le C 2^{-j\epsilon} \|f\|^2_{L^2_w}$$

for some $\epsilon > 0$, where q, w are as in Theorem 2. Since

$$I_j(f) \le \left(\sum_{N\in\mathbb{Z}} \left|(\delta - Q_{2^N}) * H_{N+j} f\right|^q\right)^{1/q}$$

and $q \geq 2$, we have

$$\|I(f)\|_{L^2_w(L^q)} \leq \sum_{j=0}^{\infty} \left\| \left(\sum_{N\in\mathbb{Z}} \|(\delta - Q_{2^N}) * H_{N+j} f\|^q_{L^q(S^{n-1})} \right)^{2/q} \right\|^{1/2}_{L^1_w}$$

$$\leq \sum_{j=0}^{\infty} \left(\sum_{N\in\mathbb{Z}} \|(\delta - Q_{2^N}) * H_{N+j} f\|^2_{L^2_w(L^q)} \right)^{1/2}$$

$$\leq C \sum_{j=0}^{\infty} 2^{-j\epsilon/2} \|f\|_{L^2_w} \leq C\|f\|_{L^2_w},$$

where q, w are as in Theorem 2. This completes the proof of Lemma 3.

Proof of Theorem 2 for H_*. We can easily prove the pointwise inequality

$$H_* f(x, \theta) \leq C H_{**} f(x, \theta) + C M f(x, \theta) + C M f(x, -\theta).$$

Therefore, the result for H_* follows from Lemma 3 and the result for M.

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THANK YOU !