

# **Estimates for singular integrals and maximal singular integrals associated with nonisotropic dilations**

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We consider two kinds of convolution on  $\mathbb{R}^n$ :

- convolution associated with Euclidean space structure
- convolution associated with homogeneous group structure.

- Convolution associated with homogeneous group structure.

**Singular integrals on a homogeneous group:**

$$\begin{aligned} T(f)(x) &= \text{p.v.} \int f(y)L(y^{-1}x) dy \\ &= \lim_{\epsilon \rightarrow 0} \int_{r(y^{-1}x) > \epsilon} f(y)L(y^{-1}x) dy, \end{aligned}$$

**with rough kernels:**

$$L(x) = h(r(x))K(x),$$

**$K$  is homogeneous of degree  $-\gamma$  with respect to dilations  $A_t$ ,  $t > 0$ ,**

$$K(A_tx) = t^{-\gamma} K(x), \quad t > 0, \quad x \neq 0,$$

$$\begin{aligned} A_t x &= (t^{a_1}x_1, t^{a_2}x_2, \dots, t^{a_n}x_n), \quad x = (x_1, \dots, x_n), \\ 0 < a_1 &\leq a_2 \leq \dots \leq a_n, \quad \gamma = a_1 + \dots + a_n. \end{aligned}$$

**Also, we consider a maximal singular integral**

$$T_* f(x) = \sup_{N, \epsilon > 0} \left| \int_{\epsilon < r(y^{-1}x) < N} f(y) L(y^{-1}x) dy \right|.$$

**We prove  $L^p$  and weighted  $L^p$  boundedness of  $T$  and  $T_*$  under a sharp condition for the kernel**

- Convolution associated with Euclidean space structure.

Let  $\{A_t\}_{t>0}$ ,  $A_t = t^P = \exp((\log t)P)$ , be a dilation group on  $\mathbb{R}^n$ , where  $P$  is an  $n \times n$  real matrix whose eigenvalues have positive real parts.

Define

$$T(f)(x) = \text{p.v.} \int f(y) K(x - y) dy,$$

where  $K$  is homogeneous of degree  $-\gamma$ ,  $\gamma = \text{trace } P$ , with respect to dilations  $A_t$ . We prove weak type  $(1, 1)$  estimates for  $T$  on  $\mathbb{R}^2$  under the  $L \log L$  condition on the kernel.

## §1. $\mathbb{R}^n$ as a homogeneous group

## §2. Results for $L^p$ estimates for $T$ and $T_*$

## §3. Results for weighted $L^p$ estimates for $T$ and $T_*$

## §4. A basic $L^2$ estimates

## §5. Sketch of proof for $L^p$ estimates of $T$

## §6. Weak type $(1, 1)$ estimates on $\mathbb{R}^2$ .

## §1. $\mathbb{R}^n$ as a homogeneous group.

$\mathbb{R}^n$ : the  $n$  dimensional Euclidean space,  $n \geq 2$ .

We regard  $\mathbb{R}^n$  as a homogeneous group:

- multiplication is given by a polynomial mapping;
- $\exists \{A_t\}_{t>0}$ : a dilation family on  $\mathbb{R}^n$  such that

$$A_t x = (t^{a_1} x_1, t^{a_2} x_2, \dots, t^{a_n} x_n),$$

$x = (x_1, \dots, x_n)$  and  $0 < a_1 \leq a_2 \leq \dots \leq a_n$ ,

$A_t$  is an automorphism of the group structure;

- Lebesgue measure is a bi-invariant Haar measure;
- the identity is the origin  $0$ ,  $x^{-1} = -x$ .

We also write  $\mathbb{R}^n = \mathbb{H}$ .

Multiplication  $xy$  satisfies

(1)  $(ux)(vx) = ux + vx, x \in \mathbb{H}, u, v \in \mathbb{R};$

(2)

$$A_t(xy) = (A_tx)(A_ty), x, y \in \mathbb{H}, t > 0;$$

(3) if  $z = xy, z = (z_1, \dots, z_n)$ , then  $z_k = P_k(x, y),$

$$P_1(x, y) = x_1 + y_1,$$

$$P_k(x, y) = x_k + y_k + R_k(x, y) \text{ for } k \geq 2,$$

where  $R_k(x, y)$  is a polynomial of degree greater than 1 depending only on  $x_1, \dots, x_{k-1}, y_1, \dots, y_{k-1}$ .

$|x|$ : the Euclidean norm for  $x \in \mathbb{R}^n$ ,

$r(x)$ : a norm function satisfying  $r(A_t x) = t r(x)$ ,  $\forall t > 0$ ,  $\forall x \in \mathbb{R}^n$ ;

(1)  $r$  is continuous on  $\mathbb{R}^n$  and smooth in  $\mathbb{R}^n \setminus \{0\}$ ;

(2)  $r(x+y) \leq C_0(r(x) + r(y))$ ,  $r(xy) \leq C_0(r(x) + r(y))$

for some  $C_0 \geq 1$ ;

(3)  $r(x^{-1}) = r(x)$ ;

(4) If  $\Sigma = \{x \in \mathbb{R}^n : r(x) = 1\}$ , then  $\Sigma = S^{n-1}$ ,

where  $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ ;

(5)  $\exists c_1, c_2, c_3, c_4, \alpha_1, \alpha_2, \beta_1, \beta_2 > 0$  such that

$$c_1|x|^{\alpha_1} \leq r(x) \leq c_2|x|^{\alpha_2} \quad \text{if } r(x) \geq 1,$$

$$c_3|x|^{\beta_1} \leq r(x) \leq c_4|x|^{\beta_2} \quad \text{if } r(x) \leq 1.$$

- The space  $\mathbb{H}$  with a left invariant quasi-metric  $d(x, y) = r(x^{-1}y)$  is a space of homogeneous type.
- Let  $\gamma = a_1 + \cdots + a_n$ . Then,  $dx = t^{\gamma-1} dS dt$ , that is,

$$\int_{\mathbb{R}^n} f(x) dx = \int_0^\infty \int_{\Sigma} f(A_t \theta) t^{\gamma-1} dS(\theta) dt$$

where  $dS = \omega dS_0$ ,  $\omega$  is a positive  $C^\infty$  function on  $\Sigma$  and  $dS_0$  is the Lebesgue surface measure on  $\Sigma$ .

## Convolution

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(y^{-1}x) dy$$

- $(f * g) * h = f * (g * h)$
- $(f * g)^{\sim} = \tilde{g} * \tilde{f}$  if  $\tilde{f}(x) = f(x^{-1})$ .
- **Euclidean convolution**

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(x - y) dy.$$

An example.

Heisenberg group  $\mathbb{H}_1$ .

$$(x, y, u)(x', y', u') = (x + x', y + y', u + u' + (xy' - yx')/2),$$

$$(x, y, u), (x', y', u') \in \mathbb{R}^3,$$

then  $\mathbb{R}^3$  with this group law is the Heisenberg group  $\mathbb{H}_1$ ; a dilation is defined by

$$A_t(x, y, u) = (tx, ty, t^2u),$$

and a norm function is

$$r(x, y, u) = \frac{1}{\sqrt{2}} \sqrt{\sqrt{(x^2 + y^2)^2 + 4u^2} + x^2 + y^2}.$$

Also, we can adopt

$$A'_t(x, y, u) = (tx, t^2y, t^3u).$$

## §2. Results for $L^p$ estimates for $T$ and $T_*$ .

**Definition.**

- $F \in L \log L(\Sigma)$  (Zygmund class)

$\iff$

$$\int_{\Sigma} |F(x)| \log(2 + |F(x)|) dS(x) < \infty.$$

- $F \in L^q(\Sigma) \iff \|F\|_q = \left( \int_{\Sigma} |F|^q dS \right)^{1/q} < \infty.$

**Definition**  $d_s = \{h \text{ on } \mathbb{R}_+ : \|h\|_{d_s} < \infty\},$

$$\|h\|_{d_s} = \sup_{j \in \mathbb{Z}} \left( \int_{2^j}^{2^{j+1}} |h(t)|^s dt / t \right)^{1/s},$$

$\mathbb{Z}$  : the set of integers,  $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\};$

$d_\infty = L^\infty(\mathbb{R}_+).$

•  $s > t \implies d_s \subset d_t.$

**Put for  $t \in (0, 1]$ ,**

$$\omega(h, t) = \sup_{|s| < tR/2} \int_R^{2R} |h(r - s) - h(r)| dr / r,$$

**where the supremum is taken over all  $s$  and  $R$  such that  $|s| < tR/2$ .**

**Definition.** For  $\eta > 0$ , let  $\Lambda^\eta$  denote the family of functions  $h$  such that

$$\|h\|_{\Lambda^\eta} = \sup_{t \in (0, 1]} t^{-\eta} \omega(h, t) < \infty.$$

**Define a space  $\Lambda_s^\eta = d_s \cap \Lambda^\eta$  and set**

$$\|h\|_{\Lambda_s^\eta} = \|h\|_{d_s} + \|h\|_{\Lambda^\eta} \quad \text{for } h \in \Lambda_s^\eta.$$

- $\Lambda_s^{\eta_1} \subset \Lambda_s^{\eta_2}$  if  $\eta_2 \leq \eta_1$ ,  $\Lambda_{s_1}^\eta \subset \Lambda_{s_2}^\eta$  if  $s_2 \leq s_1$ .

**Definition.** Let  $\Lambda$  denote the collection of functions  $h$  on  $\mathbb{R}_+$  such that

$$h = \sum_{k=1}^{\infty} a_k h_k$$

for some functions  $h_k \in \Lambda_{1+1/k}^{1/(k+1)}$  and a sequence  $\{a_k\}$  of non-negative real numbers satisfying

$$\sup_{k \geq 1} \|h_k\|_{\Lambda_{1+1/k}^{1/(k+1)}} < \infty, \quad \sum_{k=1}^{\infty} k a_k < \infty.$$

Let  $\Omega$  be locally integrable in  $\mathbb{R}^n \setminus \{0\}$  and homogeneous of degree 0 with respect to the dilation group  $\{A_t\}$ , that is,

$$\Omega(A_t x) = \Omega(x) \quad \text{for } x \neq 0, t > 0.$$

We assume that

$$\int_{\Sigma} \Omega(\theta) dS(\theta) = 0.$$

Let

$$K(x) = \Omega(x') r(x)^{-\gamma}, \quad x' = A_{r(x)^{-1}} x \text{ for } x \neq 0,$$

where  $\gamma = a_1 + \cdots + a_n$ . Then  $K$  is a locally integrable function on  $\mathbb{R}^n \setminus \{0\}$  and

$$K(A_t x) = t^{-\gamma} K(x)$$

for all  $t > 0$  and  $x \in \mathbb{R}^n \setminus \{0\}$ .

Let

$$Tf(x) = \text{p.v.} f * L(x) = \text{p.v.} \int_{\mathbb{R}^n} f(y)L(y^{-1}x) dy,$$

where  $L(x) = h(r(x))K(x)$ ,  $h \in d_1$ .

**Theorem 1.** Let  $s > 1$ . Suppose that  $\Omega \in L^s(\Sigma)$  and  $h \in \Lambda_s^{\eta/s'}$  for some fixed positive number  $\eta$ . Then, if  $1 < p < \infty$ ,

$$\|Tf\|_p \leq C_p s(s-1)^{-1} \|h\|_{\Lambda_s^{\eta/s'}} \|\Omega\|_s \|f\|_p,$$

where the constant  $C_p$  is independent of  $s$ ,  $\Omega$  and  $h$ .

Extrapolation of Yano using Theorem 1 implies the following result.

**Theorem 2.** Suppose that  $h \in \Lambda$  and  $\Omega \in L \log L(\Sigma)$ . Then,  $T$  is bounded on  $L^p(\mathbb{R}^n)$  for all  $p \in (1, \infty)$ .

When  $h = 1$ , this is due to T. Tao 1999.

Recall

$$T_* f(x) = \sup_{N, \epsilon > 0} \left| \int_{\epsilon < r(y^{-1}x) < N} f(y) L(y^{-1}x) dy \right|.$$

**Theorem 3.** Let  $s > 1$ . Suppose that  $\Omega \in L^s(\Sigma)$  and  $h \in \Lambda_s^{\eta/s'}$  for some fixed positive number  $\eta$ . Then we have

$$\|T_* f\|_p \leq C_p s(s-1)^{-1} \|h\|_{\Lambda_s^{\eta/s'}} \|\Omega\|_s \|f\|_p$$

for all  $p \in (1, \infty)$ , where  $C_p$  is independent of  $s$ ,  $h$  and  $\Omega$ .

Extrapolation of Yano using Theorem 3 implies the following result.

**Theorem 4.** Suppose that  $\Omega \in L \log L(\Sigma)$  and  $h \in \Lambda$ . Then,

$$T_* : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n), \quad \forall p \in (1, \infty).$$

**When  $h = 1$ , Theorem 2 can be proved by interpolation between  $L^2$  estimates and weak  $(1, 1)$  estimates given by Tao 1999.**

**For  $T_*$  with  $\Omega \in L \log L$ , weak  $(1, 1)$  boundedness is yet to be proved even in the case when  $h = 1$ .**

Let  $\{A_t\}_{t>0}$ ,  $A_t = t^P = \exp((\log t)P)$ , be a dilation group on  $\mathbb{R}^n$ , where  $P$  is an  $n \times n$  real matrix whose eigenvalues have positive real parts. Let  $K$  be a locally integrable function on  $\mathbb{R}^n \setminus \{0\}$  such that  $K(A_t x) = t^{-\gamma} K(x)$ ,  $\gamma = \text{trace } P$ , for all  $t > 0$  and  $x \in \mathbb{R}^n \setminus \{0\}$ . Let  $\Omega$  be locally integrable in  $\mathbb{R}^n \setminus \{0\}$  and homogeneous of degree 0 with respect to the dilation group  $\{A_t\}$ , that is,

$$\Omega(A_t x) = \Omega(x) \quad \text{for } x \neq 0, t > 0$$

We assume that

$$K(x) = \Omega(x') r(x)^{-\gamma}, \quad x' = A_{r(x)^{-1}} x \text{ for } x \neq 0,$$

$$\int_{\Sigma} \Omega(\theta) dS(\theta) = 0.$$

Let  $L(x) = h(r(x))K(x)$ . If we define

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} f(y)L(x-y) dy,$$

using Euclidean convolution, then we can apply a method of Duoandikoetxea and Rubio de Francia via Fourier transform estimates to prove the following:

**Theorem A.** Suppose  $\Omega \in L \log L(\Sigma)$  and

$$\sup_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} |h(r)| (\log(2 + |h(r)|))^a dr / r < \infty$$

for some  $a > 2$ . Then  $T$  is bounded on  $L^p(\mathbb{R}^n)$  for all  $p \in (1, \infty)$ .

**Theorem B (Y. Chen, Y. Ding and D. Fan 2008).** Let

$$A_t x = (t^{a_1} x_1, t^{a_2} x_2, \dots, t^{a_n} x_n),$$

where  $x = (x_1, \dots, x_n)$  and  $0 < a_1 \leq a_2 \leq \dots \leq a_n$ . Suppose that  $h = 1$  and  $\Omega \in H^1(\Sigma)$ . Then  $T$  is bounded on  $L^p(\mathbb{R}^n)$  for all  $p \in (1, \infty)$ .

### §3. Results for weighted $L^p$ estimates for $T$ and $T_*$ .

#### Definition.

- $B = B(a, s)$  is called a ball in  $\mathbb{H}$  with center  $a$  and radius  $s$

$$\iff$$

$$B = \{x \in \mathbb{H} : r(a^{-1}x) < s\}$$

for  $a \in \mathbb{H}$  and  $s > 0$ .

- $w \in A_p$ ,  $1 < p < \infty$  (Muckenhoupt class on  $\mathbb{H}$ )

$$\iff$$

$$\sup_B \left( |B|^{-1} \int_B w(x) dx \right) \left( |B|^{-1} \int_B w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty.$$

•  $w \in A_1 \iff Mw \leq Cw$  a.e.

where  $M$  denotes the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{x \in B} |B|^{-1} \int_B |f(y)| dy.$$

**Theorem 5.** Let  $q > 1$ . Suppose that

$$\Omega \in L^q(\Sigma), \quad h \in \Lambda_q^\eta \text{ for some } \eta > 0.$$

Let  $1 < p < \infty$ . Then,

- (1)  $T$  and  $T_*$  are bounded on  $L^p(w)$  if  $q' \leq p < \infty$  and  $w \in A_{p/q'}$ ,  
 $q' = q/(q - 1)$ ;
- (2) if  $1 < p \leq q$  and  $w \in A_{p'/q'}$ ,  $T$  and  $T_*$  are bounded on  $L^p(w^{1-p})$ .

In the Euclidean convolution case, where Fourier transform estimates are available, this was proved independently by

J. Duoandikoetxea, 1993,

D. Watson, 1990.

## §4. A basic $L^2$ estimate.

Let  $\psi_j \in C_0^\infty(\mathbb{R})$ ,  $j \in \mathbb{Z}$ , be such that

$$\text{supp}(\psi_j) \subset \{t \in \mathbb{R} : 2^j \leq t \leq 2^{j+2}\}, \quad \psi_j \geq 0,$$

$$\sum_{j \in \mathbb{Z}} \psi_j(t) = 1 \quad \text{for } t \neq 0,$$

$$|(d/dt)^m \psi_j(t)| \leq c_m |t|^{-m} \quad \text{for } m = 0, 1, 2, \dots$$

Let

$$S_j L(x) = (\log 2)^{-1} h(r(x)) \int_0^\infty \psi_j(t) \delta_t K_0(x) dt/t, \quad \delta_t K_0(x) = t^{-\gamma} K_0(A_t^{-1}x),$$

$$K_0(x) = K(x) \chi_{D_0}(x), \quad D_0 = \{x \in \mathbb{R}^n : 1 \leq r(x) \leq 2\}.$$

Then

$$\sum_{j \in \mathbb{Z}} S_j L = L, \quad Tf = \sum_{j \in \mathbb{Z}} f * S_j L.$$

$\phi$ : a  $C^\infty$  function,  $\text{supp}(\phi) \subset B(0, 1) \setminus B(0, 1/2)$ ,  $\int \phi = 1$ ,  
 $\phi(x) = \tilde{\phi}(x)$ ,  $\phi(x) \geq 0 \forall x \in \mathbb{R}^n$ , where  $\tilde{\phi}(x) = \phi(x^{-1})$ .

Define

$$\Delta_k = \delta_{2k-1}\phi - \delta_{2k}\phi, \quad k \in \mathbb{Z},$$

where

$$\delta_t\phi(x) = t^{-\gamma}\phi(A_t^{-1}x).$$

**Lemma 1.** Let  $s > 1$ ,  $h \in \Lambda_s^{\eta/s'}$ ,  $\Omega \in L^s(\Sigma)$ . Then,

$$\|f * S_j L * \Delta_{k+j}\|_2 \leq C 2^{-\epsilon|k|/s'} \|h\|_{\Lambda_s^{\eta/s'}} \|\Omega\|_s \|f\|_2.$$

If  $h = 1$  and  $s = \infty$ , this was proved by T. Tao 1999.

## §5. Sketch of proof for $L^p$ estimates of $T$ .

Sketch of proof for  
Theorem 1'.

$$\|Tf\|_p \leq C_s \|h\|_{\Lambda_s^{\eta/s'}} \|\Omega\|_s \|f\|_p.$$

Theory of Duoandikoetxea and Rubio de Francia (1986):

- Orthogonality arguments for  $L^2$  estimates via Fourier transform estimates and Plancherel's theorem
- Littlewood-Paley theory
- Interpolation arguments

Our strategy is:

to employ a version of theory of Duoandikoetxea and Rubio de Francia adapted for the present situation;

replace use of Fourier transform estimates with Lemma 1 and apply Cotlar's lemma.

## Littlewood-Paley inequalities.

Recall

$\phi$ : a  $C^\infty$  function,  $\text{supp}(\phi) \subset B(0, 1) \setminus B(0, 1/2)$ ,  $\int \phi = 1$ ,  
 $\phi(x) = \tilde{\phi}(x)$ ,  $\phi(x) \geq 0 \forall x \in \mathbb{R}^n$ , where  $\tilde{\phi}(x) = \phi(x^{-1})$ ,

$$\Delta_k = \delta_{2k-1}\phi - \delta_{2k}\phi, \quad k \in \mathbb{Z},$$

where

$$\delta_t \phi(x) = t^{-\gamma} \phi(A_t^{-1}x).$$

Then  $\Delta_k = \tilde{\Delta}_k$ ,

$$\sum_k \Delta_k = \delta$$

where  $\delta$  is the delta function.

**Lemma 2.** Let  $w \in A_p$ ,  $1 < p < \infty$ . Then

$$\left\| \sum_k f_k * \Delta_k \right\|_{L^p(w)} \leq C_{p,w} \left\| \left( \sum_k |f_k|^2 \right)^{1/2} \right\|_{L^p(w)},$$

$$\left\| \left( \sum_k |f * \Delta_k|^2 \right)^{1/2} \right\|_{L^p(w)} \leq C_{p,w} \|f\|_{L^p(w)}.$$

**Decompose**

$$Tf = \sum_{j \in \mathbb{Z}} f * S_j L = \sum_{k_1, k_2 \in \mathbb{Z}} U_{k_1, k_2} f,$$

**where**

$$U_{k_1, k_2} f = \sum_j f * \Delta_{k_1+j} * \nu_j * \Delta_{k_2+j}, \quad \nu_j = S_j L.$$

**Lemma 3.** Let  $1 < p < \infty$ .

$$\|U_{k_1, k_2} f\|_p \leq CA 2^{-\epsilon(|k_1|+|k_2|)/s'} \|f\|_p$$

for some  $\epsilon > 0$ , where  $A = \|h\|_{\Lambda_s^{\eta/s'}} \|\Omega\|_s$ ,  $C = C(p, s)$ .

**Lemma 3 implies**

$$\begin{aligned}\|Tf\|_p &\leq \sum_{k_1, k_2} \|U_{k_1, k_2} f\|_p \leq CA \sum_{k_1, k_2} 2^{-\epsilon(|k_1|+|k_2|)/s'} \|f\|_p \\ &\leq C_s A \|f\|_p.\end{aligned}$$

To prove Lemma 3 we need

**Lemma 4.**

$$\|U_{k_1, k_2} f\|_2 \leq C 2^{-\epsilon(|k_1|+|k_2|)/s'} A \|f\|_2$$

for some  $\epsilon > 0$ .

**Proof of Lemma 4.** Let

$$S_j f = f * \Delta_{k_1+j} * \nu_j * \Delta_{k_2+j}, \quad \nu_j = S_j L.$$

Then

$$U_{k_1, k_2} f = \sum_j S_j f.$$

We prove

$$(1) \quad \|S_j S_{j'}^* f\|_2 \leq C 2^{-2\epsilon(|k_1|+|k_2|)/s'} 2^{-\delta|j-j'|} A^2 \|f\|_2$$

and

$$(2) \quad \|S_{j'}^* S_j f\|_2 \leq C 2^{-2\epsilon(|k_1|+|k_2|)/s'} 2^{-\delta|j-j'|} A^2 \|f\|_2,$$

where

$$S_{j'}^* f = f * \Delta_{k_2+j'} * \tilde{\nu}_{j'} * \Delta_{k_1+j'}, \quad \tilde{\nu}_{j'}(x) = \nu_{j'}(x^{-1}).$$

Then, Lemma 4 follows from the Cotlar-Knapp-Stein lemma.

**Proof of (2).**

By Lemma 1 and  $\|\Delta_{k_2+j} * \Delta_{k_2+j'}\|_1 \leq C 2^{-\delta|j-j'|}$ ,

$$\begin{aligned}
& \left\| f * (\Delta_{k_1+j} * \nu_j) * (\Delta_{k_2+j} * \Delta_{k_2+j'}) * (\tilde{\nu}_{j'} * \Delta_{k_1+j'}) \right\|_2 \\
& \leq C 2^{-\epsilon|k_1|/s'} A \|f * (\Delta_{k_1+j} * \nu_j) * (\Delta_{k_2+j} * \Delta_{k_2+j'})\|_2 \\
& \leq C 2^{-\epsilon|k_1|/s'} 2^{-\delta|j-j'|} A \|f * (\Delta_{k_1+j} * \nu_j)\|_2, \\
& \leq C 2^{-2\epsilon|k_1|/s'} 2^{-\delta|j-j'|} A^2 \|f\|_2.
\end{aligned}$$

Also,

$$\begin{aligned}
& \left\| f * \Delta_{k_1+j} * (\nu_j * \Delta_{k_2+j}) * (\Delta_{k_2+j'} * \tilde{\nu}_{j'}) * \Delta_{k_1+j'} \right\|_2 \\
& \leq C 2^{-\epsilon |k_2|/s'} A \|f * \Delta_{k_1+j} * (\nu_j * \Delta_{k_2+j})\|_2 \\
& \leq C 2^{-2\epsilon |k_2|/s'} A^2 \|f\|_2
\end{aligned}$$

Taking the geometric mean we have

$$\begin{aligned}
& \left\| f * \Delta_{k_1+j} * \nu_j * \Delta_{k_2+j} * \Delta_{k_2+j'} * \tilde{\nu}_{j'} * \Delta_{k_1+j'} \right\|_2 \\
& \leq C 2^{-\epsilon |k_1|/s'} 2^{-\epsilon |k_2|/s'} 2^{-\delta |j-j'|/2} A^2 \|f\|_2.
\end{aligned}$$

Similarly, we can prove (1).

### Proof of Lemma 3.

Let

$$M_L f(x) = \sup_j |f * |S_j L(x)||.$$

### Lemma 5.

$$\|M_L f\|_p \leq C_s A \|f\|_p, \quad \text{for } p > 1,$$

where  $A = \|h\|_{\Lambda_s^{\eta/s}} \|\Omega\|_s$ .

By duality we may assume  $1 < p < 2$ . Let

$$1/p = (1 - \theta)/r + \theta/2, \quad \theta \in (0, 1), \quad r \in (1, 2),$$

$$1/r - 1/2 = 1/(2u), \quad u > 1.$$

**Lemma 5 with  $p = u$  implies the vector valued inequality**

$$\left\| \left( \sum |g_k * \nu_k|^2 \right)^{1/2} \right\|_r \leq CA \left\| \left( \sum |g_k|^2 \right)^{1/2} \right\|_r, \quad \nu_k = S_k L.$$

**From this and the Littlewood-Paley theory (Lemma 2)**

$$\begin{aligned} \|U_{k_1, k_2} f\|_r &= \left\| \sum_j f * \Delta_{k_1+j} * \nu_j * \Delta_{k_2+j} \right\|_r, \quad \nu_j = S_j L, \\ &\leq C \left\| \left( \sum_j |f * \Delta_{k_1+j} * \nu_j|^2 \right)^{1/2} \right\|_r \\ &\leq CA \left\| \left( \sum_j |f * \Delta_{k_1+j}|^2 \right)^{1/2} \right\|_r \\ &\leq CA \|f\|_r. \end{aligned}$$

**Interpolating between this and the estimate in Lemma 4**

$$(\|U_{k_1, k_2} f\|_2 \leq C 2^{-\epsilon(|k_1| + |k_2|)/s'} A \|f\|_2),$$

$$\|U_{k_1, k_2} f\|_p \leq C 2^{-\epsilon\theta(|k_1| + |k_2|)/(2s')} A \|f\|_p$$

since  $1/p = (1 - \theta)/r + \theta/2$ .

**Sketch of proof of Theorem 1.**

Let  $\rho \geq 2$ . Let  $\psi_j \in C_0^\infty(\mathbb{R})$ ,  $j \in \mathbb{Z}$ , be such that

$$\text{supp}(\psi_j) \subset \{t \in \mathbb{R} : \rho^j \leq t \leq \rho^{j+2}\}, \quad \psi_j \geq 0,$$

$$\sum_{j \in \mathbb{Z}} \psi_j(t) = 1 \quad \text{for } t \neq 0,$$

$$|(d/dt)^m \psi_j(t)| \leq c_m |t|^{-m} \quad \text{for } m = 0, 1, 2, \dots,$$

where  $c_m$  is independent of  $\rho$ . Let

$$S_j L(x) = (\log 2)^{-1} h(r(x)) \int_0^\infty \psi_j(t) \delta_t K_0(x) dt/t.$$

Then

$$\sum_{j \in \mathbb{Z}} S_j L = L, \quad Tf = \sum_{j \in \mathbb{Z}} f * S_j L.$$

We choose  $\rho = 2^{s'}$ . Then, repeat the proof of Theorem 1' and check the constants carefully.

## §6. Weak type (1,1) estimates on $\mathbb{R}^2$ .

Let

$$Tf(x) = \text{p.v.} \int f(y)K(x-y) dy,$$
$$K(x) = \Omega(x')r(x)^{-\gamma}, \quad x' = A_{r(x)^{-1}}x \text{ for } x \neq 0.$$

### Theorem C (A. Seeger 1996).

Suppose that  $A_t x = tx$  and  $r(x) = |x|$ ,  $x \in \mathbb{R}^n$ ,  $n \geq 2$ ,  $\Omega \in L \log L(\Sigma)$ .

Then, the operator  $T$  is of weak type (1,1), i.e.,

$$|\{|Tf| > \lambda\}| \leq \frac{C}{\lambda} \|f\|_1, \quad \lambda > 0.$$

**Theorem D (T. Tao 1999).**

Let

$$A_t x = (t^{a_1} x_1, t^{a_2} x_2, \dots, t^{a_n} x_n),$$

where  $x = (x_1, \dots, x_n)$  and  $0 < a_1 \leq a_2 \leq \dots \leq a_n$ .

Suppose that  $\Omega \in L \log L(\Sigma)$ . Then  $T$  is of weak type  $(1, 1)$ .

In fact, T. Tao proved the weak type  $(1, 1)$  boundedness for singular integrals on general homogeneous groups.

Let  $A_t = t^P = \exp((\log t)P)$ , where  $P$  is an arbitrary  $n \times n$  real matrix whose eigenvalues have positive real parts.

Let  $K$  be a locally integrable function on  $\mathbb{R}^n \setminus \{0\}$  satisfying

$$K(A_t x) = t^{-\gamma} K(x) \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^n \setminus \{0\};$$

$$\int_{a < r(x) < b} K(x) dx = 0 \quad \text{for all } a, b \text{ with } a < b,$$

where  $\gamma = \text{trace } P$ ,  $r(x)$  is a norm function for  $A_t$ . We can define  $\Omega$ ,  $\Sigma$  and  $L \log L(\Sigma)$  similarly to the case where  $A_t$  is diagonal.

### Theorem 6.

Suppose that  $n = 2$  and  $\Omega \in L \log L(\Sigma)$ . Then, the operator  $T$  is of weak type  $(1, 1)$ .

There exists a non-singular real matrix  $Q$  such that  $Q^{-1}PQ$  is one of the following Jordan canonical forms:

$$P_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad P_2 = \begin{pmatrix} \alpha & 0 \\ 1 & \alpha \end{pmatrix}, \quad P_3 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

where  $\alpha, \beta > 0$ . So, we have three kinds of dilations

$$\begin{pmatrix} t^\alpha & 0 \\ 0 & t^\beta \end{pmatrix}, \quad t^\alpha \begin{pmatrix} 1 & 0 \\ \log t & 1 \end{pmatrix}, \quad t^\alpha \begin{pmatrix} \cos(\beta \log t) & \sin(\beta \log t) \\ -\sin(\beta \log t) & \cos(\beta \log t) \end{pmatrix}.$$

The case where  $P = P_1$  is handled by Theorem D. We have to consider the cases  $P = P_2$  and  $P = P_3$ .

A proof of Theorem 6 follows closely the methods of T. Tao, as the Fourier transform is not readily available in this context. But we need some new estimates and arguments which do not occur in the work of T. Tao. To handle the case  $P = P_3$ , we apply a trick that may have difficulty in extending to higher dimensions.

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