

Mapping properties of some square functions with Riesz potentials

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§1. Introduction.

Let

$$\mu(f)(x) = \left(\int_0^\infty |F(x+t) + F(x-t) - 2F(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

$$F(x) = \int_0^x f(y) dy,$$

be the function of Marcinkiewicz. If $1 < p < \infty$,

$$\|\mu(f)\|_p \simeq \|f\|_{L^p}, \quad f \in \mathcal{S}(\mathbb{R}),$$

where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz class of rapidly decreasing smooth functions on \mathbb{R}^n .

Let

$$P(x, t) = P_t(x) = c_n \frac{t}{(|x|^2 + t^2)^{(n+1)/2}}, \quad c_n = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}$$

be the Poisson kernel on the upper half space

$$\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty),$$

$$u(x, t) = P_t * f(x) = \int \hat{f}(\xi) e^{-2\pi t |\xi|} e^{2\pi i \langle x, \xi \rangle} d\xi.$$

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx, \quad \langle x, \xi \rangle = \sum_{k=1}^n x_k \xi_k,$$

$x = (x_1, \dots, x_n)$, $\xi = (\xi_1, \dots, \xi_n)$. Then

$$\Delta u(x, t) = 0,$$

$$\Delta = \partial_0^2 + \sum_{j=1}^n \partial_j^2, \quad \partial_0 = \partial/\partial_t, \quad \partial_j = \partial/\partial_{x_j}.$$

Let

$$Q(x) = - \int_{\mathbb{R}^n} 2\pi |\xi| e^{-2\pi|\xi|} e^{2\pi i \langle x, \xi \rangle} d\xi = [\partial_0 P_t(x)]_{t=1},$$

$$Q_j(x) = \int_{\mathbb{R}^n} 2\pi i \xi_j e^{-2\pi|\xi|} e^{2\pi i \langle x, \xi \rangle} d\xi = \partial_j P_1(x).$$

Define the Littlewood-Paley functions

$$g_Q(f)(x) = \left(\int_0^\infty |Q_t * f(x)|^2 \frac{dt}{t} \right)^{1/2},$$

$$g_{Q_j}(f)(x) = \left(\int_0^\infty |(Q_j)_t * f(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where $Q_t(x) = t^{-n} Q(x/t)$. Then

$$g_Q(f)(x) = \left(\int_0^\infty t |\partial_0 u(x, t)|^2 dt \right)^{1/2},$$

$$g_{Q_j}(f)(x) = \left(\int_0^\infty t |\partial_j u(x, t)|^2 dt \right)^{1/2},$$

and for $1 < p < \infty$

$$\|g_0(f)\|_p \simeq \|f\|_p, \quad g_0 = g_Q,$$

$$\|g_1\|_p \simeq \|f\|_p, \quad g_1 = \sum_{j=1}^n g_{Q_j}.$$

Define the Riesz potential operator I_α by

$$\widehat{I_\alpha(f)}(\xi) = (2\pi|\xi|)^{-\alpha} \widehat{f}(\xi),$$

$$I_\alpha(f)(x) = L_\alpha * f(x), \quad L_\alpha(x) = C_\alpha |x|^{\alpha-n}.$$

Theorem 1. (Hardy-Littlewood-Sobolev)

Let $1 < p < \infty$, $0 < \alpha < n/p$, $1/p - 1/q = \alpha/n$. Then

$$\|I_\alpha(f)\|_q \leq C\|f\|_p.$$

Let

$$D_\alpha(f)(x) = \left(\int_{\mathbb{R}^n} |I_\alpha(f)(x+y) - I_\alpha(f)(x)|^2 \frac{dy}{|y|^{n+2\alpha}} \right)^{1/2}.$$

Theorem 2. Let $0 < \alpha < 1$ and $p_0 = 2n/(n + 2\alpha)$.

Suppose that $p_0 > 1$. Then

(1) (E. M. Stein, 1961) D_α is bounded on $L^p(\mathbb{R}^n)$ if

$$p_0 < p < \infty;$$

(2) (C. Fefferman, 1970) D_α is of weak type (p_0, p_0) :

$$\sup_{\beta > 0} \beta^{p_0} |\{x \in \mathbb{R}^n : D_\alpha(f)(x) > \beta\}| \leq C \|f\|_{p_0}^{p_0}.$$

Theorem 3. Let $f \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz space). Then

$$g_Q(f)(x) \leq C_\alpha D_\alpha(f)(x), \quad 0 < \alpha < 1.$$

Theorem 4. Let $0 < \alpha < 1$, $p_0 = 2n/(n + 2\alpha)$ and $p_0 > 1$. Then, if $1 \leq p < p_0$, D_α is not bounded on L^p .

Theorem 5. (D. Waterman, 1959) Let $f \in \mathcal{S}(\mathbb{R})$. Then

$$g_1(f)(x) \leq C\mu(f)(x).$$

§2. L^2 boundedness of D_α .

By the Plancherel theorem

$$\begin{aligned} & \|D_\alpha(f)\|_2^2 \\ &= \int_{\mathbb{R}^n} |y|^{-n-2\alpha} \left(\int_{\mathbb{R}^n} |I_\alpha(f)(x+y) - I_\alpha(f)(x)|^2 dx \right) dy \\ &= \int |y|^{-n-2\alpha} \left(\int \left| (2\pi|\xi|)^{-\alpha} \hat{f}(\xi) \left(e^{2\pi i \langle y, \xi \rangle} - 1 \right) \right|^2 d\xi \right) dy \\ &= (2\pi)^{-2\alpha} \int |\hat{f}(\xi)|^2 \left(\int \left| e^{2\pi i \langle y, \xi' \rangle} - 1 \right|^2 |y|^{-n-2\alpha} dy \right) d\xi, \end{aligned}$$

where $\xi' = |\xi|^{-1}\xi$. We have

$$\begin{aligned} & \int_{|y| \leq 1} \left| e^{2\pi i \langle y, \xi' \rangle} - 1 \right|^2 |y|^{-n-2\alpha} dy \\ & \leq \int_{|y| \leq 1} 4\pi^2 |y|^2 |y|^{-n-2\alpha} dy < \infty \quad \text{since } \alpha < 1, \end{aligned}$$

$$\begin{aligned} & \int_{|y| \geq 1} \left| e^{2\pi i \langle y, \xi' \rangle} - 1 \right|^2 |y|^{-n-2\alpha} dy \\ & \leq \int_{|y| \geq 1} 4|y|^{-n-2\alpha} dy < \infty \quad \text{for } \alpha > 0. \end{aligned}$$

Therefore

$$\|D_\alpha(f)\|_2^2 \leq C\|\hat{f}\|_2^2 = C\|f\|_2^2.$$

§3. Proof of Theorem 3.

Let

$$\begin{aligned} U_\alpha(x, t) &= P_t * I_\alpha(f)(x) \\ &= \int \hat{f}(\xi) (2\pi|\xi|)^{-\alpha} e^{-2\pi t|\xi|} e^{2\pi i \langle x, \xi \rangle} d\xi. \end{aligned}$$

Then

$$\partial_0^2 U_\alpha(x, t) = \int \hat{f}(\xi) (2\pi|\xi|)^{-\alpha+2} e^{-2\pi t|\xi|} e^{2\pi i \langle x, \xi \rangle} d\xi,$$

where $\partial_0 = \partial/\partial_t$, and

$$\begin{aligned} & \int_0^\infty \partial_0^2 U_\alpha(x, t+s) s^{-\alpha} ds \\ &= \int \hat{f}(\xi) (2\pi|\xi|) e^{-2\pi t|\xi|} e^{2\pi i \langle x, \xi \rangle} d\xi \int_0^\infty e^{-s} s^{-\alpha} ds \\ &= -\Gamma(1-\alpha) \partial_0 u(x, t); \end{aligned}$$

Using this,

$$\begin{aligned}
 & \left(\int_0^\infty |Q_t * f(x)|^2 \frac{dt}{t} \right)^{1/2} = \left(\int_0^\infty t |\partial_0 u(x, t)|^2 dt \right)^{1/2} \\
 &= \Gamma(1 - \alpha)^{-1} \left(\int_0^\infty t \left| \int_0^\infty \partial_0^2 U_\alpha(x, t+s) s^{-\alpha} ds \right|^2 dt \right)^{1/2} \\
 &= \Gamma(1 - \alpha)^{-1} \left(\int_0^\infty t \left| \int_t^\infty \partial_0^2 U_\alpha(x, s) (s-t)^{-\alpha} ds \right|^2 dt \right)^{1/2}
 \end{aligned}$$

$$\begin{aligned}
&= C_\alpha \left(\int_0^\infty \left| \int_0^\infty t^{1/2} \chi_{[1,\infty)} \left(\frac{s}{t} \right) |s-t|^{-\alpha} \right. \right. \\
&\quad \left. \left. \partial_0^2 U_\alpha(x, s) ds \right|^2 dt \right)^{1/2} \quad (C_\alpha = \Gamma(1 - \alpha)^{-1}) \\
&= C_\alpha \left(\int_0^\infty \left| \int_0^\infty t^{3/2-\alpha} \chi_{[1,\infty)}(s) |s-1|^{-\alpha} \right. \right. \\
&\quad \left. \left. \partial_0^2 U_\alpha(x, st) ds \right|^2 dt \right)^{1/2} \\
&\leq C_\alpha \int_1^\infty (s-1)^{-\alpha} \\
&\quad \left(\int_0^\infty t^{2(3/2-\alpha)} |\partial_0^2 U_\alpha(x, st)|^2 dt \right)^{1/2} ds
\end{aligned}$$

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$$\begin{aligned}
&= C_\alpha \int_1^\infty (s-1)^{-\alpha} s^{-(3/2-\alpha)-1/2} \\
&\quad \left(\int_0^\infty t^{3-2\alpha} |\partial_0^2 U_\alpha(x, t)|^2 dt \right)^{1/2} ds \\
&= C_\alpha \left(\int_1^\infty (s-1)^{-\alpha} s^{-2+\alpha} ds \right) \\
&\quad \left(\int_0^\infty t^{3-2\alpha} |\partial_0^2 U_\alpha(x, t)|^2 dt \right)^{1/2}.
\end{aligned}$$

Thus

$$\begin{aligned} & \left(\int_0^\infty |Q_t * f(x)|^2 \frac{dt}{t} \right)^{1/2} \\ & \leq c_\alpha \left(\int_0^\infty t^{3-2\alpha} |\partial_0^2 U_\alpha(x, t)|^2 dt \right)^{1/2}, \end{aligned}$$

where

$$c_\alpha = \Gamma(1 - \alpha)^{-1} \left(\int_1^\infty (s - 1)^{-\alpha} s^{-2+\alpha} ds \right).$$

Since $\int \partial_t^2 P_t = 0$,

$$\begin{aligned}\partial_t^2 U_\alpha(x, t) &= \int \partial_t^2 P_t(y) I_\alpha f(x + y) dy \\ &= \int \partial_t^2 P_t(y) (I_\alpha f(x + y) - I_\alpha f(x)) dy.\end{aligned}$$

Using

$$|\partial_t^2 P_t(y)| \leq C(t + |y|)^{-n-2},$$

we have

$$\begin{aligned}
|\partial_t^2 U_\alpha(x, t)| &\leq C \int (t + |y|)^{-n-2} |I_\alpha f(x+y) - I_\alpha f(x)| dy \\
&\leq C \int_{|y|<t} t^{-n-2} |I_\alpha f(x+z) - I_\alpha f(x)| dy \\
&\quad + C \int_{|y|\geq t} |y|^{-n-2} |I_\alpha f(x+z) - I_\alpha f(x)| dy.
\end{aligned}$$

It follows that

$$\int_0^\infty t^{3-2\alpha} |\partial_0^2 U_\alpha(x, t)|^2 dt \leq C(I + II),$$

where

$$I = \int_0^\infty t^{3-2\alpha} \left(\int_{|y|<t} t^{-n-2} |I_\alpha f(x+y) - I_\alpha f(x)| dy \right)^2 dt,$$

II

$$= \int_0^\infty t^{3-2\alpha} \left(\int_{|y|\geq t} |y|^{-n-2} |I_\alpha f(x+y) - I_\alpha f(x)| dy \right)^2 dt.$$

By the Schwarz inequality

$$\begin{aligned}
I &\leq C \int_0^\infty t^{3-2\alpha} t^{-2(n+2)} t^n \\
&\quad \int_{|y|< t} |I_\alpha f(x+y) - I_\alpha f(x)|^2 dy dt \\
&= C \int |I_\alpha f(x+y) - I_\alpha f(x)|^2 \left(\int_{|y|}^\infty t^{-1-n-2\alpha} dt \right) dy \\
&= C \int |I_\alpha f(x+y) - I_\alpha f(x)|^2 \frac{1}{n+2\alpha} |y|^{-n-2\alpha} dy \\
&= C \frac{1}{n+2\alpha} D_\alpha(f)(x)^2.
\end{aligned}$$

Also,

$$\begin{aligned}
 II &\leq C \int_0^\infty t^{3-2\alpha} \left(\int_{|y| \geq t} |y|^{-n-2} dy \right) \\
 &\quad \left(\int_{|y| \geq t} |y|^{-n-2} |I_\alpha f(x+y) - I_\alpha f(x)|^2 dy \right) dt \\
 &= C \int_0^\infty t^{3-2\alpha} t^{-2} \\
 &\quad \left(\int_{|y| \geq t} |y|^{-n-2} |I_\alpha f(x+y) - I_\alpha f(x)|^2 dy \right) dt
 \end{aligned}$$

$$\begin{aligned}
&= C \int |y|^{-n-2} |I_\alpha f(x+y) - I_\alpha f(x)|^2 \left(\int_0^{|y|} t^{1-2\alpha} dt \right) dy \\
&= C \frac{1}{2-2\alpha} \int |y|^{-n-2\alpha} |I_\alpha f(x+y) - I_\alpha f(x)|^2 dy \\
&= C_\alpha D_\alpha(f)(x)^2.
\end{aligned}$$

Therefore

$$\int_0^\infty t^{3-2\alpha} |\partial_0^2 U_\alpha(x, t)|^2 dt \leq C_\alpha D_\alpha(f)(x)^2,$$

and hence

$$\begin{aligned} \left(\int_0^\infty |Q_t * f(x)|^2 \frac{dt}{t} \right)^{1/2} &= \left(\int_0^\infty t |\partial_0 u(x, t)|^2 dt \right)^{1/2} \\ &\leq C_\alpha D_\alpha(f)(x). \end{aligned}$$

§4. Proof of Theorem 4.

We prove that if D_α is bounded on $L^p(\mathbb{R}^n)$ with $1 \leq p \leq 2$, then $p \geq p_0$.

Let $A(x) = \{y \in \mathbb{R}^n : 1/2 \leq |y - x| \leq 1\}$.
Let η be a non-zero element in $\mathcal{S}(\mathbb{R}^n)$ with
 $\text{supp}(\hat{\eta}) \subset \{r \leq |\xi| \leq s\}$, $0 < r < s$.

Then

$$\begin{aligned}
D_\alpha(\eta)(x) &\geq \left(\int_{A(0)} |I_\alpha(\eta)(x+y) - I_\alpha(\eta)(x)|^2 dy \right)^{1/2} \\
&\geq \left(\int_{A(0)} |I_\alpha(\eta)(x+y)|^2 dy \right)^{1/2} \\
&\quad - \left(\int_{A(0)} |I_\alpha(\eta)(x)|^2 dy \right)^{1/2} \\
&= \left(\int_{A(0)} |I_\alpha(\eta)(x+y)|^2 dy \right)^{1/2} - |A(0)|^{1/2} |I_\alpha(\eta)(x)|.
\end{aligned}$$

Therefore

$$\left(\int_{A(x)} |I_\alpha(\eta)(y)|^2 dy \right)^{1/2} \leq D_\alpha(\eta)(x) + C|I_\alpha(\eta)(x)|.$$

We prove

$$\begin{aligned} & \left(\int_{\mathbb{R}^n} |I_\alpha(\eta)(y)|^2 dy \right)^{p/2} \\ & \leq C \int_{\mathbb{R}^n} \left(\int_{A(x)} |I_\alpha(\eta)(y)|^2 dy \right)^{p/2} dx. \end{aligned}$$

To prove this, we consider a covering of \mathbb{R}^n :
 $\cup_{j=1}^{\infty} A(x^{(j)}) = \mathbb{R}^n$, for all $x^{(j)} \in B(c_j, \tau)$, $j = 1, 2, \dots$,
where $B(c_j, \tau) \cap B(c_k, \tau) = \emptyset$ if $j \neq k$. Then

$$\left(\int_{\mathbb{R}^n} |I_{\alpha}(\eta)(y)|^2 dy \right)^{p/2} \leq \sum_{j=1}^{\infty} \left(\int_{A(x^{(j)})} |I_{\alpha}(\eta)(y)|^2 dy \right)^{p/2}$$

for all $x^{(j)} \in B(c_j, \tau)$, $j = 1, 2, \dots$, since $p/2 \leq 1$.

Thus

$$\begin{aligned}
& \left(\int_{\mathbb{R}^n} |I_\alpha(\eta)(y)|^2 dy \right)^{p/2} \\
& \leq \sum_{j=1}^{\infty} \inf_{x^{(j)} \in B(c_j, \tau)} \left(\int_{A(x^{(j)})} |I_\alpha(\eta)(y)|^2 dy \right)^{p/2} \\
& \leq C_\tau \sum_{j=1}^{\infty} \int_{B(c_j, \tau)} \left(\int_{A(x)} |I_\alpha(\eta)(y)|^2 dy \right)^{p/2} dx \\
& \leq C_\tau \int_{\mathbb{R}^n} \left(\int_{A(x)} |I_\alpha(\eta)(y)|^2 dy \right)^{p/2} dx,
\end{aligned}$$

as claimed. Therefore

$$\|I_\alpha(\eta)\|_2 \leq C\|D_\alpha(\eta)\|_p + C\|I_\alpha(\eta)\|_p.$$

Thus if D_α is bounded on L^p ,

$$\|I_\alpha(\eta_t)\|_2 \leq C\|\eta_t\|_p + C\|I_\alpha(\eta_t)\|_p.$$

Using this and homogeneity,

$$t^{\alpha-n/2} \leq Ct^{-n+n/p} + Ct^{\alpha+n(1/p-1)} \leq Ct^{-n+n/p}$$

for all $t \in (0, 1)$, which implies that $p \geq 2n/(n + 2\alpha)$.

§5. Outline of Proof of Theorem 2 part (2).

We may assume that f is bounded and compactly supported. Let $\beta > 0$, $p_0 = 2n/(n + 2\alpha)$.

Lemma 1. (Calderón-Zygmund decomposition)

There exist a sequence $\{B(c_j, r_j)\}_{j=1}^{\infty}$ of balls contained in a bounded set, a bounded function g and a sequence $\{b_j\}_{j=1}^{\infty}$ of functions in L^{p_0} such that

$$(1) \quad f = g + b, \quad b = \sum_{j=1}^{\infty} b_j;$$

$$(2) \quad |g(x)| \leq C\beta;$$

- (3) $\|g\|_{p_0} \leq C \|f\|_{p_0};$
- (4) $b_j(x) = 0$ if $x \in B(c_j, r_j)^c$ for all j , where E^c denotes the complement of a set E ;
- (5) $\int b_j(x) dx = 0$ for all j ;
- (6) $\|b_j\|_{p_0}^{p_0} \leq C \beta^{p_0} |B(c_j, r_j)|$ for all j ;
- (7) $\sum_{j=1}^{\infty} |B(c_j, r_j)| \leq C \beta^{-p_0} \|f\|_{p_0}^{p_0};$
- (8) $\sum_j \chi_{B(c_j, 2r_j)} \leq C.$

It suffices to prove

$$|\{x \in \mathbb{R}^n : D_\alpha(g)(x) > \beta\}| \leq C\beta^{-p_0} \|f\|_{p_0}^{p_0} \quad (I)$$

(1)

and

$$|\{x \in \mathbb{R}^n : D_\alpha(b)(x) > \beta\}| \leq C\beta^{-p_0} \|f\|_{p_0}^{p_0} \quad (II).$$

(2)

The estimate (I) easily follows from the L^2 boundedness of D_α as follows. By Chebyshev's inequality along with

(2) and (3) of Lemma 1, since $1 < p_0 < 2$, we have

$$\begin{aligned} |\{x \in \mathbb{R}^n : D_\alpha(g)(x) > \beta\}| \\ &\leq \beta^{-2} \|D_\alpha(g)\|_2^2 \\ &\leq C\beta^{-2} \|g\|_2^2 \\ &\leq C\beta^{-p_0} \|g\|_{p_0}^{p_0} \\ &\leq C\beta^{-p_0} \|f\|_{p_0}^{p_0}. \end{aligned}$$

It remains to prove (II).

Let

$$v(x, t) = P_t * b(x), \quad V(x, t) = P_t * I_\alpha(b)(x).$$

Then

$$V(x, t) = \frac{1}{\Gamma(\alpha)} \int_0^\infty v(x, t+s) s^{\alpha-1} ds.$$

Let

$$J^{(1)}(x) = \int_{\mathbb{R}^n} \left| \int_0^\infty dt \int_0^{|y-x|} \partial_0 v(y, s+t) t^{\alpha-1} ds \right|^2 |y-x|^{-n-2\alpha} dy,$$

$$J^{(2)}(x) = \int_{\mathbb{R}^n} \left| \int_0^\infty dt \int_0^{|y-x|} \partial_0 v(x, s+t) t^{\alpha-1} ds \right|^2 |y-x|^{-n-2\alpha} dy,$$

$$J^{(3)}(x) = \Gamma(\alpha)^2 \int_{\mathbb{R}^n} |V(y, |y-x|) - V(x, |y-x|)|^2 |y-x|^{-n-2\alpha} dy,$$

where $\partial_0 = \partial/\partial s$. Then

$$D_\alpha(b)^2(x) \leq C(J^{(1)}(x) + J^{(2)}(x) + J^{(3)}(x)),$$

since

$$\begin{aligned} & |I_\alpha(b)(x+y) - I_\alpha(b)(x)| \\ & \leq |V(x+y, |y|) - I_\alpha(b)(x+y)| \\ & + |V(x, |y|) - I_\alpha(b)(x)| \\ & + |V(x+y, |y|) - V(x, |y|)|. \end{aligned}$$

Let $\Omega_* = \cup_j B(c_j, 4r_j)$.
 Since $|\Omega_*| \leq C\beta^{-p_0} \|f\|_{p_0}^{p_0}$, Theorem 2 part (2) follows from

$$\int_{\Omega_*^c} J^{(1)}(x) dx \leq C\beta^{2-p_0} \|f\|_{p_0}^{p_0},$$

$$\int_{\Omega_*^c} J^{(2)}(x) dx \leq C\beta^{2-p_0} \|f\|_{p_0}^{p_0},$$

$$\int_{\Omega_*^c} J^{(3)}(x) dx \leq C\beta^{2-p_0} \|f\|_{p_0}^{p_0}.$$

§6. Proof of Theorem 5.

We write

$$\mu(f)(x) = \left(\int_0^\infty |S_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where

$$S_t(f)(x) = \frac{1}{t} \int_0^t (f(x-u) - f(x+u)) du.$$

Fix x_0 and put $\psi(u) = f(x_0 - u) - f(x_0 + u)$, $\Psi(y) = \psi(e^{-y})$. By the change of variables $u = e^{-y}$, $t = e^{-x}$

$$\mu(f)(x_0)^2 = \int_{-\infty}^{\infty} \left| \int_x^{\infty} e^{x-y} \Psi(y) dy \right|^2 dx.$$

Let

$$K(x) = \begin{cases} e^x, & \text{if } x \leq 0, \\ 0, & \text{if } x > 0. \end{cases}$$

Then

$$\mu(f)(x_0)^2 = \int |(\Psi * K)(x)|^2 dx.$$

By the Plancherel theorem

$$\mu(f)(x_0)^2 = \int \left| \widehat{\Psi}(\xi) \widehat{K}(\xi) \right|^2 d\xi,$$

where

$$\widehat{K}(\xi) = \int_{-\infty}^0 e^x e^{-2\pi i x \xi} dx = \frac{1}{1 - 2\pi i \xi}.$$

On the other hand, we see that

$$\begin{aligned}
 (\partial/\partial x)u(x_0, t) &= \int_{-\infty}^{\infty} \frac{-2tu}{\pi(u^2 + t^2)^2} f(x_0 - u) du \\
 &= \int_0^{\infty} \frac{-2tu}{\pi(u^2 + t^2)^2} \psi(u) du.
 \end{aligned}$$

Therefore

$$g_1(f)(x_0)^2 = \int_0^{\infty} \left| \int_0^{\infty} \frac{-2t^2u}{\pi(1 + t^2u^2)^2} \psi(u) du \right|^2 \frac{dt}{t}.$$

Set $L(x) = \pi^{-1}(2e^{2x}/(1+e^{2x})^2)$. Then by the change

of variables $u = e^{-y}$ and $t = e^x$,

$$g_1(f)(x_0)^2 = \int |\Psi * L(x)|^2 dx = \int |\widehat{\Psi}(\xi) \widehat{L}(\xi)|^2 d\xi.$$

$$\begin{aligned}\widehat{L}(\xi) &= \int_{-\infty}^{\infty} \frac{2e^{(-2\pi i x \xi + 2x)}}{\pi(1 + e^{2x})^2} dx \\ &= \int_0^{\infty} \frac{2t^{(-2\pi i \xi + 1)}}{\pi(1 + t^2)^2} dt \\ &= \frac{-\pi i \xi}{\sin(-\pi^2 i \xi)}.\end{aligned}$$

Thus

$$|\widehat{L}(\xi)| \sim (1 + |\xi|) e^{-\pi^2 |\xi|}.$$

This completes the proof since

$$|\widehat{K}(\xi)| \sim (1 + |\xi|)^{-1}$$

and

$$\mu(f)(x_0)^2 = \int \left| \widehat{\Psi}(\xi) \widehat{K}(\xi) \right|^2 d\xi,$$

$$g_1(f)(x_0)^2 = \int \left| \widehat{\Psi}(\xi) \widehat{L}(\xi) \right|^2 d\xi.$$

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