

Functions of Marcinkiewicz type and Sobolev spaces

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§1. Introduction.

Recall the Marcinkiewicz integral:

$$\mu(f)(x) = \left(\int_0^\infty |F(x+t) + F(x-t) - 2F(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

$$F(x) = \int_0^x f(y) dy.$$

If $2/3 < p < \infty$,

$$\|\mu(f)\|_p \simeq \|f\|_{H^p}, \quad f \in \mathcal{S}_0(\mathbb{R}),$$

where $\mathcal{S}_0(\mathbb{R}^n)$ is the subspace of $\mathcal{S}(\mathbb{R}^n)$ consisting of functions f with \widehat{f} vanishing outside a compact set not

containing the origin.

This can be rephrased as

$$\|\nu(f)\|_p \simeq \|f'\|_{H^p}, \quad f \in \mathcal{S}_0(\mathbb{R}),$$

where

$$\nu(f)(x) = \left(\int_0^\infty |f(x+t) + f(x-t) - 2f(x)|^2 \frac{dt}{t^3} \right)^{1/2}.$$

$$f(x+t) + f(x-t) - 2f(x) = \int_{S^0} (f(x-t\theta) - f(x)) d\sigma(\theta),$$

where $S^0 = \{-1, 1\}$, $\sigma(\{-1\}) = 1$, $\sigma(\{1\}) = 1$.

Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then

$$f \in H^p(\mathbb{R}^n) \iff \|f\|_{H^p} = \|f^*\|_{L^p} < \infty$$

$$f^*(x) = \sup_{t>0} |\Phi_t * f(x)|,$$

where $\Phi \in \mathcal{S}(\mathbb{R}^n)$, $\int \Phi(x) dx = 1$,

$$\Phi_t(x) = t^{-n} \Phi(t^{-1}x);$$

$\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class of rapidly decreasing smooth functions on \mathbb{R}^n .

Let $n \geq 2$ and

$$D^\alpha(f)(x) = \left(\int_0^\infty \left| t^{-\alpha} \int_{S^{n-1}} (f(x - t\theta) - f(x)) d\sigma(\theta) \right|^2 \frac{dt}{t} \right)^{1/2},$$

where $d\sigma$ is the Lebesgue surface measure on S^{n-1} normalized as $\int_{S^{n-1}} d\sigma = 1$.

Let $0 < \alpha < 2$ and $S_\alpha(f) = D^\alpha(I_\alpha f)$:

$$S_\alpha(f)(x)$$

$$= \left(\int_0^\infty \left| I_\alpha(f)(x) - \int_{S^{n-1}} I_\alpha(f)(x - ty) d\sigma(y) \right|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}$$

where I_α is the Riesz potential operator defined by

$$\widehat{I_\alpha(f)}(\xi) = (2\pi|\xi|)^{-\alpha} \widehat{f}(\xi),$$

$$I_\alpha(f)(x) = L_\alpha * f(x), \quad L_\alpha(x) = C_\alpha |x|^{\alpha-n}.$$

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx, \quad \langle x, \xi \rangle = \sum_{k=1}^n x_k \xi_k.$$

Theorem A. Suppose $1 < p < \infty$. Let $f \in \mathcal{S}(\mathbb{R}^n)$.

Then

$$\|S_1(f)\|_p \simeq \|f\|_p.$$

This was used by

P. Hajłasz–Z. Liu , 2017,

to characterize the Sobolev space $W^{1,p}(\mathbb{R}^n)$.

Alternative proof of Theorem A.

Bochner-Riesz mean of order β

$$S_R^\beta(f)(x) = \int_{|\xi| < R} \widehat{f}(\xi) (1 - R^{-2}|\xi|^2)^\beta e^{2\pi i \langle x, \xi \rangle} d\xi$$

be the Bochner-Riesz mean of order β on \mathbb{R}^n .

Littlewood-Paley operator σ_β , $Re(\beta) > 0$

$$\begin{aligned}\sigma_\beta(f)(x) &= \left(\int_0^\infty \left| R \partial_R S_R^\beta(f)(x) \right|^2 \frac{dR}{R} \right)^{1/2} \\ &= \left(\int_0^\infty \left| -2\beta \left(S_R^\beta(f)(x) - S_R^{\beta-1}(f)(x) \right) \right|^2 \frac{dR}{R} \right)^{1/2},\end{aligned}$$

where $\partial_R = \partial/\partial_R$.

Theorem 1. Let $0 < \alpha < 2$, $\beta = \alpha + n/2$.

$$\sigma_\beta(f)(x) \approx D^\alpha(I_\alpha f)(x), \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

The result for $0 < \alpha < 1$ is due to Kaneko-Sunouchi 1985.

Theorem 1 with $\alpha = 1$ can be applied to give an alternative proof of Theorem A by using a property of σ_β with $\beta = 1 + n/2$.

Definition of the Sobolev space $W^{\alpha,p}(\mathbb{R}^n)$.

Let $1 < p < \infty$, $\alpha > 0$.

$$f \in W^{\alpha,p}(\mathbb{R}^n)$$

$$\iff$$

$$f \in L^p(\mathbb{R}^n),$$

$$f = J_\alpha(g) = K_\alpha * g \quad \text{for some } g \in L^p(\mathbb{R}^n),$$

where K_α is the Bessel potential:

$$\widehat{K}_\alpha(\xi) = (1 + 4\pi^2|\xi|^2)^{-\alpha/2}.$$

The norm is defined to be

$$\|f\|_{p,\alpha} = \|g\|_p = \left(\int_{\mathbb{R}^n} |g(x)|^p dx \right)^{1/p}$$

with

$$f = J_\alpha(g).$$

Let $n \geq 2$. Let $0 < \alpha < 2$. In 2012, R. Alabern, J. Mateu and J. Verdera (AMV) considered the operator

$$V_\alpha(f)(x) = \left(\int_0^\infty \left| f(x) - \mathop{\fint}_{B(x,t)} f(y) dy \right|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2},$$

where

$$\mathop{\fint}_{B(x,t)} f(y) dy = \frac{1}{|B(x,t)|} \int_{B(x,t)} f(y) dy;$$

$B(x, t)$ is a ball in \mathbb{R}^n having center x and radius t .

AMV proved

Theorem B. Let $1 < p < \infty$. Then, the following two statements are equivalent:

- (1) f belongs to $W^{1,p}(\mathbb{R}^n)$,
- (2) $f \in L^p(\mathbb{R}^n)$ and $V_1(f) \in L^p(\mathbb{R}^n)$.

Furthermore,

$$\|f\|_{p,1} \simeq \|f\|_p + \|V_1(f)\|_p.$$

Definition.

We say $\Phi \in \mathcal{M}^\alpha$, $\alpha > 0$, if

- Φ is compactly supported,
- $\Phi \in L^\infty(\mathbb{R}^n)$,
- $\int_{\mathbb{R}^n} \Phi(x) dx = 1$,
- if $\alpha \geq 1$,

(Vanishing moment)

$$\int_{\mathbb{R}^n} \Phi(x) x^\gamma dx = 0, \quad x^\gamma = x_1^{\gamma_1} \dots x_n^{\gamma_n}, \quad 1 \leq |\gamma| \leq [\alpha].$$

Let $\Phi \in \mathcal{M}^\alpha$, $\alpha > 0$,

$$G_\alpha(f)(x) = \left(\int_0^\infty |f(x) - \Phi_t * f(x)|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2},$$

$$\Phi_t(x) = t^{-n} \Phi(t^{-1}x).$$

We note that

$$F = \frac{1}{|B(0,1)|} \chi_{B(0,1)} \in \mathcal{M}^\alpha, \quad 0 < \alpha < 2.$$

$$\Phi = F \implies G_\alpha(f) = V_\alpha(f).$$

The weight class A_p , $1 < p < \infty$, of Muckenhoupt.

$$w \in A_p$$
$$\iff$$

$$\sup_{x \in B} \left(|B|^{-1} \int_B w(x) dx \right) \left(|B|^{-1} \int_B w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty.$$

The weighted L^p norm is defined as

$$\|f\|_{L_w^p} = \|f\|_{p,w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p}.$$

Definition of the weighted Sobolev space $W_w^{\alpha,p}(\mathbb{R}^n)$.

Let $1 < p < \infty$, $\alpha > 0$ and $w \in A_p$.

$$f \in W_w^{\alpha,p}(\mathbb{R}^n)$$

$$\iff$$

$$f \in L_w^p(\mathbb{R}^n),$$

$$f = J_\alpha(g) = K_\alpha * g \quad \text{for some } g \in L_w^p(\mathbb{R}^n),$$

where K_α is the Bessel potential:

$$\widehat{K}_\alpha(\xi) = (1 + 4\pi^2|\xi|^2)^{-\alpha/2}.$$

The norm is defined to be

$$\|f\|_{p,\alpha,w} = \|g\|_{p,w} = \left(\int_{\mathbb{R}^n} |g(x)|^p w(x) dx \right)^{1/p}.$$

with

$$f = J_\alpha(g).$$

Theorem 2. Let $1 < p < \infty$, $w \in A_p$, $0 < \alpha < n$.

Then

$$f \in W_w^{\alpha, p}(\mathbb{R}^n) \iff f \in L_w^p, \quad G_\alpha(f) \in L_w^p;$$

furthermore,

$$\|f\|_{p,\alpha,w} \simeq \|f\|_{p,w} + \|G_\alpha(f)\|_{p,w}.$$

H^1 Sobolev space.

$$f \in W_{H^1}^\alpha(\mathbb{R}^n)$$

$$\iff$$

$$f \in H^1(\mathbb{R}^n),$$

$$f = J_\alpha(h) = K_\alpha * h \quad \text{for some } h \in H^1(\mathbb{R}^n).$$

Define

$$\|f\|_{W_{H^1}^\alpha} = \|h\|_{H^1}, \quad f = J_\alpha(h).$$

Let

$$\psi^{(\alpha)}(x) = L_\alpha(x) - L_\alpha * \Phi(x),$$

where $\Phi \in \mathcal{M}^\alpha$, $0 < \alpha < n$.

Lusin area integral of Marcinkiewicz type.

$$\begin{aligned} S_{\psi^{(\alpha)}}(f)(x) &= \left(\int_0^\infty \int_{B(0,1)} |\psi_t^{(\alpha)} * f(x - tz)|^2 dz \frac{dt}{t} \right)^{1/2} \\ &= \left(\int_0^\infty \int_{B(x,t)} |\psi_t^{(\alpha)} * f(z)|^2 dz t^{-n} \frac{dt}{t} \right)^{1/2}. \end{aligned}$$

Also, let

$$\begin{aligned} & U_\alpha(f)(x) \\ &= \left(\int_0^\infty \int_{B(0,1)} |f(x - tz) - \Phi_t * f(x - tz)|^2 dz t^{-2\alpha} \frac{dt}{t} \right)^{1/2} \\ &= \left(\int_0^\infty \int_{B(x,t)} |f(z) - \Phi_t * f(z)|^2 dz t^{-2\alpha-n} \frac{dt}{t} \right)^{1/2}. \end{aligned}$$

Then

$$U_\alpha(f) = S_{\psi(\alpha)}(I_{-\alpha}f), \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

$S_{\psi(\alpha)}$ is closely related to

$$\mathcal{D}_\alpha(f)(x) = \left(\int_{\mathbb{R}^n} |I_\alpha(f)(x-y) - I_\alpha(f)(x)|^2 \frac{dy}{|y|^{n+2\alpha}} \right)^{1/2}.$$

Theorem C. Let $0 < \alpha < 1$, $p_0 = 2n/(n+2\alpha)$, $p_0 > 1$.

- (1) (E.M. Stein, 1961) \mathcal{D}_α is bounded on $L^p(\mathbb{R}^n)$
if $p_0 < p < \infty$;
- (2) (C. Fefferman, 1970) \mathcal{D}_α is of weak type (p_0, p_0) .

Theorem 3. Suppose that $n/2 < \alpha < n$, $\Phi \in \mathcal{M}^\alpha$ and

$$|\widehat{\Phi}(\xi)| \leq C(1 + |\xi|)^{-\beta}, \quad \alpha + \beta > n.$$

Then the following two statements are equivalent:

1. $f \in W_{H^1}^\alpha(\mathbb{R}^n)$,
2. $f \in H^1(\mathbb{R}^n)$ and $U_\alpha(f) \in L^1(\mathbb{R}^n)$.

Further, we have $\|f\|_{W_{H^1}^\alpha} \simeq \|f\|_{H^1} + \|U_\alpha(f)\|_1$.

In Theorem 3, the hypothesis $\alpha > n/2$ is optimal in the sense that if $0 < \alpha < n/2$, the estimate

$$\|U_\alpha(f)\|_1 \leq C\|f\|_{W_{H^1}^\alpha}$$

does not hold.

Weighted H^1 Sobolev space.

The weight class A_1 of Muckenhoupt.

$$w \in A_1$$



$$M(w) \leq Cw \quad \text{almost everywhere.}$$

M denotes the Hardy-Littlewood maximal operator defined as

$$M(f)(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| dy.$$

Let $w \in A_1$ and

$$H_w^1 = \{f \in L_w^1 : f^* \in L_w^1\}, \quad \|f\|_{H_w^1} = \|f^*\|_{1,w}$$

$$f^*(x) = \sup_{t>0} |\varphi_t * f(x)|, \quad \varphi \in \mathcal{S}(\mathbb{R}^n), \quad \int \varphi dx = 1.$$

$$f \in W_{H_w^1}^\alpha(\mathbb{R}^n)$$



$$f \in H_w^1(\mathbb{R}^n), \quad f = J_\alpha(h) \text{ for some } h \in H_w^1(\mathbb{R}^n).$$

Define $\|f\|_{W_{H_w^1}^\alpha} = \|h\|_{H_w^1}$.

In the one dimensional case, we have the following result.

Theorem 4. Let $w \in A_1$. Then the following two statements are equivalent:

1. $f \in W_{H_w^1}^1(\mathbb{R})$,
2. $f \in H_w^1(\mathbb{R})$ and $\nu(f) \in L_w^1(\mathbb{R})$.

Also, $\|f\|_{W_{H_w^1}^1} \simeq \|f\|_{H_w^1} + \|\nu(f)\|_{1,w}$.

§2.

Proof of Theorem 2.

§3.

Proof of Theorem 3.

§4.

Proof of Theorem 4.

§2. Proof of Theorem 2.

Theorem 2. Let $1 < p < \infty$, $w \in A_p$, $0 < \alpha < n$.

Then

$$f \in W_w^{\alpha, p}(\mathbb{R}^n) \iff f \in L_w^p, \quad G_\alpha(f) \in L_w^p;$$

furthermore,

$$\|f\|_{p,\alpha,w} \simeq \|f\|_{p,w} + \|G_\alpha(f)\|_{p,w}.$$

$$G_\alpha(f)(x) = \left(\int_0^\infty |f(x) - \Phi_t * f(x)|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}.$$

Littlewood-Paley function on \mathbb{R}^n .

$$g_\psi(f)(x) = \left(\int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2},$$

$$\psi_t(x) = t^{-n} \psi(t^{-1}x), \quad \psi \in L^1(\mathbb{R}^n),$$

$$\int_{\mathbb{R}^n} \psi(x) dx = 0.$$

The function of Marcinkiewicz is an example:

$$\mu(f)(x) = \left(\int_0^\infty |F(x+t) + F(x-t) - 2F(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

$$F(x) = \int_0^x f(y) dy.$$

This can be realized as

$$\mu(f) = g_H(f)$$

$$H(x) = \chi_{[-1,0]}(x) - \chi_{[0,1]}(x) \quad \text{the Haar function.}$$

Theorem 5. Suppose that

- (1) $\exists \epsilon > 0: \int_{|x|>1} |\psi(x)| |x|^\epsilon dx < \infty;$
- (2) $\exists u > 1: \int_{|x|<1} |\psi(x)|^u dx < \infty;$
- (3) $H_\psi(x) = \sup_{|y| \geq |x|} |\psi(y)| \in L^1(\mathbb{R}^n)$
(non-increasing radial majorant);
- (4) $\sup_{t>0} |\hat{\psi}(t\xi)| > 0, \quad \forall \xi \neq 0 \quad \text{(non-degeneracy)}.$

Then, if $w \in A_p$,

$$C_1 \|f\|_{p,w} \leq \|g_\psi(f)\|_{p,w} \leq C_2 \|f\|_{p,w}, \quad \forall p \in (1, \infty).$$

IDEA

- J. Duoandikoetxea and J. L. Rubio de Francia, 1986,
 - Hörmander, 1960.
-
- Theorem 5 \implies Theorem 2.

Proof of Theorem 2. Let $0 < \alpha < n$, $\Phi \in \mathcal{M}^\alpha$ and

$$T_\alpha(f)(x) = \left(\int_0^\infty |I_\alpha(f)(x) - \Phi_t * I_\alpha(f)(x)|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}.$$

$$\psi^{(\alpha)}(x) = L_\alpha(x) - \Phi * L_\alpha(x) \implies T_\alpha(f) = g_{\psi^{(\alpha)}}(f).$$

Since $\Phi \in \mathcal{M}^\alpha$, it is easy to see that

$$|\psi^{(\alpha)}(x)| \leq C|x|^{-n+\alpha} \quad \text{for } |x| \leq 1,$$

$$|\psi^{(\alpha)}(x)| \leq C|x|^{-n+\alpha-[\alpha]-1} \quad \text{for } |x| \geq 1.$$

These estimates implies the conditions (1), (2) and (3) of Theorem 5 for $\psi^{(\alpha)}$. Also,

$$\widehat{\psi^{(\alpha)}}(\xi) = (2\pi|\xi|)^{-\alpha}(1 - \hat{\Phi}(\xi))$$

satisfies a non-degeneracy condition (4), since $\hat{\Phi}(\xi) \rightarrow 0$ as $|\xi| \rightarrow \infty$ by the Riemann-Lebesgue lemma. Further, since

$$|\widehat{\psi^{(\alpha)}}(\xi)| \leq C|\xi|^{-\alpha+[\alpha]+1},$$

we have $\widehat{\psi^{(\alpha)}}(0) = 0$. Thus, by Theorem 5

$$\|T_\alpha(f)\|_{p,w} = \|g_{\psi^{(\alpha)}}(f)\|_{p,w} \simeq \|f\|_{p,w}.$$

Using this and

$$T_\alpha(I_{-\alpha}f) = G_\alpha(f), \quad f \in \mathcal{S}_0(\mathbb{R}^n),$$

we have

$$\|G_\alpha(f)\|_{p,w} \simeq \|I_{-\alpha}f\|_{p,w}.$$

The proof of Theorem 2 is based on this.

§3. Proof of Theorem 3.

Theorem 3. Suppose that $n/2 < \alpha < n$, $\Phi \in \mathcal{M}^\alpha$ and

$$|\widehat{\Phi}(\xi)| \leq C(1 + |\xi|)^{-\beta}, \quad \alpha + \beta > n.$$

Then the following two statements are equivalent:

1. $f \in W_{H^1}^\alpha(\mathbb{R}^n)$,
2. $f \in H^1(\mathbb{R}^n)$ and $U_\alpha(f) \in L^1(\mathbb{R}^n)$.

Further, we have $\|f\|_{W_{H^1}^\alpha} \simeq \|f\|_{H^1} + \|U_\alpha(f)\|_1$.

$$S_{\psi^{(\alpha)}}(f)(x) \\ = \left(\int_0^\infty \int_{B(x,t)} |\psi_t^{(\alpha)} * f(z)|^2 dz t^{-n} \frac{dt}{t} \right)^{1/2}.$$

$$\psi^{(\alpha)}(x) = L_\alpha(x) - L_\alpha * \Phi(x),$$

$$\Phi \in \mathcal{M}^\alpha, \quad 0 < \alpha < n.$$

$$U_\alpha(f)(x)$$

$$= \left(\int_0^\infty \int_{B(x,t)} |f(z) - \Phi_t * f(z)|^2 dz t^{-2\alpha-n} \frac{dt}{t} \right)^{1/2}.$$

$$U_\alpha(f) = S_{\psi(\alpha)}(I_{-\alpha}f), \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

Proof of Theorem 3 is based on

Theorem 6. Let $n/2 < \alpha < n$. Then

$$\|S_{\psi(\alpha)}(f)\|_1 \simeq \|f\|_{H^1}, \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

Lemma 1.

Let $\{g_m\}_{m=1}^{\infty}$ be a sequence of functions in H^1 satisfying

$$\sup_{m \geq 1} \|g_m\|_{H^1} < \infty.$$

Then there exist a subsequence $\{g_{m_k}\}_{k=1}^{\infty}$ and $g \in H^1$ such that

$$\int_{\mathbb{R}^n} g_{m_k}(x)v(x) dx \rightarrow \int_{\mathbb{R}^n} g(x)v(x) dx \quad \text{as } k \rightarrow \infty$$

for $v \in \mathcal{S}(\mathbb{R}^n)$.

Lemma 2.

$$\int_{|x|>2|y|} \left[\int_{B(0,1) \times (0,\infty)} \left| \psi_t^{(\alpha)}(x-y-tz) - \psi_t^{(\alpha)}(x-tz) \right|^2 dz \frac{dt}{t} \right]^{1/2} dx \leq C,$$

with a constant C independent of $y \in \mathbb{R}^n$.

Lemma 2 $\implies \|S_{\psi^{(\alpha)}}(f)\|_1 \leq C\|f\|_{H^1}$.

Lemma 3. If $f \in \mathcal{S}_0(\mathbb{R}^n)$ and $g \in BMO(\mathbb{R}^n)$,

$$\left| \int_{\mathbb{R}^n} f(x)g(x) dx \right| \leq C \|g\|_{BMO} \int_{\mathbb{R}^n} S_{\psi(\alpha)}(f)(x) dx.$$

Lemma 3 $\implies \|f\|_{H^1} \leq C \|S_{\psi(\alpha)}(f)\|_1.$

The proof of Theorem 2 is based on the estimates $\|g_{\psi(\alpha)}(f)\|_{p,w} \simeq \|f\|_{p,w}$. If $\|g_{\psi(\alpha)}(f)\|_1 \simeq \|f\|_{H^1}$, then we would be able to characterize $W_{H^1}^1$ by G_α . We do not know at present if the estimate $\|f\|_{H^1} \leq C \|g_{\psi(\alpha)}(f)\|_1$ is true or not.

§4. Proof of Theorem 4.

Theorem 4. Let $w \in A_1$. Then the following two statements are equivalent:

1. $f \in W_{H_w^1}^1(\mathbb{R})$,
2. $f \in H_w^1(\mathbb{R})$ and $\nu(f) \in L_w^1(\mathbb{R})$.

Also, $\|f\|_{W_{H_w^1}^1} \simeq \|f\|_{H_w^1} + \|\nu(f)\|_{1,w}$.

$$\nu(f)(x) = \left(\int_0^\infty |f(x+t) + f(x-t) - 2f(x)|^2 \frac{dt}{t^3} \right)^{1/2}.$$

The proof of Theorem 4 is based on

Lemma 4.

$$\|\mu(f)\|_{1,w} \simeq \|f\|_{H_w^1}, \quad f \in \mathcal{S}_0(\mathbb{R}).$$

Let

$$g_1(f)(x) = \left(\int_0^\infty |(\partial/\partial x)u(x,t)|^2 t dt \right)^{1/2},$$

Let $\widehat{R}(\xi) = 2\pi i \xi e^{-2\pi|\xi|}$. Then $g_1(f) = g_R(f)$ and

$$\|f\|_{H_w^1} \leq C \|g_1(f)\|_{1,w}, \quad f \in \mathcal{S}_0(\mathbb{R}), \quad w \in A_1.$$

$$g_1(f) \leq C\mu(f), \quad f \in \mathcal{S}_0(\mathbb{R}).$$

Combining results, we see that

$$\|f\|_{H_w^1} \leq C\|\mu(f)\|_{1,w}.$$

Recall that $\mu(f) = g_H(f)$,

$$H(x) = \chi_{[-1,0]}(x) - \chi_{[0,1]}(x).$$

We can show

$$\left(\int_0^\infty |H_t(x-y) - H_t(x)|^2 \frac{dt}{t} \right)^{1/2} \leq C \frac{|y|^\sigma}{|x|^{1+\sigma}}, \quad \sigma = 1/2,$$

if $2|y| < |x|$.

Using this and applying a result for vector valued singular integrals, we can prove the reverse inequality:

$$\|\mu(f)\|_{1,w} \leq C \|f\|_{H_w^1}.$$

Remark.

Discrete parameter analogue.

$$E_\alpha(f)(x) = \left(\sum_{k=-\infty}^{\infty} |f(x) - \Phi_{2^k} * f(x)|^2 2^{-2k\alpha} \right)^{1/2}.$$

Let $1 < p < \infty$, $w \in A_p$, $0 < \alpha < n$, $\Phi \in \mathcal{M}^\alpha$. Then

$$f \in W_w^{\alpha,p}(\mathbb{R}^n) \iff f \in L_w^p, \quad E_\alpha(f) \in L_w^p;$$

furthermore,

$$\|f\|_{p,\alpha,w} \simeq \|f\|_{p,w} + \|E_\alpha(f)\|_{p,w}.$$

THANK YOU

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