

# **Some weighted weak type estimates for singular integrals**

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## Singular integrals of Calderón-Zygmund.

Let  $\Omega \in L^1(S^{n-1})$  satisfy

$$\int_{S^{n-1}} \Omega(\theta) d\sigma(\theta) = 0$$

where  $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$   
 $d\sigma$  : the Lebesgue measure on  $S^{n-1}$ ,  $n \geq 2$ .

We consider singular integrals:

$$T(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - y) \frac{\Omega(y')}{|y'|^n} dy, \quad y' = \frac{y}{|y|}.$$

**§1. Singular integrals with generalized (mixed) homogeneity**

**§2. Singular integrals on homogeneous groups**

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## §1. Singular integrals with generalized (mixed) homogeneity .

Let  $\{A_t\}_{t>0}$  be a (nonisotropic) dilation group on  $\mathbb{R}^n$  defined by

$$A_t = t^P = \exp((\log t)P),$$

where  $P$  is an  $n \times n$  real matrix whose eigenvalues have positive real parts. We assume  $n \geq 2$ .

## Example.

If  $P = \text{diag}(\alpha_1, \dots, \alpha_n)$ ,

$$P = \begin{pmatrix} \alpha_1 & 0 & & 0 \\ 0 & \alpha_2 & \ddots & 0 \\ 0 & 0 & \ddots & \alpha_n \end{pmatrix},$$

then

$$A_t = \begin{pmatrix} t^{\alpha_1} & 0 & & 0 \\ 0 & t^{\alpha_2} & \ddots & 0 \\ 0 & 0 & \ddots & t^{\alpha_n} \end{pmatrix}.$$

We can define a norm function  $r$  on  $\mathbb{R}^n$  from  $\{A_t\}_{t>0}$ :

- (1)  $r(A_t x) = \text{tr}(x)$  for all  $t > 0$  and  $x \in \mathbb{R}^n$ ;
- (2)  $r(x) \geq 0$ ,  $r(x) = r(-x)$  for all  $x \in \mathbb{R}^n$ ,  
 $r(x) = 0 \iff x = 0$ ;
- (3)  $r$  is continuous on  $\mathbb{R}^n$  and infinitely differentiable in  $\mathbb{R}^n \setminus \{0\}$ ;
- (4)  $r(x + y) \leq C(r(x) + r(y))$ ;
- (5) if  $\Sigma = \{x \in \mathbb{R}^n : r(x) = 1\}$ , then  $\Sigma = \{\theta \in \mathbb{R}^n : \langle B\theta, \theta \rangle = 1\}$  for a positive symmetric matrix  $B$ , where  $\langle \cdot, \cdot \rangle$  denotes the inner product in  $\mathbb{R}^n$ .

(6)

$$\int_{\mathbb{R}^n} f(x) dx = \int_0^\infty \int_{\Sigma} f(A_t \theta) t^{\gamma-1} dS(\theta) dt$$

**with  $\gamma = \text{trace } P$  and**

$$dS = \omega dS_0,$$

**where  $\omega$  is a strictly positive  $C^\infty$  function on  $\Sigma$  and  $dS_0$  is the Lebesgue surface measure on  $\Sigma$ .**

**Let  $K$  be a locally integrable function on  $\mathbb{R}^n \setminus \{0\}$  such that**

$$K(A_t x) = t^{-\gamma} K(x), \quad \gamma = \text{trace } P \text{ (homogeneous dimension).}$$

We write

$$K(x) = \frac{\Omega(x')}{r(x)^\gamma}, \quad x' = A_{r(x)^{-1}}x \text{ for } x \neq 0,$$

where  $\Omega$  is homogeneous of degree 0 with respect to the dilation group  $\{A_t\}$ . We assume the cancellation property

$$\int_{\Sigma} \Omega(\theta) dS(\theta) = 0.$$

## Define

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - y)K(y) dy.$$

**Definition.** We denote by  $L \log L(\Sigma)$  the Zygmund class on  $\Sigma$  with the norm defined as

$$\|\Omega\|_{L \log L} = \inf \left\{ \lambda > 0 : \int_{\Sigma} |\Omega(\theta)/\lambda| \log(2 + |\Omega(\theta)/\lambda|) dS(\theta) \leq 1 \right\}.$$

- $\Omega \in L \log L(\Sigma) \implies T : L^p \rightarrow L^p$  for all  $1 < p < \infty$ .

**Theorem A (Sato, 2011).** Let  $n = 2$  and  $\Omega \in L \log L(\Sigma)$ . Then  $T$  is of weak type  $(1, 1)$  on  $\mathbb{R}^2$ .

For  $\Omega \in L^1(S^{n-1})$ ,  $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ , let

$$M_\Omega(f)(x) = \sup_{t>0} t^{-n} \int_{|y|<t} |f(x-y)| |\Omega(y')| dy, \quad y' = y/|y|.$$

Put  $M_{\Omega,s}(f) = [M_\Omega(|f|^s)]^{1/s}$ ,  $s > 0$ .

Let  $M_\beta(f) = [M(|f|^\beta)]^{1/\beta}$ , where  $M$  is the Hardy-Littlewood maximal operator.

**Theorem B (Vargas, 1996).** Let  $\Omega \in L^q(S^1)$ ,  $q > 1$  and  $K(x) = \Omega(x')/|x|^2$  on  $\mathbb{R}^2$ . For a weight  $w$  define

$$W = \|\Omega\|_q^{1/\beta'} M_\beta M_{\tilde{\Omega}, \beta} M_\beta(w) + \|\Omega\|_q M_\beta(w),$$

where  $\beta \in (1, \infty)$ ,  $\tilde{\Omega}(\theta) = \Omega(-\theta)$ . Then  $\exists C$  independent of  $\Omega$  such that

$$\sup_{\lambda > 0} \lambda w(\{x \in \mathbb{R}^2 : |Tf(x)| > \lambda\}) \leq C \int_{\mathbb{R}^2} |f(x)| W(x) dx,$$

where  $w(E) = \int_E w(x) dx$ .

Theorem B is generalized to higher dimensions by Fan-Sato (2004) on the basis of Seeger (1996).

We extend Theorem B to

(I) singular integrals on  $\mathbb{R}^2$  with generalized homogeneity

and

(II) singular integrals on homogeneous groups.

We focus on (II).

## §2. Singular integrals on homogeneous groups.

We regard  $\mathbb{R}^n$  as a homogeneous group. We also write  $\mathbb{R}^n = \mathbb{H}$ .

- $\mathbb{H}$  is a homogeneous nilpotent Lie group;
- multiplication is given by a polynomial mapping;
- the identity is the origin 0,  $x^{-1} = -x$ ;
- $\exists \{A_t\}_{t>0}$ : a dilation family on  $\mathbb{R}^n$  such that

$$A_t x = (t^{a_1} x_1, t^{a_2} x_2, \dots, t^{a_n} x_n),$$

$$x = (x_1, \dots, x_n), 0 < a_1 \leq a_2 \leq \dots \leq a_n,$$

$A_t$  is an automorphism of the group structure

$$A_t(xy) = (A_tx)(A_ty), x, y \in \mathbb{H}, t > 0;$$

- Lebesgue measure is bi-invariant Haar measure.

- Convolution is defined as

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(y^{-1}x) dy.$$

- $\Sigma = \{r(x) = 1\} = S^{n-1}$ .

An example.

Heisenberg group  $\mathbb{H}_1$ .

$$(x, y, u)(x', y', u') = (x + x', y + y', u + u' + (xy' - yx')/2),$$

$$(x, y, u), (x', y', u') \in \mathbb{R}^3,$$

then  $\mathbb{R}^3$  with this group law is the Heisenberg group  $\mathbb{H}_1$ ; dilations are defined by

$$A_t(x, y, u) = (tx, ty, t^2u) \quad \text{2 step},$$

$$A'_t(x, y, u) = (tx, t^2y, t^3u) \quad \text{3 step}.$$

**Let**

$$K(x) = \frac{\Omega(x')}{r(x)^\gamma}, \quad x' = A_{r(x)^{-1}}x \text{ for } x \neq 0,$$

**where  $\gamma = a_1 + \cdots + a_n$  (homogeneous dimension),**

$$\Omega(A_t x) = \Omega(x) \quad \text{for } x \neq 0, t > 0;$$

**we assume**

$$\int_{\Sigma} \Omega(\theta) dS(\theta) = 0.$$

Let

$$\begin{aligned} T(f)(x) &= \text{p.v.} \int f(y) K(y^{-1}x) dy \\ &= \lim_{\epsilon \rightarrow 0} \int_{r(y^{-1}x) > \epsilon} f(y) K(y^{-1}x) dy. \end{aligned}$$

Results of T. Tao 1999:

**Theorem C.** Suppose that  $\Omega \in L \log L(\Sigma)$ . Then,  $T$  is bounded on  $L^p(\mathbb{H})$  for all  $p \in (1, \infty)$ .

**Theorem D.** Let  $\Omega \in L \log L(\Sigma)$ . Then,  $T$  is of weak type  $(1, 1)$  on  $\mathbb{H}$ .

Theorem A follows from Theorem D when the matrix  $P$  is diagonal.

## Weighted estimates on homogeneous groups.

$$f \in L^p(w) \iff \|f\|_{L^p(w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty,$$

$$f \in L^{1,\infty}(w) \iff \|f\|_{L^{1,\infty}(w)} = \sup_{\lambda > 0} \lambda w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}) < \infty,$$

where  $w(E) = \int_E w(x) dx$ .

- $B(a, s) = \{x \in \mathbb{H} : r(a^{-1}x) < s\}$ : a ball in  $\mathbb{H}$  with center  $a$  and radius  $s$ .
- $S = B(a, 2^k)$ ,  $k \in \mathbb{Z}$ , is called a dyadic ball.
- $CB(a, s) = B(a, Cs)$  for  $C > 0$ .

- the Hardy-Littlewood maximal operator:

$$Mf(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| dy.$$

- $A_p$  ( $1 \leq p < \infty$ ) denotes the Muckenhoupt class on  $\mathbb{H}$ .

**Definition.**  $\Omega$ -H-L maximal function

$$M_\Omega(f)(x) = \sup_{t>0} t^{-\gamma} \int_{r(y)< t} |f(xy^{-1})| |\Omega(y')| dy.$$

Put

$$M_s(f) = [M(|f|^s)]^{1/s}, \quad M_{\Omega,s}(f) = [M_\Omega(|f|^s)]^{1/s}.$$

**Theorem 1.** Let  $w \in A_2$ ,  $\beta \in (1, \infty)$ ,  $\Omega \in L^q(\Sigma)$ ,  $1 < q \leq \infty$ . Then,  $\exists C > 0$  independent of  $\Omega$  such that

$$\|Tf\|_{L^{1,\infty}(w)} \leq C \int_{\mathbb{H}} |f| \left( \|\Omega\|_q^{1/\beta'} M_\beta M_{\tilde{\Omega}, \beta}(w) + \|\Omega\|_q M_\beta(w) \right) dx.$$

Here  $\tilde{g}(x) = g(x^{-1})$ .

By Theorem 1 we can easily prove the weighted weak type estimates for  $T$  analogous to Theorem B.

**Corollary 1.** Suppose that  $\Omega \in L^q(\Sigma)$ ,  $1 < q \leq \infty$  and  $w^{q'} \in A_1$ . Then  $T : L^1(w) \rightarrow L^{1,\infty}(w)$ .

This follows from Theorem 1 with  $\beta$  sufficiently close to 1.

To prove Theorem 1, we use the following weighted  $L^2$ -estimates.

**Theorem 2.** Let  $\Omega \in L^q(\Sigma)$ ,  $1 < q \leq \infty$ ,  $w \in A_2$ ,  $\beta \in (1, \infty)$ . Then,  $\exists C > 0$  independent of  $q$  and  $\Omega$  such that

$$\|Tf\|_{L^2(w)} \leq Cq' \|\Omega\|_q^{1-1/(2\beta)} \left( \int_{\mathbb{H}} |f(x)|^2 M_\beta M_{\tilde{\Omega}, \beta}(w)(x) dx \right)^{1/2}.$$

To state results when  $\Omega \in L \log L(\Sigma)$ , we consider the maximal function

$$M^*(f)(x) = \sup \{ M_\Omega(f)(x) : \|\Omega\|_{L^1(\Sigma)} = 1. \}$$

Put  $M_s^*(f) = [M^*(|f|^s)]^{1/s}$ .

**Theorem 3.** Let  $\Omega \in L \log L(\Sigma)$ ,  $w \in A_2$ ,  $\beta \in (1, \infty)$ . Then

$$\|Tf\|_{L^{1,\infty}(w)} \leq C\|\Omega\|_{L \log L}\|f\|_{L^1(M_\beta M_\beta^*(w))}$$

for a constant  $C$  independent of  $\Omega$ .

To prove Theorem 3 we apply the weighted  $L^2$  estimates:

**Theorem 4.** Let  $\Omega \in L \log L(\Sigma)$ ,  $w \in A_2$ ,  $\beta \in (1, \infty)$ . Then  $\exists C > 0$  independent of  $\Omega$  such that

$$\|Tf\|_{L^2(w)} \leq C\|\Omega\|_{L \log L}\|f\|_{L^2(M_\beta M_\beta^*(w))}.$$

### §3. Proofs of weighted $L^2$ estimates (Theorems 2 and 4).

Let  $\phi$  be a non-negative, smooth function on  $\mathbb{H}$  such that  $\text{supp } \phi \subset B(0, 1) \setminus B(0, 1/2)$ ,  $\int \phi = 1$ ,  $\phi = \tilde{\phi}$ .

For  $\rho \geq 2$ , define

$$\Delta_k = \delta_{\rho^{k-1}}\phi - \delta_{\rho^k}\phi, \quad k \in \mathbb{Z}, \quad \delta_t\phi(x) = t^{-\gamma}\phi(A_t^{-1}x).$$

Then

$\text{supp}(\Delta_k) \subset B(0, \rho^k) \setminus B(0, \rho^{k-2})$ ,  $\Delta_k = \tilde{\Delta}_k$  and

$$\sum_k \Delta_k = \delta.$$

**For any  $\rho \geq 2$ ,  $\exists \{\psi_j\}_{j \in \mathbb{Z}}$ ,  $\psi_j \in C_0^\infty(\mathbb{R})$ ,  $\psi \geq 0$ , such that**

$$\text{supp}(\psi_j) \subset \{t \in \mathbb{R} : \rho^j \leq t \leq \rho^{j+2}\},$$

$$\sum_{j \in \mathbb{Z}} \psi_j(t) = 1 \quad \text{for } t > 0,$$

$$|(d/dt)^m \psi_j(t)| \leq c_m |t|^{-m} \quad \text{for } m = 0, 1, 2, \dots,$$

**where  $c_m$  is a constant independent of  $\rho$ .**

**Define the operator  $S_j$  by**

$$S_j F(x) = (\log 2)^{-1} \int_0^\infty \psi_j(t) \delta_t F(x) dt/t.$$

**Then**

$$S_j K_0(x) = r(x)^{-\gamma} \Omega(x') \Psi_j(r(x)),$$

**where**

$$K_0(x) = K(x) \chi_{D_0}(x), \quad D_0 = \{x \in \mathbb{R}^n : 1 \leq r(x) \leq 2\},$$

$$\Psi_j(s) = (\log 2)^{-1} \int_{1/2}^1 \psi_j(ts) dt/t.$$

**It follows that**

$$\sum_{j \in \mathbb{Z}} S_j K_0 = K, \quad Tf = \sum_j f * S_j K_0.$$

To prove Theorem 2, we apply a two parameter decomposition:

$$Tf = \sum_{k_1, k_2} U_{k_1, k_2} f,$$

$$U_{k_1, k_2} f = \sum_j f * \Delta_{k_1+j} * S_j K_0 * \Delta_{k_2+j}, \quad k_1, k_2 \in \mathbb{Z}.$$

**Lemma.** Let  $\Omega \in L^q(\Sigma)$ ,  $1 < q \leq \infty$  and  $\rho = 2^{q'}$ . Then,  $\exists C, \epsilon > 0$  independent of  $q$  and  $\Omega$  such that

$$\|U_{k_1, k_2} f\|_2 \leq C q' 2^{-\epsilon |k_1|} 2^{-\epsilon |k_2|} \|\Omega\|_q \|f\|_2.$$

This can be proved by a result of T. Tao (1999) when  $q = \infty$ . The result for the whole range of  $q$  is shown in S. Sato (2013).

We can prove Theorem 2 by an interpolation with change of measures between

**Lemma**

and

**weighted estimates** obtained by applying the weighted Littlewood-Paley inequalities.

**Proof of Theorem 4.**

We can prove Theorem 4 by an extrapolation using Theorem 2.

## §4. Ideas for proofs of weighted weak type estimates

### (Theorems 1 and 3).

- Koranyi-Vagi version (1971) Calderón-Zygmund decomposition:  
 $f = g + b$ ,  $b = \sum_{B \in \mathcal{F}} b_B$ ,  $\text{supp } b_B \subset CB$ ,  $\mathcal{F}$ : disjoint dyadic balls.
- The weighted  $L^2$ -estimates (Theorems 2 and 4) to handle  $Tg$ .
- Decompose

$$Tb = \sum_{s \leq C_0} \sum_{B \in \mathcal{F}} b_B * S_{k(B)+s} K_0 + \sum_{s > C_0} \sum_{B \in \mathcal{F}} b_B * S_{k(B)+s} K_0 = G + H,$$

where  $2^{k(B)} = \text{the radius of } B$ ; and  $S_j$  is defined with  $\rho = 2$ :

$$S_j K_0(x) = (\log 2)^{-1} \int_0^\infty \psi_j(t) \delta_t K_0(x) dt / t = r(x)^{-\gamma} \Omega(x') \Psi_j(r(x)).$$

We assume that  $C_0 > 0$  is sufficiently large.

- We apply  $\text{supp } G \subset \cup_{B \in \mathcal{F}} CB$ .
- We decompose

$$H = \sum_{s > C_0} \sum_{B \in \mathcal{F}} b_B * S_{k(B)+s} L_s + \sum_{s > C_0} \sum_{B \in \mathcal{F}} b_B * S_{k(B)+s} R_s = H_1 + H_2,$$

$$K_0 = L_s + R_s, \quad L_s(x) = \begin{cases} K_0(x), & \text{if } |K_0(x)| \leq 2^{\eta s}, \\ 0, & \text{otherwise.} \end{cases}$$

- Estimate for  $H_1$ . We apply interpolation with change of measures of Fan-Sato (2001) (a variant of Vargas (1996)) between
  - (1) Tao's estimate (1999) in the unweighted case
  - (2) weighted estimates obtained by a straightforward computation.

- Estimate for  $H_2$ . We apply a direct evaluation involving  $\Omega$ -H-L maximal functions  $M_\Omega(w)$ .
- To prove an analogous result for singular integrals on  $\mathbb{R}^2$  with generalized homogeneity, we apply a result of S. Sato (2011) instead of the result of T. Tao.

## §5. Weighted weak type estimates with $\Omega$ in $L \log L$ .

We consider  $\mathbb{R}^n$  with the usual addition, the isotropic dilation and the Euclidean norm.

Let  $A_1(\mathbb{R}_+)$  be the  $A_1$  class on  $\mathbb{R}_+ = (0, \infty)$ .

We recall a weight class introduced by Duoandikoetxea (1993). Define

$$\tilde{A}_1(\mathbb{R}^n)$$

$$= \{w(x) = u(|x|) : u \text{ is in } A_1(\mathbb{R}_+) \text{ and is decreasing or } u^2 \in A_1(\mathbb{R}_+)\}.$$

Theorem 3 implies:

**Corollary 2.**

Let  $\Omega \in L \log L(S^{n-1})$ ,

$$T(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - y) \frac{\Omega(y')}{|y|^n} dy.$$

Then

$$w \in \tilde{A}_1(\mathbb{R}^n) \implies \|Tf\|_{L^{1,\infty}(w)} \leq C \|\Omega\|_{L \log L} \|f\|_{L^1(w)}.$$

Results for power weights  $w(x) = |x|^\alpha$ ,  $-1 < \alpha < 0$ , follow from this.

# **THANK YOU !**

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