

Boundedness of Littlewood-Paley operators

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§1. Background results.

We consider the Littlewood-Paley function on \mathbb{R}^n defined by

$$S_\psi(f)(x) = \left(\int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where

$$\psi_t(x) = t^{-n} \psi(t^{-1}x), \quad \psi \in L^1(\mathbb{R}^n),$$

$$\int_{\mathbb{R}^n} \psi(x) dx = 0.$$

Examples.

Let

$$P_t(x) = c_n \frac{t}{(|x|^2 + t^2)^{(n+1)/2}}$$

be the Poisson kernel for the upper half space $\mathbb{R}^n \times (0, \infty)$. Define $S_\psi(f)$ with

$$\psi(x) = \left(\frac{\partial}{\partial t} P_t(x) \right)_{t=1},$$

then $S_\psi(f)$ is the Littlewood-Paley g function.

The Marcinkiewicz integral.

$$\mu(f)(x) = \left(\int_0^\infty |F(x+t) + F(x-t) - 2F(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

$$F(x) = \int_0^x f(y) dy.$$

This can be realized as

$$\mu(f) = S_\psi(f)$$

with

$$\psi(x) = \chi_{[-1,0]}(x) - \chi_{[0,1]}(x) \quad \text{the Haar function.}$$

Theorem A. (Benedek, Calderón and Panzone, 1962)

Suppose that

$$|\psi(x)| \leq C(1 + |x|)^{-n-\epsilon} \quad \text{for some } \epsilon > 0,$$

$$\int_{\mathbb{R}^n} |\psi(x - y) - \psi(x)| dx \leq C|y|^\epsilon \quad \text{for some } \epsilon > 0.$$

then the operator S_ψ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$.

We can easily see that the two classical examples above fulfill the conditions assumed in Theorem A.

Theorem B. Suppose that

$$|\psi(x)| \leq C(1 + |x|)^{-n-1}.$$

Then S_ψ is bounded on $L^2(\mathbb{R}^n)$.

- Coifman and Meyer, p. 148, Au delà des opérateurs pseudo-différentiels, Astérisque no. 57, Soc. Math. France, 1978.
- J.-L. Journé, pp. 81-82, Calderón-Zygmund Operators, Pseudo-differential Operators and the Cauchy Integral of Calderón, Lecture Notes in Math. vol. 994, Springer-Verlag, 1983.

Theorem C. (Sato, 1998.) Let $1 < p < \infty$. Suppose that

$$|\psi(x)| \leq C(1 + |x|)^{-n-\epsilon} \quad \text{for some } \epsilon > 0.$$

Then

$$S_\psi : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n);$$

also

$$S_\psi : L_w^p(\mathbb{R}^n) \rightarrow L_w^p(\mathbb{R}^n),$$

where $w \in A_p$ (the Muckenhoupt class).

We assume that ψ is compactly supported.

Definition. $f \in L(\log L)^\alpha(\mathbb{R}^n)$, $\alpha > 0$,

$$\iff$$

$$\int_{\mathbb{R}^n} |f(x)| [\log(2 + |f(x)|)]^\alpha dx < \infty.$$

$\Omega \in L(\log L)^\alpha(S^{n-1})$

$$\iff$$

$$\int_{S^{n-1}} |\Omega(\theta)| [\log(2 + |\Omega(\theta)|)]^\alpha d\sigma(\theta) < \infty,$$

where $d\sigma$ denotes the Lebesgue surface measure on S^{n-1} .

Theorem D. (Fan-Sato, 2002.) Suppose that $\psi \in L^q(\mathbb{R}^n)$ for some $q > 1$. Then

$$S_\psi : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n), \quad 2 \leq p < \infty.$$

Theorem E. (Sato, 2008.) Let $2 \leq p < \infty$. Then

$$\psi \in L(\log L)^{1/2}(\mathbb{R}^n) \implies S_\psi : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n).$$

For $p < 2$, the following result is known.

Theorem F. (Duoandikoetxea, 2012.) Let

$1 < q \leq 2$ and $\psi \in L^q(\mathbb{R}^n)$. Then

$$\frac{1}{p} < \frac{1}{2} + \frac{1}{q'} \implies S_\psi : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n).$$

If $1 < q < 2$ and $1/p > 1/2 + 1/q'$, then there exists $\psi \in L^q(\mathbb{R}^n)$ such that S_ψ is not bounded on $L^p(\mathbb{R}^n)$.

- L. C. Cheng, 2007. Let $1 < p < 2, 1 < q < 2$. Then

$$\frac{1}{p} - \frac{1}{2} < \frac{q-1}{2} \implies S_\psi : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n).$$

- Fan-Sato, 2002. Let

$$\psi^{(\alpha)}(x) = \begin{cases} \alpha(1 - |x|)^{\alpha-1} \operatorname{sgn}(x), & x \in (-1, 1), \\ 0, & \text{otherwise.} \end{cases}$$

If $1 < p < 2$, $1 < q < 2$ and $1/q' < \alpha < 1/p - 1/2$, then $\psi^{(\alpha)} \in L^q(\mathbb{R})$ and $S_{\psi^{(\alpha)}}$ is not bounded on L^p .

Marcinkiewicz integral. (Stein, 1958.) Let

$$\psi(x) = |x| \frac{\Omega(x')}{|x|^n} \chi_{(0,1]}(|x|) \quad \text{for } x \in \mathbb{R}^n \setminus \{0\},$$

where $\Omega \in L^1(S^{n-1})$,

$$\int_{S^{n-1}} \Omega \, d\sigma = 0.$$

Then, $S_\psi(f)$ coincides with the Marcinkiewicz integral

$$\mu_\Omega(f) = \left(\int_0^\infty \left| \int_{|y| < t} f(x - y) \frac{\Omega(y')}{|y|^{n-1}} dy \right|^2 \frac{dt}{t^3} \right)^{1/2}.$$

Theorem G. (Al-Salman-Al-Qassem-Cheng-Pan, 2002.)

Let $1 < p < \infty$.

$$\Omega \in L(\log L)^{1/2}(S^{n-1}) \implies S_\psi : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n).$$

The case $p = 2$ of Theorem G is due to T. Walsh, 1972.

§2. Littlewood-Paley functions on homogeneous groups.

We can also consider Littlewood-Paley functions on homogeneous groups.

We regard \mathbb{R}^n as a homogeneous group. We also write $\mathbb{R}^n = \mathbb{H}$.

- multiplication is given by a polynomial mapping;
- $\exists \{A_t\}_{t>0}$: a dilation family on \mathbb{R}^n such that

$$A_t x = (t^{a_1}x_1, t^{a_2}x_2, \dots, t^{a_n}x_n),$$

$x = (x_1, \dots, x_n)$, $0 < a_1 \leq a_2 \leq \dots \leq a_n$,

A_t is an automorphism of the group structure;

$$A_t(xy) = (A_tx)(A_ty), x, y \in \mathbb{H}, t > 0;$$

- Lebesgue measure is a bi-invariant Haar measure;
- the identity is the origin 0, $x^{-1} = -x$.

Multiplication xy satisfies

(1) $(ux)(vx) = ux + vx, x \in \mathbb{H}, u, v \in \mathbb{R};$

(2) if $z = xy, z = (z_1, \dots, z_n), z_k = P_k(x, y)$, then

$$P_1(x, y) = x_1 + y_1,$$

$$P_k(x, y) = x_k + y_k + R_k(x, y) \quad \text{for } k \geq 2,$$

where $R_k(x, y)$ is a polynomial depending only on $x_1, \dots, x_{k-1}, y_1, \dots, y_{k-1}$.

$|x|$: the Euclidean norm for $x \in \mathbb{R}^n$,

$r(x)$: a norm function satisfying $r(A_t x) = t r(x)$, $\forall t > 0$, $\forall x \in \mathbb{R}^n$;

(1) r is continuous on \mathbb{R}^n and smooth in $\mathbb{R}^n \setminus \{0\}$;

(2) $r(x + y) \leq C_0(r(x) + r(y))$, $r(xy) \leq C_0(r(x) + r(y))$

for some $C_0 \geq 1$;

(3) $r(x) \geq 0$, $r(x) = r(-x)$ for all $x \in \mathbb{R}^n$,

$r(x) = 0 \iff x = 0$;

(4) $\Sigma = \{x \in \mathbb{R}^n : r(x) = 1\} = S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$;

- If $\gamma = a_1 + \cdots + a_n$ (the homogeneous dimension of \mathbb{H}), then $dx = t^{\gamma-1} dS dt$, that is,

$$\int_{\mathbb{R}^n} f(x) dx = \int_0^\infty \int_{\Sigma} f(A_t \theta) t^{\gamma-1} dS(\theta) dt$$

with $dS = \omega d\sigma$, where ω is a strictly positive C^∞ function on Σ and $d\sigma$ is the Lebesgue surface measure on Σ .

Convolution

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(y^{-1}x) dy$$

- $(f * g) * h = f * (g * h)$
- $(f * g)^{\sim} = \tilde{g} * \tilde{f}$ if $\tilde{f}(x) = f(x^{-1})$.

An example.

Heisenberg group \mathbb{H}_1 .

$$(x, y, u)(x', y', u') = (x + x', y + y', u + u' + (xy' - yx')/2),$$

$$(x, y, u), (x', y', u') \in \mathbb{R}^3,$$

then \mathbb{R}^3 with this group law is the Heisenberg group \mathbb{H}_1 ;
a dilation is defined by

$$A_t(x, y, u) = (tx, ty, t^2u), \quad \text{2-steps}$$

and a norm function is

$$r(x, y, u) = \frac{1}{\sqrt{2}} \sqrt{\sqrt{(x^2 + y^2)^2 + 4u^2} + x^2 + y^2}.$$

Also, we can adopt

$$A'_t(x, y, u) = (tx, t^2y, t^3u), \quad \text{3-steps.}$$

We consider the Littlewood-Paley function on \mathbb{H} defined by

$$S_\psi(f)(x) = \left(\int_0^\infty |f * \psi_t(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where ψ is in $L^1(\mathbb{H})$ and

$$\psi_t(x) = t^{-\gamma} \psi(A_t^{-1}x) = \delta_t \psi(x).$$

Let Ω be locally integrable in $\mathbb{R}^n \setminus \{0\}$ such that

$$\Omega(A_tx) = \Omega(x) \quad \text{for } x \neq 0, t > 0,$$

$$\int_{\Sigma} \Omega(\theta) dS(\theta) = 0.$$

We write $S_\Psi = \mu_\Omega$ if

$$\Psi(x) = r(x)^a \frac{\Omega(x')}{r(x)^\gamma} \chi_{(0,1]}(r(x)), \quad a > 0,$$

where $x' = A_{r(x)^{-1}}x$ for $x \neq 0$.

Theorem H. (Ding-Wu, 2009.) Let $\Omega \in L \log L(\Sigma)$. Then μ_Ω is bounded on $L^p(\mathbb{H})$ for $p \in (1, 2]$ and is of weak type $(1, 1)$.

We can prove the following (a joint work with Y. Ding).

Theorem 1. Suppose that $\Omega \in L(\log L)^{1/2}(\Sigma)$. Then

$$\mu_\Omega : L^p(\mathbb{H}) \rightarrow L^p(\mathbb{H}) \quad \text{for all } p \in (1, \infty).$$

Decompose $\Psi(x) = \sum_{k<0} 2^{ka} \Psi^{(k)}(x)$, $k \in \mathbb{Z}$, where

$$\Psi^{(k)}(x) = 2^{-ka} r(x)^a \chi_{(1,2]}(2^{-k}r(x)) \frac{\Omega(x')}{r(x)^\gamma}.$$

Note that

$$S_{\Psi^{(k)}} f(x) = S_{\Psi_{2^{-k}}^{(k)}} f(x) = S_{\Psi^{(0)}} f(x),$$

and hence

$$S_\Psi f(x) \leq \sum_{k<0} 2^{ka} S_{\Psi_{2^{-k}}^{(k)}} f(x) = c_a S_{\Psi^{(0)}} f(x).$$

This observation suggests to consider a function of the form

$$\Psi(x) = \ell(r(x)) \frac{\Omega(x')}{r(x)^\gamma} \quad \text{for} \quad \Psi^{(0)}(x) = r(x)^a \chi_{(1,2]}(r(x)) \frac{\Omega(x')}{r(x)^\gamma},$$

where $\ell \in \Lambda_\infty^\eta$ for some $\eta > 0$ (ℓ is bounded and $\ell \in \Lambda^\eta$) and $\text{supp } \ell \subset [1, 2]$.

For $t \in (0, 1]$, define

$$\omega(h, t) = \sup_{|s| < tR/2} \int_R^{2R} |h(r - s) - h(r)| dr / r,$$

where the supremum is taken over all s and R such that $|s| < tR/2$.

Definition of Λ_∞^η . h on $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$.

$$h \in \Lambda_\infty^\eta \iff$$

$$h \in L^\infty(\mathbb{R}_+), \quad \|h\|_{\Lambda^\eta} = \sup_{t \in (0, 1]} t^{-\eta} \omega(h, t) < \infty.$$

Let $\|h\|_{\Lambda_\infty^\eta} = \|h\|_\infty + \|h\|_{\Lambda^\eta}$.

Theorem 2. Let

$$\Psi(x) = \ell(r(x)) \frac{\Omega(x')}{r(x)^\gamma},$$

where $\ell \in \Lambda_\infty^\eta$ for some $\eta > 0$, $\text{supp } \ell \subset [1, 2]$, $\Omega \in L(\log L)^{1/2}(\Sigma)$. Then S_Ψ is bounded on $L^p(\mathbb{H})$ for all $p \in (1, \infty)$.

We prove Theorem 2 via extrapolation arguments.

Theorem 3. Let $\Omega \in L^s(\Sigma)$, $s \in (1, 2]$, $1 < p < \infty$. Then

$$\|S_\Psi f\|_p \leq C_p(s - 1)^{-1/2} \|\Omega\|_s \|f\|_p,$$

where the constant C_p is independent of s and Ω .

Let $\Omega \in L(\log L)^{1/2}(\Sigma)$. We can decompose:

$$\Omega = \sum_{m=1}^{\infty} b_m \Omega_m,$$

where $b_m \geq 0$, $\sum_{m=1}^{\infty} m^{1/2} b_m < \infty$, $\sup_{m \geq 1} \|\Omega_m\|_{1+1/m} \leq 1$.
Accordingly,

$$\Psi = \sum_{m=1}^{\infty} \Psi_m, \quad \Psi_m = b_m \ell(r(x)) \frac{\Omega_m(x')}{r(x)^\gamma}.$$

Let $1 < p < \infty$. By Theorem 3 we have

$$\|S_{\Psi_m} f\|_p \leq C_p m^{1/2} b_m \|\Omega_m\|_{1+1/m} \|f\|_p \leq C_p m^{1/2} b_m \|f\|_p,$$

which implies

$$\|S_\Psi f\|_p \leq \sum_{m=1}^{\infty} \|S_{\Psi_m} f\|_p \leq C_p \left(\sum_{m=1}^{\infty} m^{1/2} b_m \right) \|f\|_p.$$

This completes the proof of Theorem 2.

§3. Vector valued inequalities.

Define a maximal function

$$M_\psi(f)(x) = \sup_{t>0} |f * |\psi|_t(x)| .$$

Lemma 1. Recall

$$\Psi(x) = \ell(r(x)) \frac{\Omega(x')}{r(x)^\gamma}.$$

Suppose that Ω is in $L^1(\Sigma)$. Then

$$\|M_\Psi f\|_p \leq C_p \|\Omega\|_1 \|f\|_p \quad \text{for } p > 1.$$

For $\theta \in \Sigma$, we define

$$M_\theta f(x) = \sup_{s>0} \frac{1}{s} \int_0^s |f(x(A_t\theta)^{-1})| dt.$$

Lemma 2. (M. Christ, 1985.) There exists a constant C_p independent of θ such that

$$\|M_\theta f\|_p \leq C_p \|f\|_p$$

for $p > 1$.

Proof of Lemma 1.

By a change of variables, we have

$$\begin{aligned} f * |\Psi|_t(x) &= \int f(xy^{-1})|\Psi|_t(y) dy \\ &= \int_1^2 \int_{\Sigma} f(x(A_{st}\theta)^{-1})|\Omega(\theta)\ell(s)| dS(\theta) \frac{ds}{s}. \end{aligned}$$

Thus

$$M_{\Psi}f(x) \leq C\|\ell\|_{\infty} \int_{\Sigma} M_{\theta}f(x)|\Omega(\theta)| dS(\theta).$$

This estimate and Minkowski's inequality imply the conclusion.

Let

$$\mathcal{H} = L^2((0, \infty), dt/t).$$

For each $k \in \mathbb{Z}$ and $\rho \geq 2$ we consider an operator T_k defined by

$$(T_k(f)(x))(t) = T_k(f)(x, t) = f * \psi_t(x) \chi_{[1, \rho)}(\rho^{-k}t).$$

The operator T_k maps functions on \mathbb{H} to \mathcal{H} -valued functions on \mathbb{H} and we see that

$$|T_k(f)(x)|_{\mathcal{H}} = \left(\int_{\rho^k}^{\rho^{k+1}} |f * \psi_t(x)|^2 \frac{dt}{t} \right)^{1/2} = \left(\int_1^{\rho} |f * \psi_{\rho^k t}(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

By Lemmas 1, we have the following vector valued inequality.

Lemma 3. Let $1 < s < \infty$. Then

$$\left\| \left(\sum_k |T_k(f_k)|_{\mathcal{H}}^2 \right)^{1/2} \right\|_s \leq C(\log \rho)^{1/2} \|\Omega\|_1 \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_s,$$

where

$$T_k(f)(x, t) = f * \Psi_t(x) \chi_{[1, \rho)}(\rho^{-k} t),$$

$$\Psi(x) = \ell(r(x)) \frac{\Omega(x')}{r(x)^\gamma}.$$

§4. Outline of proof of Theorem 3.

Let ϕ be a C^∞ function supported in $\{1/2 < r(x) < 1\}$. We assume that

$\int \phi = 1$, $\phi(x) = \tilde{\phi}(x)$, $\phi(x) \geq 0$ for all $x \in \mathbb{R}^n$.

For $\rho \geq 2$, we define

$$\Delta_k = \delta_{\rho^{k-1}}\phi - \delta_{\rho^k}\phi, \quad k \in \mathbb{Z},$$

where $\delta_t\phi(x) = t^{-\gamma}\phi(A_t^{-1}x)$.

Then $\text{supp } \Delta_k \subset \{\rho^{k-1}/2 < r(x) < \rho^k\}$ and $\Delta_k = \tilde{\Delta}_k$ and

$$\sum_k \Delta_k = \delta,$$

where δ is the delta function.

We decompose

$$f * \Psi_t(x) = \sum_{j \in \mathbb{Z}} F_j(x, t),$$

where

$$F_j(x, t) = \sum_{k \in \mathbb{Z}} f * \Delta_{j+k} * \Psi_t(x) \chi_{[\rho^k, \rho^{k+1})}(t).$$

Define

$$\begin{aligned} U_j f(x) &= \left(\int_0^\infty |F_j(x, t)|^2 \frac{dt}{t} \right)^{1/2} = \left(\sum_{k \in \mathbb{Z}} \int_1^\rho |f * \Delta_{j+k} * \Psi_{\rho^k t}|^2 \frac{dt}{t} \right)^{1/2} \\ &= \left(\sum_k |T_k(f * \Delta_{j+k})|_{\mathcal{H}}^2 \right)^{1/2}. \end{aligned}$$

Lemma 4. Let $1 < s \leq 2$ and $\rho = 2^{s'}$. Then we have

$$\|U_j f\|_2 \leq C(s-1)^{-1/2} \min\left(1, 2^{-\epsilon(|j|-c)}\right) \|\Omega\|_s \|f\|_2,$$

where the constant C is independent of s and $\Omega \in L^s(\Sigma)$.

Let $\psi_j \in C_0^\infty(\mathbb{R})$, $j \in \mathbb{Z}$, be such that

$$\text{supp}(\psi_j) \subset \{t \in \mathbb{R} : \rho^j \leq t \leq \rho^{j+2}\}, \quad \psi_j \geq 0,$$

$$\log 2 \sum_{j \in \mathbb{Z}} \psi_j(t) = 1 \quad \text{for } t > 0,$$

$$|(d/dt)^m \psi_j(t)| \leq c_m |t|^{-m} \quad \text{for } m = 0, 1, 2, \dots,$$

where c_m is a constant independent of $\rho \geq 2$.

Let

$$K_0(x) = \frac{\Omega(x')}{r(x)^\gamma} \chi_{[1,2]}(r(x))$$

and decompose

$$\frac{\Omega(x')}{r(x)^\gamma} = \sum_{j \in \mathbb{Z}} S_j(x),$$

$$S_j(x) = \int_0^\infty \psi_j(t) \delta_t K_0(x) \frac{dt}{t} = \frac{\Omega(x')}{r(x)^\gamma} \int_{1/2}^1 \psi_j(tr(x)) \frac{dt}{t}.$$

We note that S_j is supported in $\{\rho^j \leq r(x) \leq 2\rho^{j+2}\}$.

Let

$$L_m^{(t)}(x) = \ell(t^{-1}r(x))S_m(x).$$

Recall

$$\Psi(x) = \ell(r(x))\frac{\Omega(x')}{r(x)^\gamma}, \quad \Psi_t(x) = \ell(t^{-1}r(x))\frac{\Omega(x')}{r(x)^\gamma}.$$

Then by the support condition we have

$$\Psi_t(x)\chi_{[\rho^k, \rho^{k+1})}(t) = \sum_{m=k-3}^{k+3} L_m^{(t)}(x)\chi_{[\rho^k, \rho^{k+1})}(t)$$

and

$$F_j(x, t) = \sum_{k \in \mathbb{Z}} \sum_{m=k-3}^{k+3} f * \Delta_{j+k} * L_m^{(t)}(x)\chi_{[\rho^k, \rho^{k+1})}(t).$$

Lemma 5. Let $1 < s \leq 2$, $-3 \leq m \leq 3$, $\sigma_k = 1$ or -1 . Define $R_j^{(t)}$ by

$$R_j^{(t)} f(x) = \sum_{k \in \mathbb{Z}} \sigma_k f * \Delta_{j+k} * L_{k+m}^{(\rho^k t)}(x).$$

Then, if $\rho = 2^{s'}$,

$$\|R_j^{(t)} f\|_2 \leq C \min \left(1, 2^{-\epsilon(|j|-c)} \right) \|\Omega\|_s \|f\|_2,$$

where C is independent of $j \in \mathbb{Z}$, $t \in [1, \rho)$ and $\{\sigma_k\}$.

To prove this we use the following estimates.

Lemma 6. Let $1 < s \leq 2$, $\rho = 2^{s'}$ and $-3 \leq m \leq 3$. Then

$$\|f * \Delta_{j+k} * L_{k+m}^{(\rho^k t)}\|_2 \leq C \min \left(1, 2^{-\epsilon(|j|-c)} \right) \|\Omega\|_s \|f\|_2,$$

where the constant C is independent of $j, k \in \mathbb{Z}$ and $t \in [1, \rho)$.

This can be proved by using $(TT^*)^M$ methods of T. Tao (1999). We now prove Lemma 5 assuming Lemma 6. Let

$$G_{j,j'} f = \sum_{k \in \mathbb{Z}} \sigma_k f * \Delta_{j+k} * L_{k+m}^{(\rho^k t)} * \Delta_{j'+k}$$

By Lemma 5

$$\begin{aligned}
& \left\| f * \Delta_{j+k} * L_{k+m}^{(\rho^k t)} * \Delta_{j'+k} * \Delta_{j'+k'} * \tilde{L}_{k'+m}^{(\rho^{k'} t)} * \Delta_{j+k'} \right\|_2 \\
& \leq C \|\Omega\|_s^2 \min(1, 2^{-\epsilon(|j|-c)}) \min(1, 2^{-\epsilon(|j'|-c)}) \\
& \quad \min(1, \rho^{-\epsilon(|k-k'|-c)/2}) \|f\|_2.
\end{aligned}$$

We have a similar estimate for

$$\left\| f * \Delta_{j'+k'} * \tilde{L}_{k'+m}^{(\rho^{k'} t)} * \Delta_{j+k'} * \Delta_{j+k} * L_{k+m}^{(\rho^k t)} * \Delta_{j'+k} \right\|_2.$$

Thus, the Cotlar-Knapp-Stein lemma implies

$$\|G_{j,j'} f\|_2 \leq C \|\Omega\|_s \min(1, 2^{-\epsilon(|j|-c)/2}) \min(1, 2^{-\epsilon(|j'|-c)/2}) \|f\|_2,$$

and hence

$$\|R_j^{(t)} f\|_2 \leq \sum_{j' \in \mathbb{Z}} \|G_{j,j'} f\|_2 \leq C \|\Omega\|_s \min(1, 2^{-\epsilon(|j|-c)/2}) \|f\|_2.$$

Proof of Lemma 4. By Lemma 5 and the Khintchine inequality we see that

$$\left\| \left(\sum_{k \in \mathbb{Z}} |f * \Delta_{j+k} * L_{k+m}^{(\rho^k t)}|^2 \right)^{1/2} \right\|_2^2 \leq C \min \left(1, 2^{-2\epsilon(|j|-c)} \right) \|\Omega\|_s^2 \|f\|_2^2.$$

This estimate is uniform in $t \in [1, \rho)$. Thus, integration over $[1, \rho)$

with respect to the measure dt/t gives

$$\begin{aligned} \|U_j f\|_2^2 &= \int_1^\rho \left\| \left(\sum_{k \in \mathbb{Z}} |f * \Delta_{j+k} * \sum_{m=-2}^2 L_{k+m}^{(\rho^k t)}|^2 \right)^{1/2} \right\|_2^2 \frac{dt}{t} \\ &\leq C \log \rho \min \left(1, 2^{-2\epsilon(|j|-c)} \right) \|\Omega\|_s^2 \|f\|_2^2, \end{aligned}$$

which proves Lemma 4.

Proof of Theorem 3. Let $1 < p < \infty$. $\rho = 2^{s'}$. By Lemma 3 and the Littlewood-Paley inequality:

$$\left\| \left(\sum_k |f * \Delta_k|^2 \right)^{1/2} \right\|_r \leq C_r \|f\|_r, \quad 1 < r < \infty,$$

where C_r is independent of ρ , we have

$$\begin{aligned}
\|U_j(f)\|_r &= \left\| \left(\sum_k |T_k(f * \Delta_{j+k})|_{\mathcal{H}}^2 \right)^{1/2} \right\|_r \\
&\leq C(\log \rho)^{1/2} \|\Omega\|_1 \left\| \left(\sum_k |f * \Delta_k|^2 \right)^{1/2} \right\|_r \\
&\leq C(\log \rho)^{1/2} \|\Omega\|_1 \|f\|_r
\end{aligned}$$

for all $r \in (1, \infty)$. By Lemma 4

$$\|U_j f\|_2 \leq C(\log \rho)^{1/2} \min \left(1, 2^{-\epsilon(|j|-c)} \right) \|\Omega\|_s \|f\|_2.$$

Thus interpolation will give

$$\|U_j f\|_p \leq C(\log \rho)^{1/2} \min \left(1, 2^{-\epsilon(|j|-c)} \right) \|\Omega\|_s \|f\|_p$$

with some $\epsilon > 0$, which implies

$$\|S_\Psi f\|_p \leq \sum_j \|U_j f\|_p \leq C_p(s-1)^{-1/2} \|\Omega\|_s \|f\|_p,$$

since

$$S_\Psi f \leq \sum_j U_j f.$$

This completes the proof of Theorem 3.

THANK YOU

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