Some results in harmonic analysis related to singular integrals with mixed homogeneity

Shuichi Sato

Kanazawa University

§1. Singular integrals of Calderón-Zygmund and R. Fefferman

§2. Singular integrals with generalized (mixed) homogeneity

§3. Singular integrals on homogeneous groups

§4. Singular integrals and maximal functions along curves and the method of rotations with weight

§1. Let $\Omega \in L^1(S^{n-1})$ satisfy

$$\int_{S^{n-1}} \Omega(\theta)\, d\sigma(\theta) = 0$$

where $\quad S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$
$d\sigma$: the Lebesgue measure on S^{n-1}, $\quad n \geq 2$.

We consider singular integrals of the form:

$$T(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y)K(y)\, dy, \quad K(x) = h(|x|)\frac{\Omega(x')}{|x|^n}, \;\; x' = \frac{x}{|x|}.$$

Homogeneous kernels.

When $Tf = \mathrm{p.v.}\, f * K$, $K(x) = \dfrac{\Omega(x')}{|x|^n}$, write $T = T_\Omega$.

$$K(tx) = t^{-n} K(x), \quad tx = (tx_1, \ldots, tx_n) \quad \text{isotropic dilation.}$$

If Ω is odd,

$$T_\Omega f(x) = \frac{1}{2} \int_{S^{n-1}} Hf(x, \theta) \Omega(\theta)\, d\sigma(\theta),$$

$$Hf(x, \theta) = \mathrm{p.v.} \int_{-\infty}^{\infty} f(x - t\theta)\, \frac{dt}{t}.$$

The method of rotations of Calderón-Zygmund (1956) implies:

• Ω is in $L^1(S^{n-1})$ and odd $\implies T_\Omega : L^p \to L^p$ for all $1 < p < \infty$;

- $\Omega \in L \log L(S^{n-1}) \quad \Longrightarrow T_\Omega : L^p \to L^p$ for all $1 < p < \infty$.

Definition.

$$F \in L \log L(S^{n-1})$$

$$\Longleftrightarrow$$

$$\int_{S^{n-1}} |F(\theta)| \log(2 + |F(\theta)|) \, d\sigma(\theta) < \infty.$$

The case where h is not constant; inhomogeneous kernels.

Recall

$$T(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y)K(y)\,dy,$$

$$K(x) = h(|x|)\frac{\Omega(x')}{|x|^n}, \qquad x' = \frac{x}{|x|}.$$

If h is not constant, then the method of rotations of Calderón-Zygmund is not applicable in general.

A result of R. Fefferman, 1979.

$$h \in L^\infty,\ \Omega \in Lip(S^{n-1}) \implies T : L^p \to L^p,\ 1 < p < \infty.$$

§2. Singular integrals with generalized (mixed) homogeneity .

Let $\{A_t\}_{t>0}$ be a (nonisotropic) dilation group on $\mathbb{R}^n$ defined by

$$A_t = t^P = \exp((\log t)P),$$

where P is an $n \times n$ real matrix whose eigenvalues have positive real parts. We assume $n \geq 2$.

Example.

If $P = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$,

$$P = \begin{pmatrix} \alpha_1 & 0 & & 0 \\ 0 & \alpha_2 & & 0 \\ & & \ddots & \\ 0 & 0 & & \alpha_n \end{pmatrix},$$

then

$$A_t = \begin{pmatrix} t^{\alpha_1} & 0 & & 0 \\ 0 & t^{\alpha_2} & & 0 \\ & & \ddots & \\ 0 & 0 & & t^{\alpha_n} \end{pmatrix}.$$

We can define a norm function r on $\mathbb{R}^n$ from $\{A_t\}_{t>0}$. We assume the following:

(1) $r(A_t x) = tr(x)$ for all $t > 0$ and $x \in \mathbb{R}^n$;

(2) $r(x) \geq 0, r(x) = r(-x)$ for all $x \in \mathbb{R}^n$,
$r(x) = 0 \iff x = 0$;

(3) r is continuous on $\mathbb{R}^n$ and infinitely differentiable in $\mathbb{R}^n \setminus \{0\}$;

(4) $r(x + y) \leq C(r(x) + r(y))$;

(5)
$$\Sigma = \{x \in \mathbb{R}^n : r(x) = 1\} = S^{n-1},$$
where $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

Let K be a locally integrable function on $\mathbb{R}^n \setminus \{0\}$ such that

$$K(A_t x) = t^{-\gamma} K(x), \qquad \gamma = \text{trace } P \text{ (homogeneous dimension)}.$$

We write

$$K(x) = \frac{\Omega(x')}{r(x)^\gamma}, \qquad x' = A_{r(x)^{-1}} x \text{ for } x \neq 0,$$

where Ω is homogeneous of degree 0 with respect to the dilation group $\{A_t\}$. We assume that

$$\int_{a < r(x) < b} K(x)\, dx = 0 \quad \text{for all } 0 < a < b.$$

Definition **The space Δ_s, $s \geq 1$, is defined as**

$$\Delta_s = \{h \text{ on } \mathbb{R}_+ : \|h\|_{\Delta_s} < \infty\},$$

$$\|h\|_{\Delta_s} = \sup_{j \in \mathbb{Z}} \left(\int_{2^j}^{2^{j+1}} |h(t)|^s \frac{dt}{t} \right)^{1/s},$$

where

$\mathbb{Z}$: **the set of integers,** $\quad \mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$;

● $s > t \implies \Delta_s \subset \Delta_t$.

Definition

For h on $\mathbb{R}_+$ and $a > 0$, let

$$L_a(h) = \sup_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} |h(r)| \left(\log(2 + |h(r)|)\right)^a \frac{dr}{r}.$$

Define

$$\mathcal{L}_a = \{h : L_a(h) < \infty\}.$$

- $a < b \implies \mathcal{L}_b \subset \mathcal{L}_a.$

- $\bigcup_{s>1} \Delta_s \subsetneq \bigcap_{a>0} \mathcal{L}_a.$

L^p estimates.

Let

$$L(x) = h(r(x))K(x),$$

$$Tf(x) = \mathrm{p.v.} \int_{\mathbb{R}^n} f(y)L(x-y)\,dy.$$

Theorem 1. (Sato, 2009.) Let $\Omega \in L\log L(S^{n-1})$ and $h \in \mathcal{L}_a$ for some $a > 2$. Then

$$\|T(f)\|_{L^p} \leq C_p \|f\|_{L^p}$$

for all $p \in (1, \infty)$.

Proposition. Suppose that $\Omega \in L^q(S^{n-1})$, $h \in \Delta_s$, $q, s \in (1, 2]$. Then

$$\|T(f)\|_{L^p} \leq C_p(q-1)^{-1}(s-1)^{-1}\|\Omega\|_{L^q(S^{n-1})}\|h\|_{\Delta_s}\|f\|_{L^p}$$

for all $p \in (1, \infty)$, where the constant C_p is independent of q, s, Ω and h.

Theorem 1 follows from Proposition and the extrapolation of Yano by suitably decomposing

$$h = \sum_{k=1}^{\infty} a_k h_k, \quad \Omega = \sum_{m=1}^{\infty} b_m \Omega_m,$$

where $h_k \in \Delta_{1+\frac{1}{k}}$, $\Omega_m \in L^{1+\frac{1}{m}}$.

Previous results.

(1) E. B. Fabes and N. Rivière (1966).

(2) J. Duoandikoetxea and J. L. Rubio de Francia (1986).
T is bounded on L^p, $1 < p < \infty$, if $\Omega \in L^q$ for some $q > 1$ and $h \in \Delta_2$.

(3) Al-Salman-Pan (2002).

Let

$$T f(x) = \mathrm{p.v.} \int f(y) K(x-y)\, dy, \quad K(x) = \frac{\Omega(x')}{r(x)^\gamma}.$$

Theorem 2. (Sato, 2011.) **Suppose that** $n = 2$ **and** $\Omega \in L \log L(\Sigma)$. **Then, the operator** T **is of weak type** $(1,1)$, **i.e.,**

$$\left| \left\{ x \in \mathbb{R}^2 : |T f(x)| > \lambda \right\} \right| \le \frac{C}{\lambda} \| f \|_1, \quad \lambda > 0.$$

Previous results.

Theorem A (A. Seeger 1996). Suppose that $A_t x = tx$ and $r(x) = |x|$, $x \in \mathbb{R}^n$, $n \geq 2$, $\quad \Omega \in L \log L(\Sigma)$. Then, the operator T is of weak type $(1, 1)$.

IDEA: Fourier transform estimates + microlocal analysis;

Calderón-Zygmund decomposition.

Theorem B (T. Tao 1999). Let $A_t x = (t^{a_1} x_1, t^{a_2} x_2, \ldots, t^{a_n} x_n)$, $x = (x_1, \ldots, x_n)$, $0 < a_1 \leq a_2 \leq \cdots \leq a_n$. Suppose that $\Omega \in L \log L(\Sigma)$. Then T is of weak type $(1, 1)$.

In fact, T. Tao proved the weak type $(1, 1)$ boundedness of singular integrals on general homogeneous groups.

IDEA: $(TT^*)^M$ estimates via convolution;

Calderón-Zygmund decomposition;
Covering lemma of Vitali type,
John-Nirenberg inequality for BMO.

About the proof of Theorem 2. There exists a non-singular real matrix Q such that $Q^{-1}PQ$ is one of the following Jordan canonical forms:

$$P_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad P_2 = \begin{pmatrix} \alpha & 0 \\ 1 & \alpha \end{pmatrix}, \quad P_3 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

where $\alpha, \beta > 0$. Accordingly, we have three kinds of dilations

$$\begin{pmatrix} t^\alpha & 0 \\ 0 & t^\beta \end{pmatrix}, \quad t^\alpha \begin{pmatrix} 1 & 0 \\ \log t & 1 \end{pmatrix}, \quad t^\alpha \begin{pmatrix} \cos(\beta \log t) & \sin(\beta \log t) \\ -\sin(\beta \log t) & \cos(\beta \log t) \end{pmatrix}.$$

§3. Singular integrals on homogeneous groups.

We regard $\mathbb{R}^n$ as a homogeneous group. We also write $\mathbb{R}^n = \mathbb{H}$.

- $\mathbb{H}$ a homogeneous nilpotent Lie group;
- multiplication is given by a polynomial mapping;
- the identity is the origin 0, $x^{-1} = -x$;
- $\exists \{A_t\}_{t>0}$: a dilation family on $\mathbb{R}^n$ such that

$$\boxed{A_t x = (t^{a_1}x_1, t^{a_2}x_2, \ldots, t^{a_n}x_n),}$$

$x = (x_1, \ldots, x_n)$, $0 < a_1 \leq a_2 \leq \cdots \leq a_n$,
A_t is an automorphism of the group structure

$$\boxed{A_t(xy) = (A_t x)(A_t y), \ x, y \in \mathbb{H}, \ t > 0;}$$

- Lebesgue measure is bi-invariant Haar measure.

Convolution is defined as

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(y^{-1}x)\,dy.$$

An example.   Heisenberg group $\mathbb{H}_1$.

$$(x, y, u)(x', y', u') = (x + x', y + y', u + u' + (xy' - yx')/2),$$

$$(x, y, u), (x', y', u') \in \mathbb{R}^3,$$

then $\mathbb{R}^3$ with this group law is the Heisenberg group $\mathbb{H}_1$; a dilation is defined by

$$A_t(x, y, u) = (tx, ty, t^2 u) \qquad \text{2 steps,}$$

and a norm function is

$$r(x, y, u) = \frac{1}{\sqrt{2}} \sqrt{\sqrt{(x^2 + y^2)^2 + 4u^2} + x^2 + y^2}.$$

Also, we can adopt

$$A'_t(x, y, u) = (tx, t^2 y, t^3 u) \qquad \textbf{3 steps.}$$

Let

$$K(x) = \frac{\Omega(x')}{r(x)^{\gamma}}, \qquad x' = A_{r(x)^{-1}}x \text{ for } x \neq 0,$$

where $\gamma = a_1 + \cdots + a_n$ (**homogeneous dimension**),

$$\Omega(A_t x) = \Omega(x) \qquad \text{for } x \neq 0, \, t > 0;$$

we assume

$$\int_{a<r(x)<b} K(x)\,dx = 0 \quad \text{for all } 0 < a < b.$$

Let

$$T(f)(x) = \text{p.v.} \int f(y) K(y^{-1}x) \, dy$$

$$= \lim_{\epsilon \to 0} \int_{r(y^{-1}x) > \epsilon} f(y) K(y^{-1}x) \, dy.$$

Also, we consider the maximal singular integral

$$T_*(f)(x) = \sup_{N, \epsilon > 0} \left| \int_{\epsilon < r(y^{-1}x) < N} f(y) K(y^{-1}x) \, dy \right|.$$

Theorem C. (T. Tao 1999.) **Suppose that** $\Omega \in L \log L(\Sigma)$. **Then,** T **is bounded on** $L^p(\mathbb{H})$ **for all** $p \in (1, \infty)$.

T. Tao proved this by interpolation between L^2 estimates and weak $(1, 1)$ estimates.

IDEA: $(TT^*)^M$ **estimates via convolution**

Theorem 3. (Sato, 2013.) **Suppose that** $\Omega \in L \log L(\Sigma)$. **Then,**

$$T_* : L^p(\mathbb{H}) \to L^p(\mathbb{H}), \qquad \forall p \in (1, \infty).$$

IDEA: **Extrapolation**

$$\|T_*f\|_p \le C_p(s-1)^{-1}\|\Omega\|_s\|f\|_p, \quad s > 1.$$

Idea of proof.

Theory of Duoandikoetxea and Rubio de Francia (1986):

- Orthogonality arguments for L^2 estimates via Fourier transform estimates and Plancherel's theorem

- Littlewood-Paley theory

- Interpolation arguments

Our strategy is:

to employ a version of theory of Duoandikoetxea and Rubio de Francia adapted for analysis on homogeneous groups;

replace the use of Fourier transform estimates with $(TT^*)^M$ estimates via convolution (basic L^2 estimates) and apply Cotlar's lemma.

Decompose

$$\sum_{j \in \mathbb{Z}} D_j K = K, \qquad Tf = \sum_{j \in \mathbb{Z}} f * D_j K,$$

$\operatorname{supp}(D_j K) \subset \{x : \rho^j \leq r(x) \leq 2\rho^{j+2}\}.$

We choose $\rho = 2^{s'}$, $s' = s/(s-1)$, **if** $\Omega \in L^s(\Sigma)$.

   Let ϕ **be a** C^∞ **function such that** $\operatorname{supp}(\phi) \subset \{1/2 < r(x) < 1\}$, $\int \phi = 1$, $\phi(x) = \phi(x^{-1})$, $\phi(x) \geq 0$.
Define

$$\Delta_k = \delta_{\rho^{k-1}}\phi - \delta_{\rho^k}\phi, \quad k \in \mathbb{Z}, \quad \delta_t\phi(x) = t^{-\gamma}\phi(A_t^{-1}x).$$

Then

$$\boxed{\sum_k \Delta_k = \delta,}$$

where δ is the delta function.

Lemma (basic L^2 estimates). Let $s > 1$, $\Omega \in L^s(\Sigma)$, $\rho = 2^{s'}$. Then,

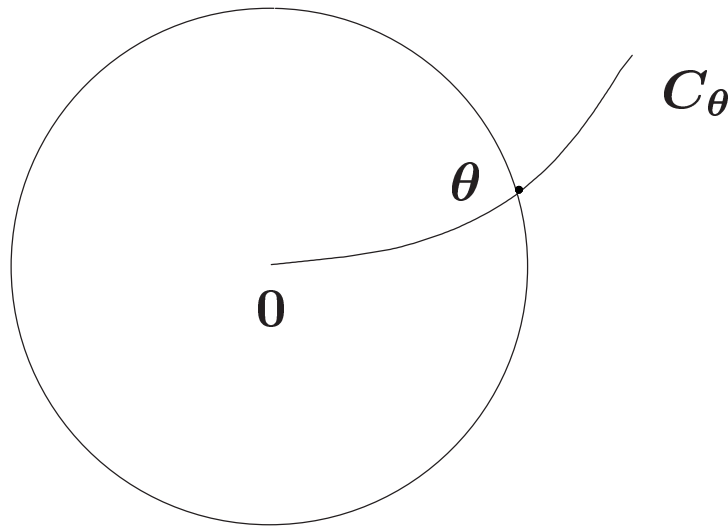$$\|f * D_j K * \Delta_{k+j}\|_2 \leq C \frac{s}{s-1} 2^{-\epsilon|k|} \|\Omega\|_s \|f\|_2.$$

When $s = \infty$, this was proved by T. Tao 1999 with $(TT^*)^n$ method.

- $\|TT^*\| = \|T\|^2$.

Let Ω be homogeneous of degree 0 on $\mathbb{R}^n \setminus \{0\}$. Define
$$C_\theta = \{A_t \theta : t > 0\}, \qquad \theta \in \Sigma.$$
Then, Ω is smooth on C_θ for every $\theta \in \Sigma$, since $\boxed{\Omega(A_t \theta) = \Omega(\theta).}$

- Convolution of smooth singular measures supported on $C_\theta, C_{\theta'} \dots$

By the lemma we have

$$\|S(f)\|_{L^p} \leq C_p(s-1)^{-1}\|\Omega\|_{L^s}\|f\|_{L^p}$$

for all $p \in (1, \infty)$, where the constant C_p is independent of $s \in (1, 2]$ and $\Omega \in L^s$, and $S = T$ or T_*. This estimate can be available in the extrapolation arguments.

The idea of considering convolution of singular measures was also used by

C. Fefferman, 1970;

M. Christ, 1985, 1988;

M. Christ and J.L. Rubio de Francia, 1988.

$$Mf(x,\theta) = \sup_{h>0} h^{-1} \left| \int_0^h f(x - A_t\theta)\, dt \right|, \quad (x,\theta) \in \mathbb{R}^n \times S^{n-1},$$

$$Hf(x,\theta) = \text{p.v.} \int_{-\infty}^{\infty} f(x - A_t\theta)\, \frac{dt}{t}, \quad A_t = (\text{sgn}\, t) A_{|t|} = -A_{|t|},$$

$$H_* f(x,\theta) = \sup_{0<\epsilon<R} \left| \int_{\epsilon<|t|<R} f(x - A_t\theta)\, \frac{dt}{t} \right|.$$

Let w be a weight function. We recall that

$$\|F\|_{L^p_w(L^q)} = \left(\int_{\mathbb{R}^n} \left(\int_{S^{n-1}} |F(x, \theta)|^q \, d\sigma(\theta) \right)^{p/q} w(x) \, dx \right)^{1/p}$$

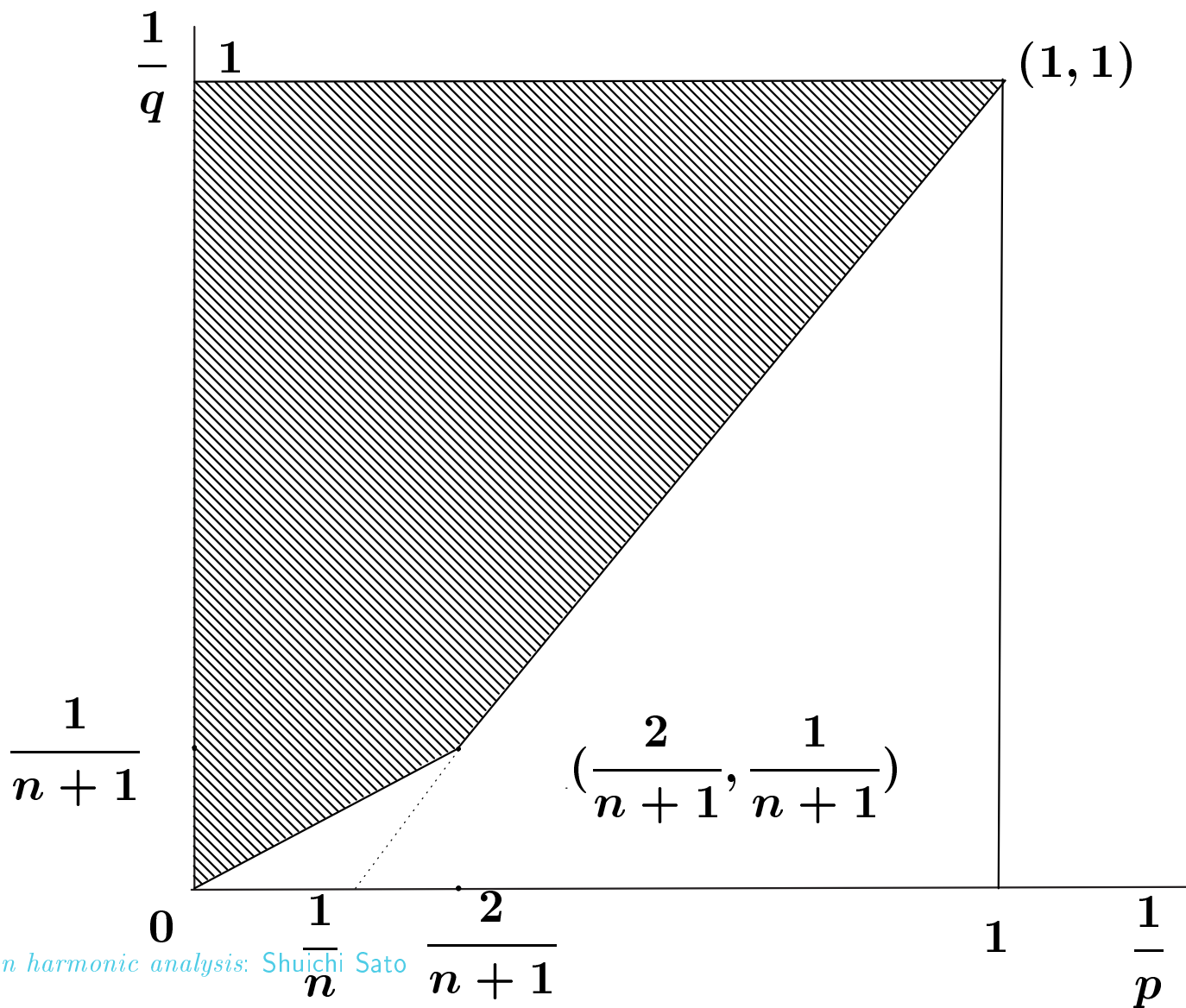**for functions $F \in L^p_w(L^q(S^{n-1}))$, where $d\sigma$ denotes the Lebesgue surface measure on S^{n-1}.**
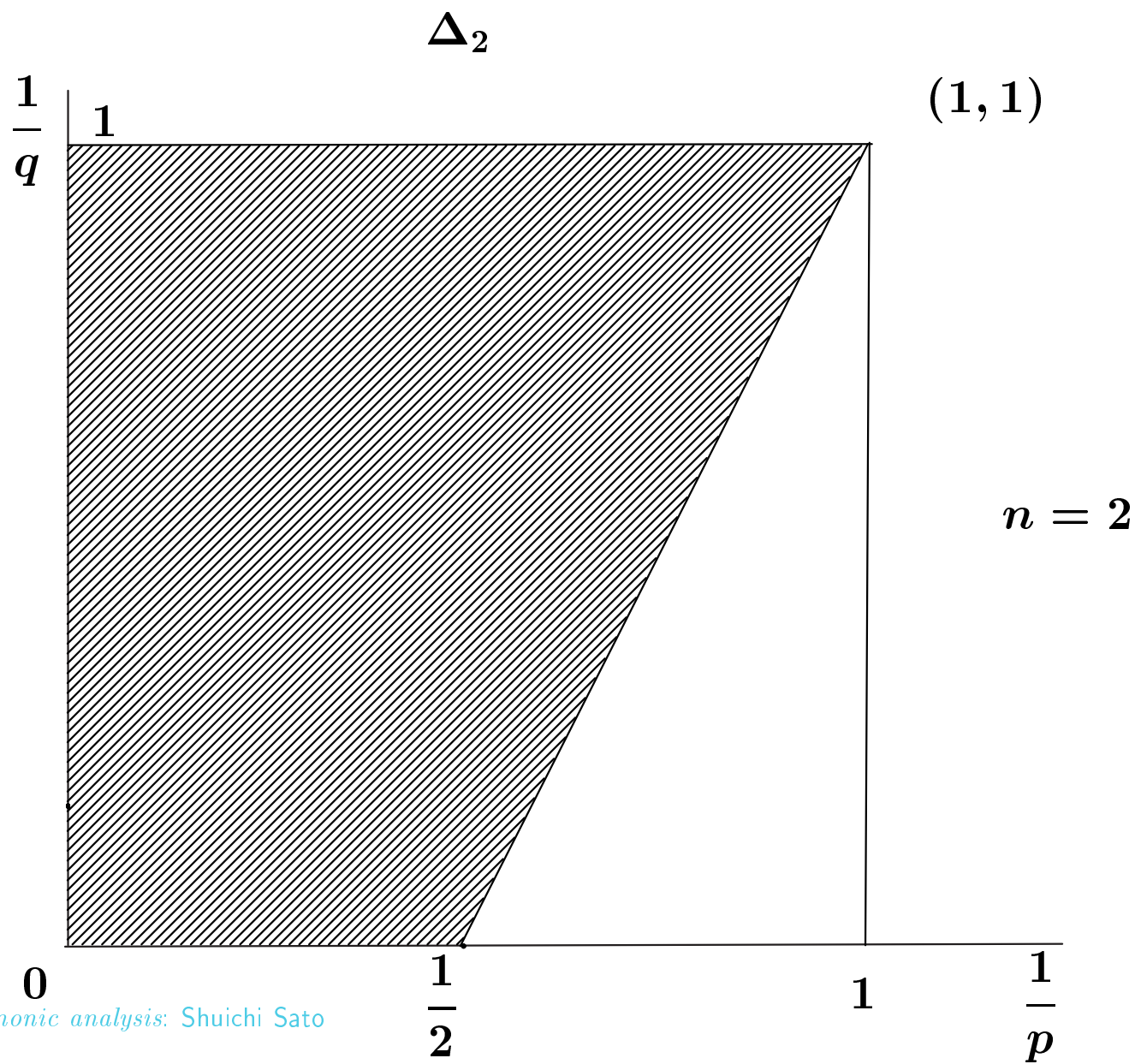
Also, we write

$$\|f\|_{L^p_w} = \|f w^{1/p}\|_{L^p} = \|f w^{1/p}\|_p$$

for $f \in L^p_w(\mathbb{R}^n)$.

$$\Delta_n, \quad n \geq 3$$

$\frac{1}{q}$ 1 $(1,1)$

$\frac{1}{n+1}$

$\left(\dfrac{2}{n+1}, \dfrac{1}{n+1}\right)$

0 $\dfrac{1}{n}$ $\dfrac{2}{n+1}$ 1 $\dfrac{1}{p}$

Δ_2

$\dfrac{1}{q}$

1

$(1,1)$

$n=2$
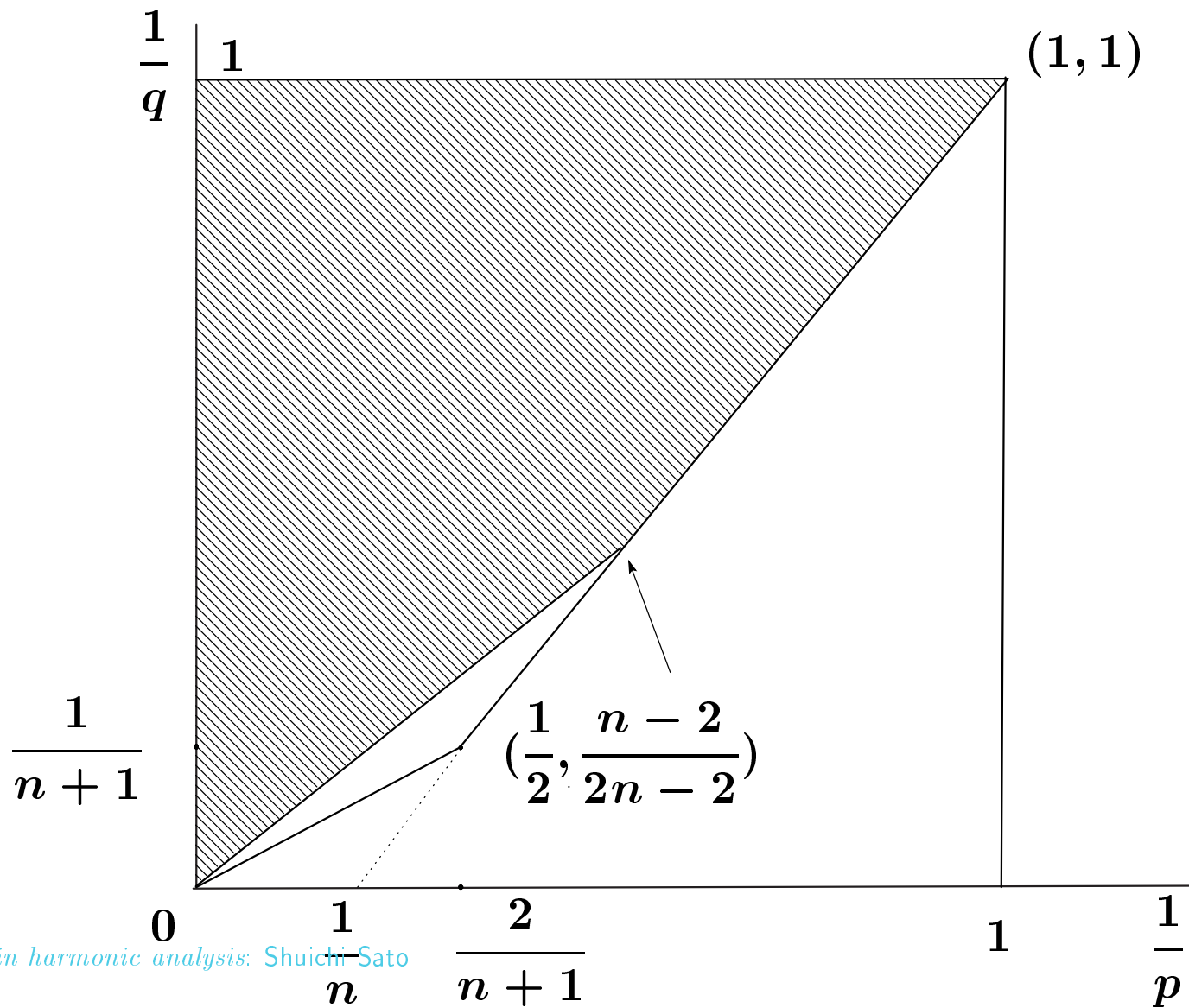
0
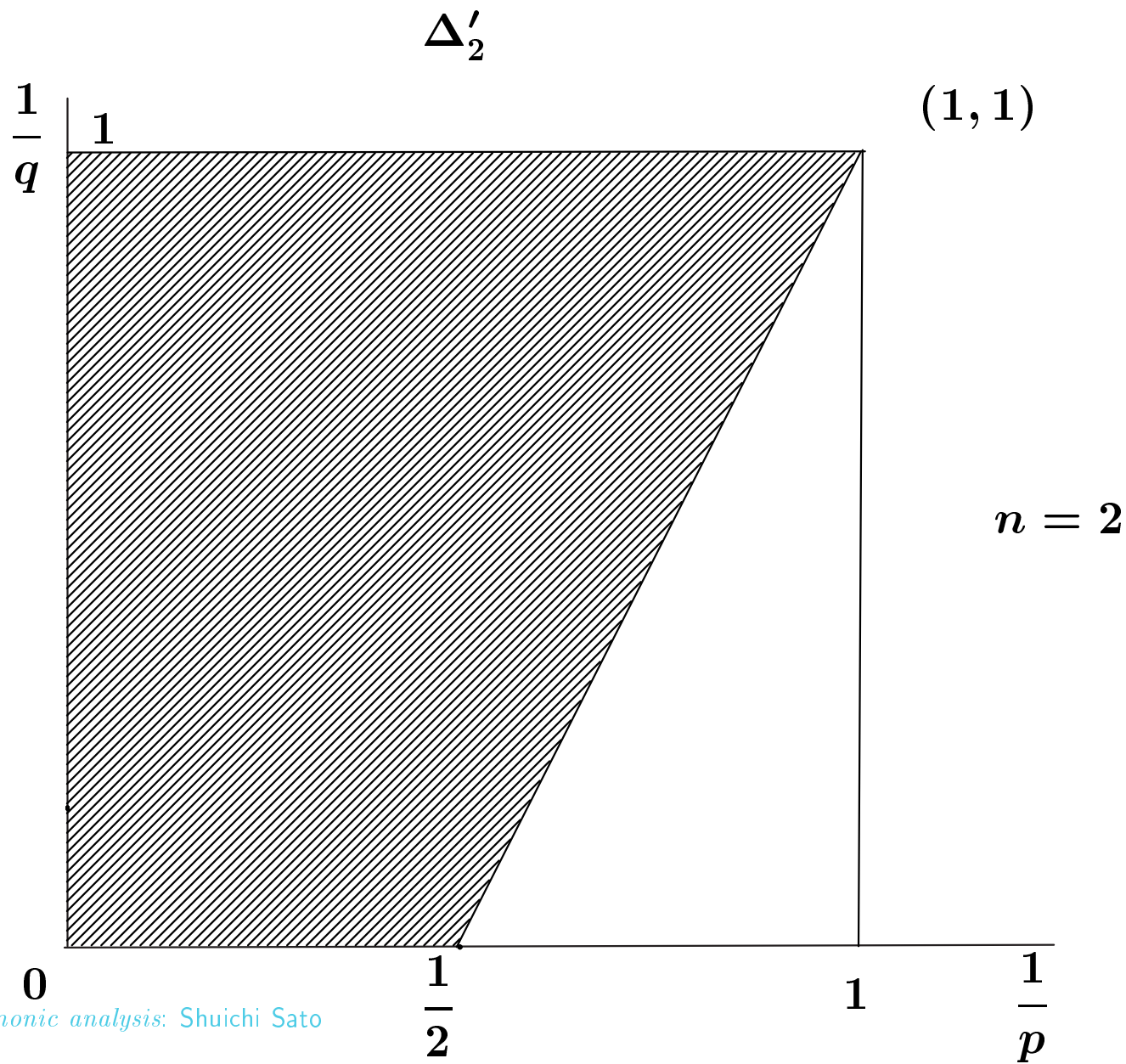
$\dfrac{1}{2}$

1

$\dfrac{1}{p}$

Theorem D. (M. Christ, J. Duoandikoetxea and J. L. Rubio de Francia, 1986.) Suppose $A_t = tE$ (the identity matrix). Then

$$\left(\frac{1}{p}, \frac{1}{q}\right) \in \Delta_n \implies M, H, H_* : L^p(\mathbb{R}^n) \to L^p(L^q).$$

$\Delta'_n, \quad n \geq 3$

$(1,1)$

$\dfrac{1}{q}$

1

$\dfrac{1}{n+1}$

$(\dfrac{1}{2}, \dfrac{n-2}{2n-2})$

0

$\dfrac{1}{n}$

$\dfrac{2}{n+1}$

1

$\dfrac{1}{p}$

Δ_2'

$(1,1)$

$\dfrac{1}{q}$ 1

$n = 2$

0     $\dfrac{1}{2}$ 1 $\dfrac{1}{p}$

Theorem A for M, H was extended to the case of nonisotropic dilations by Bez (2008) as follows.

Theorem E.

(1)
$$\left(\frac{1}{p}, \frac{1}{q}\right) \in \Delta_n \Longrightarrow M : L^p(\mathbb{R}^n) \to L^p(L^q);$$

(2)
$$\left(\frac{1}{p}, \frac{1}{q}\right) \in \Delta_n' \Longrightarrow H : L^p(\mathbb{R}^n) \to L^p(L^q).$$

IDEA: To apply X-ray like transforms of P. Gressman, 2006,

and decay estimates for certain trigonometric integrals of Bez, 2008.

Theorem 4. (Sato, 2012.) Suppose that $\left(\frac{1}{p}, \frac{1}{q}\right) \in \Delta'_n$. Then,

$$H_* : L^p(\mathbb{R}^n) \to L^p(L^q).$$

● This is optimal when $n = 2$ and improves on a result of Lung-Kee Chen (1988).

Theorem 5. (Weighted estimates.) (Sato, 2012.)

$$M, H, H_* : L^2_w(\mathbb{R}^n) \to L^2_w(L^2), \qquad w \in \mathcal{A}_1.$$

● $r(x)^\alpha \in \mathcal{A}_1$ if $-\gamma < \alpha \leq 0$, $\gamma = \mathrm{trace} P$.

IDEA: To apply weighted L-P theory for vector valued functions

and decay estimates for certain trigonometric integrals of Bez, 2008.

==Applications.==

By the method of rotations of Calderón-Zygmund, Theorems 4 and 5 can be applied to singular integrals with ==a variable kernel== **of the form:**

$$Tf(x) = \text{p.v.} \int K(x,y) f(x-y)\, dy = \lim_{\epsilon \to 0} \int_{r(y) \geq \epsilon} K(x,y) f(x-y)\, dy$$

and the maximal singular integral

$$T_* f(x) = \sup_{\epsilon, R > 0} \left| \int_{\epsilon \leq r(y) \leq R} K(x,y) f(x-y)\, dy \right|.$$

THANK YOU !

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