# Multiple singular integrals on product of homogeneous groups

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**We consider singular integral operators and maximal singular integral operators with homogeneous kernels on the product space of homogeneous groups.**

**We prove the $L^p$ boundedness of the singular integrals for $p \in (1, \infty)$ under the $L(\log L)^2$ integrability condition of the kernels on the product of unit spheres.**

**Our methods will give different proofs for some previous results, where singular integrals are defined by Euclidean convolution, since our proofs will not use Fourier transform estimates explicitly.**

**My talk is based on a joint work with Yong Ding (Beijing Normal University).**

**§1.** $\mathbb{R}^{d}$ **as a homogeneous group**

**§2.** $L^p$ **estimates for (one parameter) singular integrals on** $\mathbb{R}^{d}$

**§3.** $L^p$ **estimates for singular integrals on product domains**

**§4.** **Orthogonality in** $L^2$ **via convolution**

**§5.** **Sketch of proof of** $L^p$ **boundeness of multiple singular integrals**

## §1. $\mathbb{R}^d$ as a homogeneous group.

$\mathbb{R}^d$: the $d$ dimensional Euclidean space, $d \geq 2$.
We regard $\mathbb{R}^d$ as a homogeneous group:

- multiplication is given by a polynomial mapping;
- $\exists \{A_t\}_{t>0}$: a dilation family on $\mathbb{R}^d$ such that

$$\boxed{A_t x = (t^{a_1} x_1, t^{a_2} x_2, \dots, t^{a_d} x_d),}$$

  $x = (x_1, \dots, x_d)$, $0 < a_1 \leq a_2 \leq \cdots \leq a_d$,
  $A_t$ is an automorphism of the group structure;
- Lebesgue measure is a bi-invariant Haar measure;
- the identity is the origin $0$, $x^{-1} = -x$.

We also write $\mathbb{R}^d = \mathbb{H}$.

Multiplication $xy$ satisfies

**(1)** $(ux)(vx) = ux + vx$, $x \in \mathbb{H}$, $u, v \in \mathbb{R}$;

**(2)**

$$\boxed{A_t(xy) = (A_t x)(A_t y),\ x, y \in \mathbb{H},\ t > 0;}$$

**(3)** if $z = xy$, $z = (z_1, \dots, z_d)$, $z_k = P_k(x, y)$, then

$$P_1(x, y) = x_1 + y_1,$$
$$P_k(x, y) = x_k + y_k + R_k(x, y) \quad \text{for } k \geq 2,$$

where $R_k(x, y)$ is a polynomial depending only on $x_1, \dots, x_{k-1}, y_1, \dots, y_{k-1}$.

$|x|$: the Euclidean norm for $x \in \mathbb{R}^d$,

$r(x)$: a norm function satisfying $r(A_t x) = t r(x)$, $\forall t > 0$, $\forall x \in \mathbb{R}^d$;

**(1)** $r$ is continuous on $\mathbb{R}^d$ and smooth in $\mathbb{R}^d \setminus \{0\}$;

**(2)** $r(x+y) \le C_0(r(x) + r(y)), \quad r(xy) \le C_0(r(x) + r(y))$

for some $C_0 \ge 1$;

**(3)** $r(x^{-1}) = r(x)$;

**(4)** If $\Sigma_d = \{x \in \mathbb{R}^d : r(x) = 1\}$, then $\Sigma_d = S^{d-1}$,

where $S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$;

**(5)** $\exists c_1, c_2, c_3, c_4, \alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ such that

$$c_1|x|^{\alpha_1} \le r(x) \le c_2|x|^{\alpha_2} \quad \text{if } r(x) \ge 1,$$

$$c_3|x|^{\beta_1} \le r(x) \le c_4|x|^{\beta_2} \quad \text{if } r(x) \le 1.$$

- The space $\mathbb{H}$ with a left invariant quasi-metric $d(x, y) = r(x^{-1}y)$ is a space of homogeneous type.
- if $\gamma = a_1 + \cdots + a_d$ (the homogeneous dimension of $\mathbb{H}$), then $dx = t^{\gamma-1}\, dS_d\, dt$, that is,

$$\int_{\mathbb{R}^d} f(x)\, dx = \int_0^\infty \int_{\Sigma_d} f(A_t\theta) t^{\gamma-1}\, dS_d(\theta)\, dt$$

with $dS_d = \omega\, d\sigma_d$, where $\omega$ is a strictly positive $C^\infty$ function on $\Sigma_d$ and $d\sigma_d$ is the Lebesgue surface measure on $\Sigma_d$.

## Convolution

$$f * g(x) = \int_{\mathbb{R}^d} f(y) g(y^{-1} x)\, dy$$

- $(f * g) * h = f * (g * h)$
- $(f * g)^{\tilde{}} = \tilde{g} * \tilde{f}$ **if** $\tilde{f}(x) = f(x^{-1})$.

**An example.** **Heisenberg group $\mathbb{H}_1$.**

$$(x, y, u)(x', y', u') = (x + x', y + y', u + u' + (xy' - yx')/2),$$

$$(x, y, u), (x', y', u') \in \mathbb{R}^3,$$

**then $\mathbb{R}^3$ with this group law is the Heisenberg group $\mathbb{H}_1$;**
**a dilation is defined by**

$$A_t(x, y, u) = (tx, ty, t^2 u),$$

**and a norm function is**

$$r(x,y,u) = \frac{1}{\sqrt{2}}\sqrt{\sqrt{(x^2+y^2)^2+4u^2}+x^2+y^2}.$$

**Also, we can adopt**

$$A'_t(x,y,u) = (tx, t^2y, t^3u).$$

## §2. $L^p$ estimates for singular integrals on $\mathbb{R}^d$.

**Definition.**

- $F \in L \log L(\Sigma_d)$ (Zygmund class)

$$\Longleftrightarrow$$

$$\int_{\Sigma_d} |F(x)| \log(2 + |F(x)|)\, dS_d(x) < \infty.$$

- $F \in L^q(\Sigma_d) \Longleftrightarrow \|F\|_q = \left( \int_{\Sigma_d} |F|^q \, dS_d \right)^{1/q} < \infty.$

Let $\Omega$ be locally integrable in $\mathbb{R}^d \setminus \{0\}$ and homogeneous of degree $0$ with respect to the dilation group $\{A_t\}$, that is,

$$\Omega(A_t x) = \Omega(x) \qquad \text{for } x \neq 0,\ t > 0.$$

We assume that

$$\int_{\Sigma_d} \Omega(\theta)\, dS_d(\theta) = 0.$$

Let

$$K(x) = \Omega(x')r(x)^{-\gamma}, \qquad x' = A_{r(x)^{-1}}x \text{ for } x \neq 0,$$

where $\gamma = a_1 + \cdots + a_d$. Then $K$ is a locally integrable function on

$\mathbb{R}^d \setminus \{0\}$ **and**

$$K(A_t x) = t^{-\gamma} K(x) \qquad \text{for all } t > 0 \text{ and } x \in \mathbb{R}^d \setminus \{0\}.$$

**Let**

$$Tf(x) = \mathrm{p.v.} f * K(x) = \mathrm{p.v.} \int_{\mathbb{R}^d} f(y) K(y^{-1}x)\, dy.$$

**Theorem $A$ (T. Tao, 1999).** Suppose that $\Omega \in L\log L(\Sigma_d)$. Then, $T$ is bounded on $L^p(\mathbb{R}^d)$ for all $p \in (1,\infty)$.

We also consider the maximal singular integral operator

$$T_* f(x) = \sup_{\epsilon>0}\left|\int_{r(y)>\epsilon} f(xy^{-1})K(y)\,dy\right|.$$

Then the following result is known.

**Theorem $B$.** Suppose that $\Omega \in L\log L(\Sigma_d)$. Then, $T_*$ is bounded on $L^p(\mathbb{R}^d)$ for $p \in (1,\infty)$.

Theorem $A$ for $p \in (1, 2]$ can be proved by interpolation between $L^2$ estimates and weak $(1, 1)$ estimates for $T$ with $\Omega \in L \log L$; both estimates are given by T. Tao (1999); the result for $p \in [2, \infty)$ follows by duality.

For $T_*$ with $\Omega \in L \log L$, neither weak $(1, 1)$ boundedness nor $L^2$ boundedness was known.

We can prove Theorem B and give a different proof of Theorem $A$ via extrapolation arguments;
our proof of Theorem $A$ will not depend on the weak $(1, 1)$ boundedness of $T$ and will be applicable to some other operators for which weak $(1, 1)$ boundedness is not known.

An analogue of a theory of Duoandikoetxea and Rubio de Francia (1986) for homogeneous groups was developed by S. Sato (2010),

**where the use of Fourier transform estimates was replaced by a variant of the $L^2$ orthogonality estimates given by T. Tao (1999).**

**The theory enables us to prove Theorem B and to give a different proof of Theorem A.**

**Here I would like to talk that the theory extends to the case of product spaces of homogeneous groups.**

**Consequently, we can obtain analogues of Theorems A and B for multiple singular integrals with rough kernels.**

## Idea of proof of Theorem $B$.

- Extrapolation on $\Omega$ using

**Proposition.** Let $1 < p < \infty$, $1 < s \le 2$ and $\Omega \in L^s(\Sigma_d)$. Then, there exists a constant $C_p$ independent of $s$ and $\Omega$ such that

$$\|T_*^\Omega f\|_p \le C_p (s-1)^{-1} \|\Omega\|_s \|f\|_p, \quad T_*^\Omega = T_*.$$

We can prove Theorem $B$ from Proposition by decomposing $\Omega \in L \log L$ as

$$\Omega = \sum_{k=1}^{\infty} c_k \Omega_k,$$

where $\sup_{k \ge 1} \|\Omega_k\|_{1+1/k} \le 1$, $c_k \ge 0$, $\sum_{k=1}^{\infty} k c_k < \infty$.

$$
\begin{aligned}
\|T_*^{\Omega} f\|_p &\le \sum_k c_k \|T_*^{\Omega_k} f\|_p \\
&\le \sum_k c_k\, C_p \left( \inf_{s\in(1,2]} (s-1)^{-1} \|\Omega_k\|_s \right) \|f\|_p \\
&\le \sum_k c_k\, C_p\, k \|\Omega_k\|_{1+1/k} \|f\|_p \\
&\le C_p \left( \sum_k k c_k \right) \|f\|_p .
\end{aligned}
$$

**Theory of Duoandikoetxea and Rubio de Francia (1986):**

- **Orthogonality arguments for $L^2$ estimates via Fourier transform estimates and Plancherel's theorem**
- **Littlewood-Paley theory**
- **Interpolation arguments**

**Our strategy is:**

**to employ a version of theory of Duoandikoetxea and Rubio de Francia adapted for analysis on homogeneous groups;**

**replace the use of Fourier transform estimates with $(TT^*)^M$ estimates ($L^2$ orthogonality estimates for convolution) and apply Cotlar's lemma.**

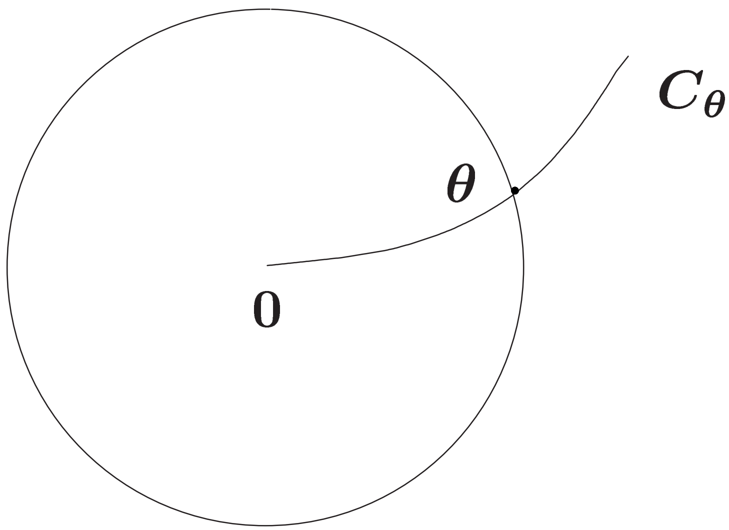
## $(TT^*)^M$ method.

- $\|TT^*\| = \|T\|^2$.

Let $\Omega$ be homogeneous of degree $0$ on $\mathbb{R}^d \setminus \{0\}$. Define

$$C_\theta = \{A_t\theta : t > 0\}, \qquad \theta \in \Sigma_d.$$

Then, $\Omega$ is smooth on $C_\theta$ for every $\theta \in \Sigma_d$ since $\Omega(A_t\theta) = \Omega(\theta)$.



## Orthogonality estimates in $L^2$ via convolution.

**Let $\psi_j \in C_0^\infty(\mathbb{R})$, $j \in \mathbb{Z}$, be such that**

$$\operatorname{supp}(\psi_j) \subset \{t \in \mathbb{R} : \rho^j \le t \le \rho^{j+2}\}, \quad \psi_j \ge 0, \quad \rho \ge 2,$$

$$\log 2 \sum_{j \in \mathbb{Z}} \psi_j(t) = 1 \quad \text{for } t \ne 0,$$

$$|(d/dt)^m \psi_j(t)| \le c_m |t|^{-m} \quad \text{for } m = 0, 1, 2, \dots,$$

**where $c_m$ is independent of $\rho$.**

**Let** $\delta_t K_0(x) = t^{-\gamma} K_0(A_t^{-1}x)$, $\quad K_0(x) = K(x)\chi_{I_0}(x)$, $I_0 = \{x \in \mathbb{R}^d : 1 \le r(x) \le 2\}$,

$$\begin{aligned} S_j K_0(x) &= \int_0^\infty \psi_j(t)\delta_t K_0(x)\, dt/t \\ &= \Omega(x')r(x)^{-\gamma} \int_{1/2}^1 \psi_j(tr(x))\, dt/t. \end{aligned}$$

**Then,** $\mathrm{supp}(S_j K_0) \subset \{x : \rho^j \le r(x) \le 2\rho^{j+2}\}$ **and**

$$\sum_{j\in\mathbb{Z}} S_j K_0 = K, \qquad Tf = \sum_{j\in\mathbb{Z}} f * S_j K_0.$$

**We choose** $\boxed{\rho = 2^{s'}}$ **if** $\Omega \in L^s(\Sigma_d)$.

Let $\phi$ be a $C^\infty$ function such that $\mathrm{supp}(\phi) \subset \{1/2 < r(x) < 1\}$, $\int \phi = 1$, $\phi(x) = \phi(x^{-1})$, $\phi(x) \geq 0$.
Define

$$\Delta_k = \delta_{\rho^{k-1}}\phi - \delta_{\rho^k}\phi, \quad k \in \mathbb{Z}, \quad \delta_t\phi(x) = t^{-\gamma}\phi(A_t^{-1}x).$$

Then

$$\boxed{\sum_k \Delta_k = \delta,}$$

where $\delta$ is the delta function.

**Lemma ($L^2$ orthogonality estimates).** Let $s > 1$, $\Omega \in L^s(\Sigma_d)$, $\rho = 2^{s'}$. Then,

$$\|f * S_j K_0 * \Delta_{k+j}\|_2 \le C \frac{s}{s-1} 2^{-\epsilon|k|} \|\Omega\|_s \|f\|_2.$$

For $s = \infty$, this was proved by T. Tao 1999 with $(TT^*)^d$ method.

## §3. $L^p$ estimates for singular integrals on product domain.

We consider the product space

$$\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \qquad n = n_1 + n_2;$$

$$\mathbb{R}^{n_1} = \mathbb{H}_1, \qquad \mathbb{R}^{n_2} = \mathbb{H}_2$$

are homogeneous groups with dilations $A_t^{(1)}$, $A_t^{(2)}$ and norm functions $r_1, r_2$, respectively.

Let $\Omega \in L^1(\Sigma_{n_1} \times \Sigma_{n_2})$ satisfy

$$\int_{\Sigma_{n_1}} \Omega(u,v)\, dS_{n_1}(u) = \int_{\Sigma_{n_2}} \Omega(u,v)\, dS_{n_2}(v) = 0,$$

$$\forall (u,v) \in \Sigma_{n_1} \times \Sigma_{n_2}.$$

Define the singular integral

$$
\begin{aligned}
Tf(x,y) &= \text{p.v. } f * K(x,y) \\
&= \text{p.v.} \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} f(xu^{-1}, yv^{-1}) K(u,v)\, du\, dv,
\end{aligned}
$$

where

$$
K(u,v) = r_1(u)^{-\gamma_1} r_2(v)^{-\gamma_2} \Omega(u', v'),
$$

$$u' = A^{(1)}_{r_1(u)^{-1}} u,\ v' = A^{(2)}_{r_2(v)^{-1}} v;$$

$\gamma_1$ and $\gamma_2$ are the homogeneous dimensions of $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively.

**Theorem 1.** Suppose that $\Omega \in L(\log L)^2(\Sigma_{n_1} \times \Sigma_{n_2})$. Then,

$$T : L^p(\mathbb{H}_1 \times \mathbb{H}_2) \to L^p(\mathbb{H}_1 \times \mathbb{H}_2) \qquad \text{for all } p \in (1, \infty).$$

Also, we consider the maximal singular integral

$$T_* f(x, y) = \sup_{\substack{\epsilon_1 > 0, \\ \epsilon_2 > 0}} \left| \int_{\substack{r_1(u) > \epsilon_1, \\ r_2(u) > \epsilon_2}} f(xu^{-1}, yv^{-1}) K(u, v)\, du\, dv \right| .$$

**Theorem 2.**

$$T_* : L^p(\mathbb{H}_1 \times \mathbb{H}_2) \to L^p(\mathbb{H}_1 \times \mathbb{H}_2) \qquad \text{for all } p \in (1, \infty),$$

whenever $\Omega \in L(\log L)^2(\Sigma_{n_1} \times \Sigma_{n_2})$.

## Previous works.


- R. Fefferman and E. M. Stein, Singular integrals on product spaces,
  Adv. in Math. 45 (1982), 117–143.
- J. Duoandikoetxea, Multiple singular integrals and maximal functions along hypersurfaces,
  Ann. Inst. Fourier 36 (1986), 185–206.
- H. Al-Qassem and Y. Pan, $L^p$ boundedness for singular integrals with rough kernels on product domains,
  Hokkaido Math. J. 31 (2002), 555–613.
- A. Al-Salman, H. Al-Qassem and Y. Pan, Singular integrals on product domains,
  Indiana Univ. Math. J., 55 (2006), 369–387.

- Theorems 1 and 2 are extensions of results of A. Al-Salman, H. Al-Qassem and Y. Pan (2006) to the case of singular integrals on product of homogeneous groups.
- The optimality of the kernel class $L(\log L)^2$, in the case of Euclidean convolution, can be found in A. Al-Salman, H. Al-Qassem and Y. Pan (2006).
- Our methods give different proofs for previous results, where singular integrals are defined by Euclidean convolution, since our proofs of Theorems 1 and 2 do not use Fourier transform estimates explicitly.

To prove Theorems 1 and 2, we apply extrapolation arguments via the following estimates.

**Proposition 1.** Let $1 < p < \infty$, $1 < s \le 2$ and $\Omega \in L^s(\Sigma_{n_1} \times \Sigma_{n_2})$. Then, $\exists\ C_p$ independent of $s$ and $\Omega$ such that

$$\|Tf\|_p \le C_p (s-1)^{-2} \|\Omega\|_s \|f\|_p.$$

**Proposition 2.** Let $p, s$ and $\Omega$ be as in Proposition 1. Then

$$\|T_* f\|_p \le C_p (s-1)^{-2} \|\Omega\|_s \|f\|_p$$

for some constant $C_p$ independent of $s$ and $\Omega$.

**Proposition 1 $\Longrightarrow$ Theorem 1, Proposition 2 $\Longrightarrow$ Theorem 2.**
**Decomposing $\Omega \in L(\log L)^2(\Sigma_{n_1} \times \Sigma_{n_2})$ as**

$$\Omega = \sum_{k=1}^{\infty} c_k \Omega_k, \qquad c_k \geq 0$$

**where**

$$\int_{\Sigma_{n_1}} \Omega_k(u,v)\, dS_{n_1}(u) = \int_{\Sigma_{n_2}} \Omega_k(u,v)\, dS_{n_2}(v) = 0,$$

$$\sup_{k \geq 1} \|\Omega_k\|_{1+1/k} \leq 1,$$

$$\sum_{k=1}^{\infty} k^2 c_k < \infty.$$

## §4. Orthogonality in $L^2$ via convolution.

We write

$$x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^n, x^{(1)} \in \mathbb{R}^{n_1}, x^{(2)} \in \mathbb{R}^{n_2}, n = n_1 + n_2.$$

Let $\phi^{(i)}$, $i = 1, 2$, be a $C^\infty$ function on $\mathbb{R}^{n_i}$ such that

$$\mathrm{supp}(\phi^{(i)}) \subset \left\{ x^{(i)} \in \mathbb{R}^{n_i} : \frac{1}{2} < r_i(x^{(i)}) < 1 \right\},$$

$$\int \phi^{(i)} = 1, \quad \phi^{(i)} = \tilde{\phi}^{(i)}, \quad \phi^{(i)} \geq 0$$

where $\tilde{\phi}^{(i)}(x^{(i)}) = \phi^{(i)}((x^{(i)})^{-1})$. Set

$$\Delta_k^{(i)} = \delta_{\rho^{k-1}}^{(i)} \phi^{(i)} - \delta_{\rho^k}^{(i)} \phi^{(i)}, \quad k \in \mathbb{Z},$$

where $\delta_t^{(i)}\phi^{(i)}(x^{(i)}) = t^{-\gamma_i}\phi^{(i)}((A_t^{(i)})^{-1}x^{(i)})$, $\rho \geq 2$.

Then

$$\Delta_k^{(i)} = \tilde{\Delta}_k^{(i)}, \quad \sum_k \Delta_k^{(i)} = \delta^{(i)},$$

where $\delta^{(i)}$ is the delta function on $\mathbb{R}^{n_i}$.

Choose $\psi_j \in C_0^\infty(\mathbb{R})$, $j \in \mathbb{Z}$, satisfying

$$\mathrm{supp}(\psi_j) \subset \{t \in \mathbb{R} : \rho^j \leq t \leq \rho^{j+2}\}, \quad \psi_j \geq 0,$$

$$(\log 2)\sum_{j\in\mathbb{Z}} \psi_j(t) = 1 \quad \text{for } t \neq 0,$$

$$|(d/dt)^m \psi_j(t)| \leq c_m |t|^{-m} \quad \text{for } m = 0, 1, 2, \ldots,$$

where $c_m$ is a constant independent of $\rho \geq 2$.

Let $\delta_{s,t} = \delta_s^{(1)} \otimes \delta_t^{(2)}$. Define

$$S_{j,k}F(x) = \int_0^\infty \int_0^\infty \psi_j(s)\psi_k(t)\delta_{s,t}F(x)\,\frac{ds}{s}\,\frac{dt}{t},$$

where $F \in L^1(\mathbb{R}^n)$, $\mathrm{supp}(F) \subset D_0$, $D_0 = D_0^{(1)} \times D_0^{(2)}$,

$$D_0^{(i)} = \{x^{(i)} \in \mathbb{R}^{n_i} : 1 \le r_i(x^{(i)}) \le 2\}.$$

Let $K_0(x) = K(x)\chi_{D_0}(x)$. Then

$$\sum_{j,k\in\mathbb{Z}} S_{j,k}K_0 = K.$$

Let $\Phi^{(i)}$ be a non-negative smooth function on $\mathbb{R}^{n_i}$ such that

$$\int \Phi^{(i)}(x^{(i)})\, dx^{(i)} = 1, \tilde{\Phi}^{(i)} = \Phi^{(i)}, \mathrm{supp}(\Phi^{(i)}) \subset \left\{ r_i(x^{(i)}) < 1 \right\}.$$

For $F \in L^1(\mathbb{R}^n)$ with $\mathrm{supp}(F) \subset D_0$, define the operator $U_\sigma = U_\sigma(F)$ by

$$U_\sigma f = U_\sigma(F)(f) = \sum_{j,k} \sigma_{j,k} f * \nu_{j,k},$$

where

$$\nu_{j,k}(x) = \nu_{j,k}(F)(x) = S_{j,k}F(x) - \Phi^{(1)}_{j,k}(x) - \Phi^{(2)}_{j,k}(x) + \Phi_{j,k}(x),$$

$$\Phi^{(1)}_{j,k}(x) = \Phi^{(1)}_{j,k}(F)(x) = \left(\int S_{jk}F(x)\,dx^{(1)}\right)\delta^{(1)}_{\rho^j}\Phi^{(1)}(x^{(1)}),$$

$$\Phi^{(2)}_{j,k}(x) = \Phi^{(2)}_{j,k}(F)(x) = \left(\int S_{jk}F(x)\,dx^{(2)}\right)\delta^{(2)}_{\rho^k}\Phi^{(2)}(x^{(2)}),$$

$$\Phi_{j,k}(x) = \Phi_{j,k}(F)(x) = \left(\int S_{jk}F(x)\,dx\right)\delta_{\rho^j,\rho^k}\Phi(x), \quad \Phi = \Phi^{(1)} \otimes \Phi^{(2)},$$

and $\sigma = \{\sigma_{j,k}\}$ is an arbitrary sequence such that $\sigma_{j,k} = 1$ or $-1$. Then

$$\int \nu_{j,k}(x)\,dx^{(i)} = 0, \quad S_{j,k}K_0 = \nu_{j,k}(K_0)$$

$$U_\sigma(K_0)(f) = Tf \quad \text{if } \sigma_{j,k} = 1 \text{ for all } j, k.$$

For $s \geq 1$, let

$$L^s(D_0) = \{F \in L^s(\mathbb{H}_1 \times \mathbb{H}_2) : \operatorname{supp} F \subset D_0\}.$$

**Lemma 1.** Suppose that $F \in L^s(D_0)$, $s \in (1,2]$. Let $\nu_{j_1,j_2} = \nu_{j_1,j_2}(F)$,

$$a(t) = \min(1, \rho^{-t}), \qquad t \in \mathbb{R}.$$

Then, for $j_i, k_i \in \mathbb{Z}$, $i = 1, 2$, we have

$$\|f * \nu_{j_1,j_2} * \Delta_{k_1,k_2}\|_2 \leq C(\log \rho)^2 \left(\prod_{i=1}^{2} a(\epsilon(|j_i - k_i| - c)/s')\right) \|F\|_s \|f\|_2$$

for some positive constants $C, \epsilon$ and $c$ independent of $\rho$, $s$ and $F$, where $\Delta_{k_1,k_2} = \Delta_{k_1}^{(1)} \otimes \Delta_{k_2}^{(2)}$ and $s' = s/(s-1)$.

Put $S = S_{0,0}F$. By the $T^*T$ method, one of the key estimates to prove Lemma 1 is the following:

$$\left\| f * \left( \Delta_{k_1,k_2} * \tilde{S} * S * \Delta_{k_1,k_2} \right)_*^{n_1} \right\|_2 \le C(\log \rho)^{4n_1} \rho^{\epsilon(k_1+k_2+c)/s'} \|F\|_s^{2n_1} \|f\|_2,$$

for some $\epsilon, c > 0$, where

$$g_*^m = \underbrace{g * \cdots * g}_{m \text{ times}}$$

and we may assume $n_1 \ge n_2$ without loss of generality.

## §5. Proof of Proposition 1.

**Lemma 2 (Littlewood-Paley inqualities).** Let $1 < p < \infty$ and $\Delta_{k_1,k_2} = \Delta^{(1)}_{k_1} \otimes \Delta^{(2)}_{k_2}$. Then

$$\left\| \sum_{k_1,k_2} f_{k_1,k_2} * \Delta_{k_1,k_2} \right\|_p \le C_p \left\| \left( \sum_{k_1,k_2} |f_{k_1,k_2}|^2 \right)^{1/2} \right\|_p,$$

$$\left\| \left( \sum_{k_1,k_2} |f * \Delta_{k_1,k_2}|^2 \right)^{1/2} \right\|_p \le C_p \|f\|_p,$$

where the constant $C_p$ is independent of $\rho \ge 2$.

We write $U_\sigma = U_\sigma(F)$ with $\rho = 2^{s'}$ and
$U_\sigma f = \sum_{k^{(1)},k^{(2)}} U_{k^{(1)},k^{(2)}} f, \quad k^{(i)} = (k_1^{(i)}, k_2^{(i)}) \in \mathbb{Z}^2,$

$$U_{k^{(1)},k^{(2)}} f = \sum_{j\in\mathbb{Z}^2} \sigma_j f * \Delta_{k^{(1)}+j} * \nu_j * \Delta_{k^{(2)}+j}, \quad \nu_j = \nu_j(F), \quad j = (j_1, j_2);$$

$$\nu_{r,s}(x) = \nu_{r,s}(F)(x) = S_{r,s}F(x) - \Phi^{(1)}_{r,s}(x) - \Phi^{(2)}_{r,s}(x) + \Phi_{r,s}(x),$$

$$\Phi^{(1)}_{r,s}(x) = \Phi^{(1)}_{r,s}(F)(x) = \left(\int S_{r,s}F(x)\, dx^{(1)}\right)\delta^{(1)}_{\rho^r}\Phi^{(1)}(x^{(1)}),$$

$$\Phi^{(2)}_{r,s}(x) = \Phi^{(2)}_{r,s}(F)(x) = \left(\int S_{r,s}F(x)\, dx^{(2)}\right)\delta^{(2)}_{\rho^s}\Phi^{(2)}(x^{(2)}),$$

$$\Phi_{r,s}(x) = \Phi_{r,s}(F)(x) = \left(\int S_{r,s}F(x)\, dx\right)\delta_{\rho^r,\rho^s}\Phi(x), \quad \Phi = \Phi^{(1)} \otimes \Phi^{(2)}.$$

**Fix $k^{(1)}, k^{(2)} \in \mathbb{Z}^2$. By Lemma 1 with $\rho = 2^{s'}$ and duality,**

$$\|f * \Delta_k * \nu_j\|_2 \le C(s-1)^{-2}\|F\|_s\|f\|_2 \prod_{i=1}^{2} \lambda(\epsilon(|k_i - j_i| - c)),$$

**where**

$$\lambda(t) = \min(1, 2^{-t}), \qquad t \in \mathbb{R}.$$

**Applying this and Lemma 1, to $\nu_j$ and $\tilde{\nu_j}$, and noting that**

$$\|\Delta_{k^{(2)}+j} * \Delta_{k^{(2)}+j'}\|_1 \le C \prod_{i=1}^{2} \lambda(\epsilon(|j_i - j_i'| - c)), \quad j' = (j_1', j_2'),$$

$\|\Delta_k\|_1 \le C$, **we have**

$$\left\| f * (\Delta_{k^{(1)}+j} * \nu_j) * (\Delta_{k^{(2)}+j} * \Delta_{k^{(2)}+j'}) * (\tilde{\nu}_{j'} * \Delta_{k^{(1)}+j'}) \right\|_2$$

$$\le CA^2 \|f\|_2 \prod_{i=1}^{2} \lambda(2\epsilon(|k_i^{(1)}| - c))\lambda(\epsilon(|j_i - j_i'| - c)),$$

$$A = (s-1)^{-2}\|F\|_s,$$

$$\left\| f * \Delta_{k^{(1)}+j} * (\nu_j * \Delta_{k^{(2)}+j}) * (\Delta_{k^{(2)}+j'} * \tilde{\nu}_{j'}) * \Delta_{k^{(1)}+j'} \right\|_2$$

$$\le CA^2 \|f\|_2 \prod_{i=1}^{2} \lambda(2\epsilon(|k_i^{(2)}| - c)).$$

Taking the geometric mean, we have

$$\left\| f * \left( \Delta_{k^{(1)}+j} * \nu_j * \Delta_{k^{(2)}+j} \right) * \left( \Delta_{k^{(2)}+j'} * \tilde{\nu}_{j'} * \Delta_{k^{(1)}+j'} \right) \right\|_2$$
$$\leq CA^2 \|f\|_2 \prod_{i=1}^{2} \left( \prod_{m=1}^{2} \lambda(\epsilon(|k_i^{(m)}| - c)) \right) \lambda(\epsilon(|j_i - j_i'| - c)/2).$$

We can treat

$$\|f * \left( \Delta_{k^{(2)}+j'} * \tilde{\nu}_{j'} * \Delta_{k^{(1)}+j'} \right) * \left( \Delta_{k^{(1)}+j} * \nu_j * \Delta_{k^{(2)}+j} \right) \|_2$$

similarly. Thus, by the Cotlar-Knapp-Stein lemma

$$\left\| U_{k^{(1)},k^{(2)}} f \right\|_2 \leq CA \|f\|_2 \prod_{m=1}^{2} \prod_{i=1}^{2} \lambda(\epsilon(|k_i^{(m)}| - c)/2)$$

uniformly in $\sigma$. This implies that

$$\|U_\sigma f\|_2 \le \sum_{k^{(1)},k^{(2)}} \|U_{k^{(1)},k^{(2)}} f\|_2 \le CA\|f\|_2, \quad A = (s-1)^{-2}\|F\|_s.$$

By the bootstrap argument of Duoandikoetxea-Rubio de Francia (1986), we can prove that

$$\|U_\sigma f\|_p \le C_p A \|f\|_p, \quad A = (s-1)^{-2}\|F\|_s \quad p \in (1,2],$$

for all $F \in L^s(D_0)$, where $C_p$ is independent of $\sigma$, $F$ and $s$.

To carry out the bootstrap argument, we use the Littlewood-Paley theory (Lemma 2) and the Khintchine inequality, and need to consider general $U_\sigma(F)$, even though we would like to have the result for the particular case

$$U_\sigma(K_0)(f) = Tf, \qquad \sigma_{j,k} = 1 \text{ for all } j, k.$$

Thus we have

$$\|Tf\|_p \le C(s-1)^{-2}\|\Omega\|_s\|f\|_p \quad \text{for } p \in (1, 2];$$

a duality argument will imply the conclusion for $p \in [2, \infty)$.

**THANK YOU**

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