Cesàro and Riesz means of critical order on certain function spaces

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§1. Fourier series; 1 dimensional case.

Let

$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}, \qquad a_n = \int_{Q_1} f(x) e^{-2\pi i n x}\, dx$$

be the Fourier series for $f \in L^1(Q_1)$, $Q_1 = (-1/2, 1/2]$ and

$$T_N f(x) = \sum_{|n|<N} a_n e^{2\pi i n x}$$

the partial sum.

According to J. Arias-de-Reyna (2002), we define a space $\mathfrak{Q}\mathcal{A}(Q_1)$.

Definition.

$$f \in \mathfrak{Q}\mathcal{A}(Q_1)$$

$$\Longleftrightarrow$$

there exists a sequence $\{f_j\}$ of bounded functions such that

$$f = \sum_{j=1}^{\infty} f_j, \quad N(\{f_j\}) := \sum_{j=1}^{\infty}(1+\log j)\|f_j\|_1 \log\left(\frac{e\|f_j\|_\infty}{\|f_j\|_1}\right) < \infty;$$

$$\|f\|_{\mathfrak{Q}\mathcal{A}} = \inf N(\{f_j\}),$$

where the infimum is taken over all possible $\{f_j\}$.

Then, the space $\mathfrak{Q}\mathcal{A}$ is a subspace of $L \log L$ and is a logconvex quasi-Banach

space of N. J. Kalton (1981), where logconvex means

$$\exists C > 0 : \quad \left\| \sum_{j=1}^{\infty} f_j \right\|_{\mathfrak{Q}\mathcal{A}} \leq C \sum_{j=1}^{\infty} (1 + \log j) \| f_j \|_{\mathfrak{Q}\mathcal{A}}.$$

space of N. J. Kalton (1981), where logconvex means

$$\exists C > 0 : \quad \left\| \sum_{j=1}^{\infty} f_j \right\|_{\mathcal{QA}} \le C \sum_{j=1}^{\infty} (1 + \log j) \|f_j\|_{\mathcal{QA}}.$$

Define $T_*(f)(x) = \sup_N |T_N(f)(x)|$. Then:

Theorem A (J. Arias-de-Reyna, 2002). There exists a positive constant C such that

$$\|T_*(f)\|_{1,\infty} = \sup_{\lambda > 0} \lambda |\{x \in Q_1 : T_*(f)(x) > \lambda\}| \le C\|f\|_{\mathcal{QA}};$$

consequently,

$$\lim_{R \to \infty} T_N(f)(x) = f(x) \quad a.e. \qquad \text{for } f \in \mathcal{QA}(Q_1).$$

It is known that $L \log L \log \log \log L$ is a proper subspace of $\mathcal{QA}$. So, Theorem A implies the following.

Theorem B (Antonov, 1996). If $f \in L \log L \log \log \log L(Q_1)$, then

$$\lim_{N \to \infty} T_N(f)(x) = f(x) \quad a.e.$$

The convergence a.e. for $f \in L \log L \log \log L(Q_1)$ was proved by P. Sjölin (1968).

- For $f \in L^2(Q_1)$, there is a result of L. Carleson (1966) which shows that $\{T_N f\}$ converges a.e.

- R. Hunt (1968) proved the restricted weak type estimates:

$$\boxed{\star \quad \sup_{\lambda>0} \lambda |\{x \in Q_1 : T_*(\chi_A)(x) > \lambda\}|^{1/p} \le Cp^2(p-1)^{-1}|A|^{1/p},}$$

for $1 < p < \infty$, where χ_A denotes the characteristic function of a set $A \subset Q_1$.

- By ★ R. Hunt (1968) proved the convergence a.e. of $\{T_N f\}$ for $f \in L(\log L)^2(Q_1)$.

- P. Sjölin proved that ★ can be used to prove the convergence a.e. for the class $L \log L \log\log L(Q_1)$.

- Applying ★ more effectively, N. Yu. Antonov (1996) proved that $\{T_N f\}$ converges a.e. if $f \in L \log L \log\log\log L(Q_1)$.

I would like to talk about analogues of Theorem A for

(1) **the Cesàro means of the critical order $1/2$ for spherical harmonics expansions of functions on the unit sphere of $\mathbb{R}^3$;**

(2) **the Bochner-Riesz means of order $(d-1)/2$ for multiple Fourier series of periodic functions on $\mathbb{R}^d$, $d \geq 2$.**

§2. Bochner-Riesz means of multiple Fourier series.

Let

$$Q_d = \{x \in \mathbb{R}^d : -1/2 < x_i \le 1/2, i = 1, 2, \dots, d\}, \quad x = (x_1, \dots, x_d),$$

be the fundamental cube in the d-dimensional Euclidean space $\mathbb{R}^d$. For $f \in L^1(Q_d)$ we consider the Fourier series

$$f(x) \sim \sum a_n e^{2\pi i \langle n, x\rangle}, \quad n = (n_1, n_2, \dots, n_d) \in \mathbb{Z}^d,$$

where $\langle n, x\rangle = n_1 x_1 + \cdots + n_d x_d$ and

$$a_n = \int_{Q_d} f(x) e^{-2\pi i \langle n, x\rangle}\, dx, \quad dx = dx_1 \dots dx_d,$$

is the Fourier coefficient. The Bochner-Riesz means of order δ of the series are defined by

$$T_R^\delta(f)(x) = \sum_{|n|<R} \left(1 - \frac{|n|^2}{R^2}\right)^\delta a_n e^{2\pi i\langle n,x\rangle}, \quad |n| = (n_1^2 + \cdots + n_d^2)^{1/2}.$$

Definition (Arias-de-Reyna).

$$f \in \mathfrak{Q}\mathcal{A}(Q_d)$$

$$\Longleftrightarrow$$

there exists a sequence $\{f_j\}$ of bounded functions such that

$$f = \sum_{j=1}^\infty f_j, \quad N(\{f_j\}) := \sum_{j=1}^\infty (1 + \log j)\|f_j\|_1 \log\left(\frac{e\|f_j\|_\infty}{\|f_j\|_1}\right) < \infty;$$

$$\|f\|_{\mathfrak{Q}\mathcal{A}} = \inf N(\{f_j\}),$$

where the infimum is taken over all possible $\{f_j\}$.

The space $\mathfrak{Q}\mathcal{A}$ is a logconvex quasi-Banach space and a subspace of $L \log L$.

Define $T_*^\delta(f)(x) = \sup_{R>0} |T_R^\delta(f)(x)|$. Let $\alpha = (d-1)/2$ (the critical index).

Theorem 1. There exists a positive constant C such that

$$\|T_*^\alpha(f)\|_{1,\infty} = \sup_{\lambda>0} \lambda |\{x \in Q_d : T_*^\alpha(f)(x) > \lambda\}| \leq C\|f\|_{\mathcal{QA}};$$

consequently,

$$\lim_{R\to\infty} T_R^\alpha(f)(x) = f(x) \quad a.e. \qquad \text{for } f \in \mathcal{QA}(Q_d).$$

Since $L \log L \log \log \log L \subset \mathcal{QA}$, Theorem 1 implies the following.

Theorem 2. If $f \in L \log L \log \log \log L(Q_d)$, then

$$\lim_{R\to\infty} T_R^\alpha(f)(x) = f(x) \quad a.e.$$

The convergence a.e. for $f \in L \log L \log \log L(Q_d)$ was proved by G. Sunouchi (1985).

To prove Theorem 1 we use the following estimates:

Lemma 1. Let $1 < p \leq 2$. Then, there exists a constant C independent of p such that

$$\sup_{\lambda>0} \lambda |\{x \in Q_d : T_*^{\alpha}(f)(x) > \lambda\}|^{1/p} \leq C(p-1)^{-1}\|f\|_p.$$

Lemma 1 was proved in G. Sunouchi (1985) by using the following two results and analytic interpolation:

Lemma 2 (E. M. Stein, 1958). Suppose $f \in L^1(Q_d)$ and $\sigma > \alpha$. Then

$$\|T_*^\delta(f)\|_{1,\infty} \le A_\sigma e^{\pi|\tau|}(\sigma - \alpha)^{-1}\|f\|_1, \quad \delta = \sigma + i\tau, \sigma, \tau \in \mathbb{R},$$

where A_σ remains bounded as $\sigma \to \alpha$.

Lemma 3 (E. M. Stein, 1958). Suppose that $f \in L^2(Q_d)$. Then

$$\|T_*^\delta(f)\|_2 \le A_\sigma e^{\pi|\tau|}\|f\|_2, \quad \sigma > 0.$$

- Theorem 1 can be proved in the same way as Theorem A by applying Lemma 1, as we shall briefly see below.

§3. Cesàro means of spherical harmonics expansions.

We have analogous results for the Cesàro means of spherical harmonics expansions.

$\mathcal{H}_k$: the space of the spherical harmonics of degree k on Σ_d,

$\Sigma_d = \{x \in \mathbb{R}^{d+1} : |x| = 1\}$: the unit sphere in $\mathbb{R}^{d+1}$.

We recall that the space $\mathcal{H}_k$ consists of the restrictions to Σ_d of harmonic homogeneous polynomials of degree k.

Let

$$H_k f(x) = \int_{\Sigma_d} Z_x^{(k)}(y) f(y)\, d\mu(y),$$

where $d\mu$ is the Lebesgue surface measure on Σ_d normalized as $|\Sigma_d| = \mu(\Sigma_d) = 1$, and $Z_x^{(k)} \in \mathcal{H}_k$ is the zonal harmonic of degree k with pole $x \in \Sigma_d$:

$$\begin{aligned} Z_x^{(k)}(y) &= \left(\frac{2k}{d-1} + 1\right) \frac{\Gamma(d/2)\Gamma(d+k-1)}{\Gamma(d-1)\Gamma(k+d/2)} P_k^{((d-2)/2,(d-2)/2)}(\langle x, y\rangle) \\ &= \left(\frac{2k}{d-1} + 1\right) P_k^{((d-1)/2)}(\langle x, y\rangle). \end{aligned}$$

Here $P_k^{(\alpha,\beta)}$ is the Jacobi polynomial and $P_k^{(\lambda)}$ is the Gegenbauer polynomial defined by $(1-2tr+r^2)^{-\lambda} = \sum_{k=0}^{\infty} P_k^{(\lambda)}(t) r^k$. We consider the spherical harmonics expansion

$$f \sim \sum_{k=0}^{\infty} H_k f$$

and the Cesàro means of order δ defined by

$$S_n^\delta f = \frac{1}{A_n^{(\delta)}} \sum_{k=0}^{n} A_{n-k}^{(\delta)} H_k f, \quad n = 0, 1, 2, \dots, \quad \delta = \sigma + i\tau,$$

where

$$A_k^{(\delta)} = \frac{\Gamma(k+\delta+1)}{\Gamma(k+1)\Gamma(\delta+1)} = \binom{k+\delta}{k}, \quad \sigma > -1$$

Let $S_*^\delta(f)(x) = \sup_{n>0} |S_n^\delta(f)(x)|$.

We define the space $\mathfrak{QA}(\Sigma_d)$ analogously to $\mathfrak{QA}(Q_d)$.

Theorem 3. There exists a positive constant C such that

$$\sup_{\lambda>0} \lambda|\{x \in \Sigma_2 : S_*^{1/2}(f)(x) > \lambda\}| \le C\|f\|_{\mathcal{QA}}$$

for $f \in \mathcal{QA}(\Sigma_2)$, which implies

$$\lim_{n\to\infty} S_n^{1/2}(f)(x) = f(x) \quad a.e. \qquad \text{for } f \in \mathcal{QA}(\Sigma_2).$$

Theorem 3 implies the following result as Theorem 1 implies Theorem 2.

Theorem 4. If $f \in L \log L \log \log \log L(\Sigma_2)$, then

$$\lim_{n\to\infty} S_n^{1/2} f(x) = f(x) \quad a.e.$$

The convergence a.e. of $\{S_n^{1/2}f\}$ for $f \in L^p(\Sigma_2)$, $p > 1$, can be found in A. Bonami and J.-L. Clerc (1973).

The proof of Theorem 3 is similar to that of Theorem 1, if we have the following estimates:

Lemma 4. Let $1 < p \leq 2$. Then, we have

$$\sup_{\lambda>0} \lambda |\{x \in \Sigma_2 : S_*^{1/2}(f)(x) > \lambda\}|^{1/p} \leq C(p-1)^{-1}\|f\|_p$$

for a positive constant C independent of p.

To prove Lemma 4 we need the following two results.

Lemma 5. Suppose that $f \in L^1(\Sigma_2)$ and $\alpha < \sigma < 1$, where $\alpha = 1/2$. Then

$$\|S_*^\delta(f)\|_{1,\infty} \le A_\sigma e^{B\tau^2}(\sigma - \alpha)^{-1}\|f\|_1, \quad \delta = \sigma + i\tau.$$

The constant A_σ remains bounded as $\sigma \to \alpha$.

When δ is real, Lemma 5 is known for all d (A. Bonami and J.-L. Clerc,1973, L. Colzani, M. H. Taibleson and G. Weiss, 1984).

Lemma 6 (A. Bonami-J.-L. Clerc, 1973). Suppose that $f \in L^2(\Sigma_2)$. Then

$$\|S_*^\delta(f)\|_2 \le A_\sigma e^{B_\sigma \tau^2}\|f\|_2, \quad \sigma > 0.$$

A_σ and B_σ are bounded on any compact subinterval of $(0, \infty)$.

Using Lemmas 5 and 6, we can prove Lemma 4 by analytic interpolation of Sagher (1969).

§4. Proof of Theorem 1.

Lemma 7. Let $\subset E \subset Q_d$. Then

$$\|T_*^\alpha(\chi_E)\|_{1,\infty} \le C|E| \log\left(\frac{e}{|E|}\right).$$

Proof. For $\lambda > 0$, let $m(\lambda) = \inf_{1<p\le 2} \lambda^{-p}(p-1)^{-p}$. Then, by Lemma 1:

$$\sup_{\lambda>0} \lambda^p |\{x \in Q_d : T_*^\alpha(f)(x) > \lambda\}| \le C(p-1)^{-p}\|f\|_p^p,$$

we have

$$|\{x \in Q_d : T_*^\alpha(\chi_E)(x) > \lambda\}| \le C \min\left(1, m(\lambda)|E|\right).$$

This will imply the conclusion, since

$$m(\lambda) \lesssim \frac{1}{\lambda} \log\left(2 + \frac{1}{\lambda}\right).$$

Lemma 8. If $f \in L^\infty(Q_d)$, then

$$\|T_*^\alpha(f)\|_{1,\infty} \le C\|f\|_1 \log\left(\frac{e\|f\|_\infty}{\|f\|_1}\right).$$

If $f = A\chi_E$, $A > 0$, $E \subset Q_d$, then Lemma 7 implies Lemma 8. Let $\mathfrak{R} = \{R_1, R_2, \dots, R_N\}$ be any finite set of positive numbers. Define

$$T_{\mathfrak{R}}^\alpha(f) = \sup_{1\le j\le N} |T_{R_j}^\alpha(f)|.$$

The transition from $A\chi_E$ to a general f can be carried out by

$$\boxed{\inf_E \|T_{\mathfrak{R}}^\alpha(f - \|f\|_\infty \chi_E)\|_1 = 0,}$$

where the infimum is taken over all E satisfying $|E|\|f\|_\infty = \|f\|_1$.
This is the idea of Antonov (1996), which can be applied to T_R^α and $S_n^{1/2}$.

Proof of Theorem 1.

Suppose $f \in \mathcal{QA}(Q_d)$ and

$$f = \sum_{j=1}^{\infty} f_j, \quad N(\{f_j\}) := \sum_{j=1}^{\infty}(1+\log j)\|f_j\|_1 \log\left(\frac{e\|f_j\|_\infty}{\|f_j\|_1}\right) < \infty.$$

Since $L^{1,\infty}$ is a logconvex quasi-Banach space (N. J. Kalton, 1981), by Lemma 8 we have

$$\begin{aligned}
\|T_*^\alpha(f)\|_{1,\infty} &\leq C\sum_{j=1}^{\infty}(1+\log j)\|T_*^\alpha(f_j)\|_{1,\infty} \\
&\leq C\sum_{j=1}^{\infty}(1+\log j)\|f_j\|_1 \log\left(\frac{e\|f_j\|_\infty}{\|f_j\|_1}\right) \\
&= CN(\{f_j\}).
\end{aligned}$$

Taking the infimum we get the conclusion: $\|T_*^\alpha(f)\|_{1,\infty} \leq C\|f\|_{\mathcal{QA}}$.

§5. Proof of Lemma 5 .

Lemma 4. Let $1 < p \le 2$. Then, we have

$$\sup_{\lambda>0} \lambda |\{x \in \Sigma_2 : S_*^{1/2}(f)(x) > \lambda\}|^{1/p} \le C(p-1)^{-1}\|f\|_p$$

for a positive constant C independent of p.

Lemma 5. Suppose that $f \in L^1(\Sigma_2)$ and $\alpha < \sigma < 1$, where $\alpha = 1/2$. Then

$$\|S_*^{\delta}(f)\|_{1,\infty} \le A_\sigma e^{B\tau^2}(\sigma-\alpha)^{-1}\|f\|_1, \quad \delta = \sigma + i\tau.$$

The constant A_σ remains bounded as $\sigma \to \alpha$.

Let

$$S_n^{(\delta,\lambda)}(\cos v) = (A_n^{(\delta)})^{-1} \sum_{k=0}^{n} A_{n-k}^{(\delta)} 2(k+\lambda) P_k^{(\lambda)}(\cos v),$$

where

$$0 < \lambda < 1, \quad 0 \le v \le \pi, \quad 0 < \sigma < 1, \quad \delta = \sigma + i\tau.$$

Then, $S_n^{(\delta,1/2)}(\langle x, y\rangle)$ is the kernel of the operator S_n^δ:

$$S_n^\delta f(x) = \int_{\Sigma_2} S_n^{(\delta,1/2)}(\langle x, y\rangle) f(y)\, d\mu(y).$$

Let

$$i_n^{(\delta,\lambda)}(v) = \frac{\lambda \sin(\delta\pi)}{\pi} \int_0^1 \frac{u^{n+\delta+2\lambda}}{(1-u)^\delta (1-2u\cos v + u^2)^{\lambda+1}}\, du,$$

$$\mathcal{I}_n^{(\delta,\lambda)}(v) = \frac{\exp\left(-i\left[(n+\lambda+(\delta+1)/2)v - (\lambda+\delta+1)\pi/2\right]\right)}{(2\sin v)^\lambda (2\sin(v/2))^{\delta+1}} \frac{\sin(\lambda\pi)}{\pi}$$

$$\times \int_0^1 \frac{u^{-\lambda}(1-u)^{n+\delta+2\lambda}}{(1-u\tau(v/2))^{\delta+1}(1-u\tau(v))^\lambda}\, du,$$

$$\mathcal{J}_n^{(\delta,\lambda)}(v) = \frac{\exp\left(i\left[(n+\lambda+(\delta+1)/2)v - (\lambda+\delta+1)\pi/2\right]\right)}{(2\sin v)^\lambda (2\sin(v/2))^{\delta+1}} \frac{\sin(\lambda\pi)}{\pi}$$

$$\times \int_0^1 \frac{u^{-\lambda}(1-u)^{n+\delta+2\lambda}}{(1-u\tau(-v/2))^{\delta+1}(1-u\tau(-v))^\lambda}\, du,$$

where $\tau(v) = (1 + i\cot v)/2$.

Then, by E. Kogbetliantz (1924) it follows that

$$\frac{1}{2}A_n^{(\delta)}S_n^{(\delta,\lambda)}(\cos v) = (n+\lambda)\mathcal{I}_n^{(\delta,\lambda)}(v) - (\delta+1)\mathcal{I}_{n-1}^{(\delta+1,\lambda)}(v) + i_{n+1}^{(\delta,\lambda)}(v) + i_n^{(\delta,\lambda)}(v)$$
$$+ (n+\lambda)\mathcal{J}_n^{(\delta,\lambda)}(v) - (\delta+1)\mathcal{J}_{n-1}^{(\delta+1,\lambda)}(v)$$

We also need the following result.

Lemma 9 (R. Askey and I. I. Hirschiman, 1963). Let $\sigma > -1$, $\tau \in \mathbb{R}$. Then

$$|A_n^{(\sigma+i\tau)}| \geq |A_n^{(\sigma)}|, \qquad |A_n^{(\sigma+i\tau)}| \leq e^{c(\sigma)\tau^2} A_n^{(\sigma)},$$

where

$$c(\sigma) = \frac{1}{2}\sum_{k=1}^{\infty}(\sigma+k)^{-2}.$$

Let $\langle x, y\rangle = \cos v$, $x, y \in \Sigma_2$. Then

$$\left|S_n^{(\delta,\lambda)}(\langle x, y\rangle)\right|$$
$$\leq \begin{cases} Ce^{B\tau^2}(n+1)^{\lambda-\sigma}((n+1)^{-1}+|x-y|)^{-\lambda-\sigma-1} & \text{if } \langle x, y\rangle \geq 0, \\ Ce^{B\tau^2}(n+1)^{\lambda-\sigma}((n+1)^{-1}+|x+y|)^{-\lambda-\sigma-1} & \text{if } \langle x, y\rangle \leq 0. \end{cases}$$

Since $S_n^\delta f(x) = \int_{\Sigma_2} S_n^{(\delta,1/2)}(\langle x, y\rangle) f(y)\, d\mu(y)$, we see that

$$S_*^\delta(f)(x) \leq A_\sigma e^{B\tau^2}\left(\sigma - \frac{1}{2}\right)^{-1}(Mf(x) + Mf(-x)), \quad \delta = \sigma + i\tau,$$

where

$$Mf(x) = \sup_{r>0} |B(x, r)|^{-1} \int_{B(x,r)} |f(y)|\, d\mu(y),$$

where $B(x, r) = \{y \in \Sigma_2 : |y - x| < r\}$, $x \in \Sigma_2$.

By the $L^1 - L^{1,\infty}$ boundedness of the maximal operator M we get the conclusion of Lemma 5:

$$\|S_*^\delta(f)\|_{1,\infty} \le A_\sigma e^{B\tau^2}(\sigma - \alpha)^{-1}\|f\|_1, \quad \delta = \sigma + i\tau.$$

§5. Proof of Lemma 4.

Recall

Lemma 4. Let $1 < p \le 2$. Then, we have

$$\sup_{\lambda>0} \lambda |\{x \in \Sigma_2 : S_*^{1/2}(f)(x) > \lambda\}|^{1/p} \le C(p-1)^{-1}\|f\|_p$$

for a positive constant C independent of p.

Let $1 < p < 2$,

$$1/p = (1-\theta)/2 + \theta, \qquad \alpha = (1-\theta)c + \theta b,$$

where

$$c = \alpha - (1/2)(1/p - 1/2), \qquad b = \alpha + (1/2)(1 - 1/p), \qquad \alpha = 1/2.$$

We note that

$$\theta = 2(1/p - 1/2), \qquad 1/4 \le c \le \alpha, \qquad \alpha \le b \le 3/4.$$

Define

$$T_z f = S_0^{\delta(z)} f, \quad \delta(z) = (1-z)c + zb, \quad z = \sigma + i\tau, \quad 0 \le \sigma \le 1.$$

Here S_0^δ is a linear operator approximating S_*^δ defined by

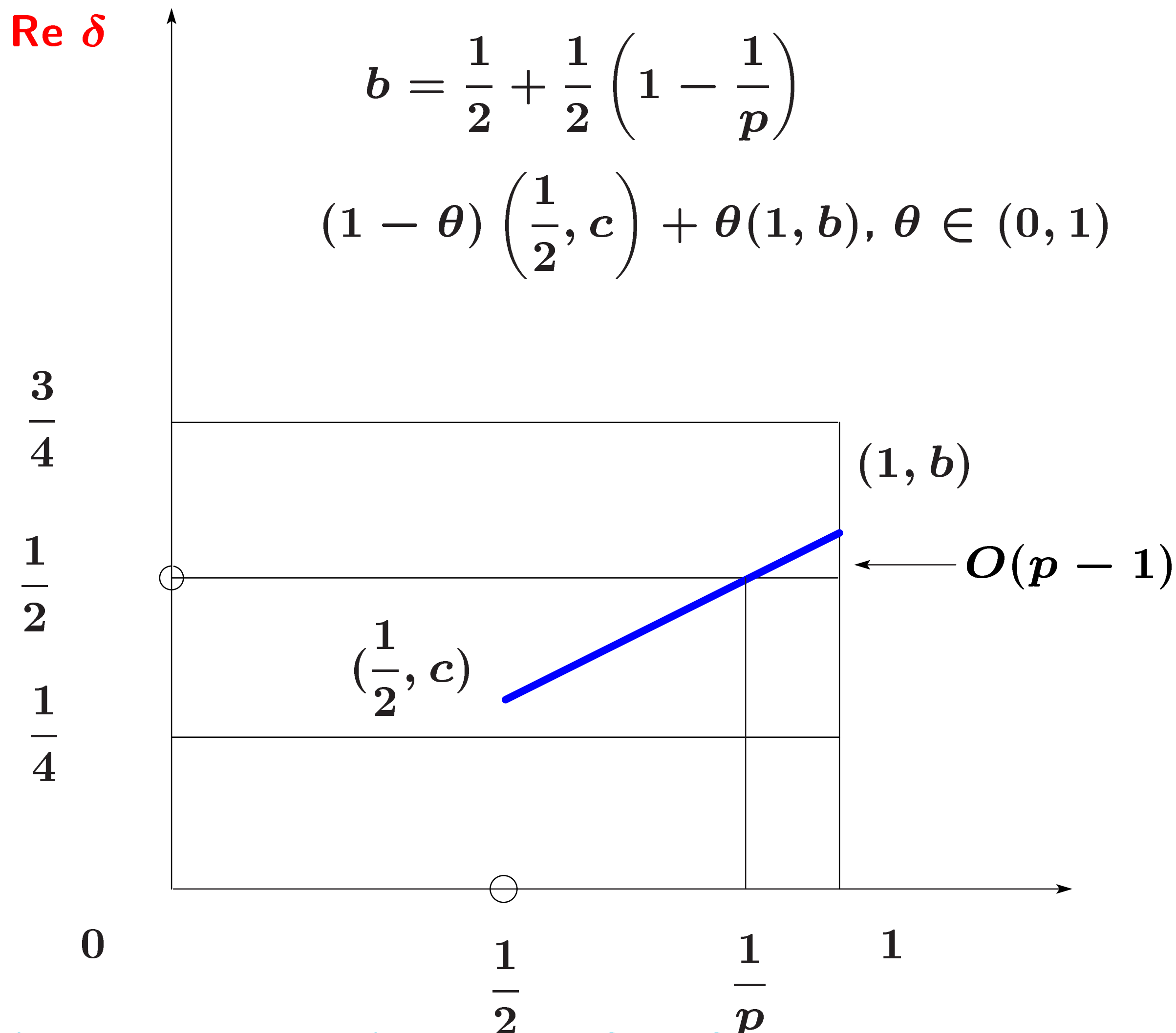
$$S_0^\delta f(x) = S_{n(x)}^\delta f(x),$$

where $n(x)$ is a suitable non-negative mapping from Σ_2 to $\mathbb{Z}_+$, so that $\{T_z\}$ is an admissible analytic family of linear operators.

$$c = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{p} - \frac{1}{2}\right)$$

$$b = \frac{1}{2} + \frac{1}{2}\left(1 - \frac{1}{p}\right)$$

$$(1-\theta)\left(\frac{1}{2}, c\right) + \theta(1, b),\ \theta \in (0, 1)$$



We apply the analytic interpolation theorem on the Lorentz spaces $L^{p,q}$ due to Sagher (1969). Recall

Lemma 5. Suppose that $f \in L^1(\Sigma_2)$ and $\alpha < \sigma < 1$, where $\alpha = 1/2$. Then

$$\|S_*^\delta(f)\|_{1,\infty} \le A_\sigma e^{B\tau^2}(\sigma - \alpha)^{-1}\|f\|_1, \quad \delta = \sigma + i\tau.$$

The constant A_σ remains bounded as $\sigma \to \alpha$.

Lemma 6. Suppose that $f \in L^2(\Sigma_2)$. Then

$$\|S_*^\delta(f)\|_2 \le A_\sigma e^{B_\sigma \tau^2}\|f\|_2, \quad \sigma > 0.$$

A_σ and B_σ are bounded on any compact subinterval of $(0, \infty)$.

Lemma 6 implies

$$\|S_0^{\delta(i\tau)} f\|_{2,2} = \|T_{i\tau} f\|_{2,2} \le A_c e^{B_c \tau^2}\|f\|_{2,2}, \quad \delta(i\tau) = c + i\tau(b - c).$$

By Lemma 5 we have

$$\|S_0^{\delta(1+i\tau)}f\|_{1,\infty} = \|T_{1+i\tau}f\|_{1,\infty} \le A_b(p-1)^{-1}e^{B\tau^2}\|f\|_{1,1},$$

$$\delta(1+i\tau) = b + i\tau(b-c).$$

Interpolating between these estimates, we get

$$\|S_0^{\alpha}f\|_{p,p'} = \|T_\theta f\|_{p,p'} \le A_\theta\|f\|_{p,p},$$

where

$$A_\theta \le C(p-1)^{-\theta} \le C(p-1)^{-1}.$$

Therefore

$$\|S_0^{\alpha}f\|_{p,\infty} \le C\|S_0^{\alpha}f\|_{p,p'} \le C(p-1)^{-1}\|f\|_p,$$

which implies Lemma 4.

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