Some non-regular convolution operators

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§1. Singular integrals of Calderón-Zygmund and R. Fefferman

§2. Singular Radon transforms

§3. Singular integrals on homogeneous groups

§4. Nonisotropic dilations and weak type $(1, 1)$ estimates

§1. Let $\Omega \in L^1(S^{n-1})$ satisfy

$$\int_{S^{n-1}} \Omega(\theta)\, d\sigma(\theta) = 0$$

where $\quad S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$
$d\sigma$: the Lebesgue measure on $S^{n-1}, \quad n \geq 2.$

We consider singular integrals of the form:

$$T(f)(x) = \mathrm{p.v.} \int_{\mathbb{R}^n} f(x-y)K(y)\, dy,$$

$$K(x) = h(|x|)\frac{\Omega(x')}{|x|^n}, \qquad x' = x/|x|.$$

Definition. Let $F \in L^1(S^{n-1})$.

(1) $F \in L\log L(S^{n-1})$ (Zygmund class)

$$\Longleftrightarrow$$

$$\int_{S^{n-1}} |F(\theta)| \log(2 + |F(\theta)|)\, d\sigma(\theta) < \infty.$$

(2) $F \in H^1(S^{n-1})$ (Hardy space) $\Longleftrightarrow \|F\|_{H^1} = \|P^+F\|_{L^1} < \infty$,

where

$$P^+F(\theta) = \sup_{0 \le r < 1} \left| \int_{S^{n-1}} F(\omega) P_{r\theta}(\omega)\, d\sigma(\omega) \right|$$

$$P_{r\omega}(\theta) = c_n \frac{1 - r^2}{|r\omega - \theta|^n} \quad \text{(Poisson kernel)}, \quad 0 \le r < 1,\ \omega,\ \theta \in S^{n-1}.$$

- $L\log L(S^{n-1})$ is a proper subspace of $H^1(S^{n-1})$.

The case of homogeneous kernels.

When $Tf = \mathrm{p.v.}\, f * K$, $K = \Omega/|x|^n$, write $T = T_\Omega$. Then,

$$(T_\Omega f)\hat{}\,(\xi) = m(\xi')\hat{f}(\xi),$$

where

$$m(\xi') = -\int_{S^{n-1}} \Omega(\theta) F(\xi', \theta)\, d\sigma(\theta),$$

$$F(\xi', \theta) = \left[i\frac{\pi}{2}\,\mathrm{sgn}(\langle \xi', \theta\rangle) + \log|\langle \xi', \theta\rangle| \right].$$

This implies

- $\Omega \in L\log L(S^{n-1}) \Longrightarrow T_\Omega : L^2 \to L^2$.

If Ω is odd,

$$T_\Omega f(x) = \frac{\pi}{2}\int_{S^{n-1}} H_\theta f(x)\Omega(\theta)\, d\sigma(\theta), \quad H_\theta f(x) = \mathrm{p.v.}\,\frac{1}{\pi}\int_{-\infty}^{\infty} f(x-t\theta)\, dt/t.$$

The method of rotations of Calderón-Zygmund (1956) implies:

- Ω is in $L^1(S^{n-1})$ and odd $\quad\Longrightarrow T_\Omega : L^p \to L^p$ for all $1 < p < \infty$;
- $\Omega \in L\log L(S^{n-1}) \quad\Longrightarrow T_\Omega : L^p \to L^p$ for all $1 < p < \infty$.

Furthermore,

- $\Omega \in H^1(S^{n-1}) \quad\Longrightarrow T_\Omega : L^p \to L^p$ for all $1 < p < \infty$.

This was proved by Coifman-Weiss, Connett, Ricci-Weiss (1977–1979) by applying developed versions of the Calderón-Zygmund method of rotations.

This improves upon the previous result since $L\log L(S^{n-1})$ is a proper subspace of $H^1(S^{n-1})$.

The case where h is not constant.

Recall

$$T(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y)K(y)\,dy,$$

$$K(x) = h(|x|)\frac{\Omega(x')}{|x|^n}, \qquad x' = x/|x|.$$

If h is not constant, then the method of rotations of Calderón-Zygmund is not applicable in general.

The operator

$$S(f)(x) = \int_{\mathbb{R}^n} f(x-y)\frac{\exp(i|y|)}{(1+|y|^2)^{n/2}}\,dy$$

is included in the class of operators T as a particular example.

Definition The space Δ_s, $s \geq 1$, is defined as

$$\Delta_s = \{h \text{ on } \mathbb{R}_+ : \|h\|_{\Delta_s} < \infty\},$$

$$\|h\|_{\Delta_s} = \sup_{j \in \mathbb{Z}} \left(\int_{2^j}^{2^{j+1}} |h(t)|^s \, dt/t \right)^{1/s},$$

where

$\mathbb{Z}$: the set of integers, $\quad \mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$;

$\Delta_\infty = L^\infty(\mathbb{R}_+)$.

- $s > t \Longrightarrow \Delta_s \subset \Delta_t$.

(1) $h \in L^\infty$, $\Omega \in Lip(S^{n-1}) \quad \Longrightarrow T : L^p \to L^p$, $1 < p < \infty$.

R. Fefferman, 1979.

(2) $h \in L^\infty$, $\exists q > 1 : \Omega \in L^q(S^{n-1}) \quad \Longrightarrow T : L^p \to L^p$, $1 < p < \infty$.

J. Namazi, 1986.

(3) $h \in \Delta_2$, $\exists q > 1 : \Omega \in L^q(S^{n-1}) \quad \Longrightarrow T : L^p \to L^p$, $1 < p < \infty$.

J. Duoandikoetxea and J. L. Rubio de Francia (D-R), 1986.

(4) $\Omega \in H^1(S^{n-1})$, $\exists s > 1 : h \in \Delta_s$

$\Longrightarrow T : L^p \to L^p$ if $|1/p - 1/2| < \min(1/2, 1/s')$, $s' = s/(s-1)$.

D. Fan and Y. Pan, 1997.

(5) $\Omega \in L \log L(S^{n-1})$, $\exists s > 1 : h \in \Delta_s \quad \Longrightarrow T : L^p \to L^p$, $1 < p < \infty$.

A. Al-Salman and Y. Pan, 2002.

Theorem 1. Suppose that $\Omega \in L^q(S^{n-1})$, $h \in \Delta_s$, $q, s \in (1, 2]$. Then

$$\|T(f)\|_{L^p} \le C_p (q-1)^{-1} (s-1)^{-1} \|\Omega\|_{L^q(S^{n-1})} \|h\|_{\Delta_s} \|f\|_{L^p}$$

for all $p \in (1, \infty)$, where the constant C_p is independent of q, s, Ω and h.

Definition

For h on $\mathbb{R}_+$ and $a > 0$, let

$$L_a(h) = \sup_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} |h(r)| \left(\log(2 + |h(r)|)\right)^a \, dr/r.$$

Define

$$\mathfrak{L}_a = \{h : L_a(h) < \infty\}.$$

- $a < b \Longrightarrow \mathfrak{L}_b \subset \mathfrak{L}_a$.
- $\bigcup_{s>1} \Delta_s \subsetneqq \bigcap_{a>0} \mathfrak{L}_a$.

Definition

Let $\mathcal{M}_a$, $a > 0$, be the collection of functions h on $\mathbb{R}_+$ such that

$\exists \{h_k\}_{k=1}^{\infty}$, $\exists \{a_k\}_{k=1}^{\infty}$ (a sequence of non-negative real numbers),

$$\boxed{h = \sum_{k=1}^{\infty} a_k h_k,} \quad \sup_{k \geq 1} \|h_k\|_{\Delta_{1+1/k}} \leq 1, \quad \sum_{k=1}^{\infty} k^a a_k < \infty.$$

Define

$$\|h\|_{\mathcal{M}_a} = \inf_{\{a_k\}} \sum_{k=1}^{\infty} k^a a_k.$$

Proposition.

(1) $\|\cdot\|_{\mathcal{M}_a}$ is a norm on the space $\mathcal{M}_a$;
(2) $h \in \mathcal{L}_{a+b}$ for some $b > 1 \Longrightarrow h \in \mathcal{M}_a$;
(3) $h \in \mathcal{M}_a \Longrightarrow h \in \mathcal{L}_a$.

Lemma 1. Suppose $F \in L^1(S^{n-1})$ and $a > 0$. Then, the following two statements are equivalent:

(1) $\int_{S^{n-1}} |F| \left(\log(2 + |F|)\right)^a \, d\sigma < \infty$ and $\int_{S^{n-1}} F \, d\sigma = 0$;

(2) there exist a sequence $\{F_m\}_{m=1}^{\infty}$ of functions on S^{n-1} and a sequence $\{b_m\}_{m=1}^{\infty}$ of non-negative real numbers such that

$$\boxed{F = \sum_{m=1}^{\infty} b_m F_m,}$$

$$\sup_{m \geq 1} \|F_m\|_{1+1/m} \leq 1, \quad \int_{S^{n-1}} F_m \, d\sigma = 0, \quad \sum_{m=1}^{\infty} m^a b_m < \infty.$$

By Theorem 1 and extrapolation of Yano we have

Theorem 2. Let $\Omega \in L \log L(S^{n-1})$ and $h \in \mathcal{M}_1$. Then

$$\|T(f)\|_{L^p} \le C_p \|\Omega\|_{L \log L} \|h\|_{\mathcal{M}_1} \|f\|_{L^p}$$

for all $p \in (1, \infty)$.

Remarks.

- Al-Salman-Pan (2002) proved L^p boundedness of T under the condition that $\Omega \in L \log L(S^{n-1})$ and $h \in \Delta_s$ for some $s > 1$. Theorem 2 improves upon this result by replacing the assumption on h with $h \in \mathcal{M}_1$, which will be satisfied if $h \in \mathcal{L}_a$ for some $a > 2$.
- $\Omega \in L \log L(S^{n-1}), h \in \mathcal{L}_1 \Longrightarrow T : L^p \to L^p$ for all $p \in (1, \infty)$?

Proof of Theorem 2. Fix $p \in (1, \infty)$ and a function f with $\|f\|_p \le 1$. Set $S(h, \Omega) = \|T(f)\|_p$. Let $h \in \mathcal{M}_1$ and $\Omega \in L \log L(S^{n-1})$. Write $h = \sum_{k=1}^{\infty} a_k h_k$, as in the definition of $\mathcal{M}_1$. We may assume $\sum_{k=1}^{\infty} k a_k \lesssim \|h\|_{\mathcal{M}_1}$. Also, we have $\Omega = \sum_{m=1}^{\infty} b_m \Omega_m$, by Lemma 1 with $a = 1$, where $\sup_{m \ge 1} \|\Omega_m\|_{1+1/m} \le 1$, $\int_{S^{n-1}} \Omega_m \, d\sigma = 0$, $b_m \ge 0$, $\sum_{m=1}^{\infty} m b_m \lesssim \|\Omega\|_{L \log L}$. Now, the subadditivity of S and Theorem 1 imply

$$
\begin{aligned}
S(h, \Omega) &\le \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} a_k b_m S(h_k, \Omega_m) \\
&\le C \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} a_k b_m \inf_{q,s \in (1,2]} (q-1)^{-1} (s-1)^{-1} \|\Omega_m\|_q \|h_k\|_{\Delta_s} \\
&\le C \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} a_k b_m m k \|\Omega_m\|_{1+1/m} \|h_k\|_{\Delta_{1+1/k}} \\
&\le C \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} k m a_k b_m \le C \|h\|_{\mathcal{M}_1} \|\Omega\|_{L \log L}. \quad \square
\end{aligned}
$$

§2. Singular Radon transforms.

$$T(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - P(y))K(y)\,dy$$
$$= \lim_{\epsilon \to 0} \int_{|y|>\epsilon} f(x - P(y))K(y)\,dy,$$

where

$$K(y) = h(|y|)\Omega(y')|y|^{-n}, \qquad y' = |y|^{-1}y,$$

$n \geq 2$, $\Omega \in L^1(S^{n-1})$,

$$\int_{S^{n-1}} \Omega(\theta)\,d\sigma(\theta) = 0,$$

f: a function on $\mathbb{R}^d$,

$P(y) = (P_1(y), P_2(y), \ldots, P_d(y))$: a polynomial mapping, where each P_j is a real-valued polynomial on $\mathbb{R}^n$.

We assume that $P(-y) = -P(y)$.

Theorem 3. Suppose that $\Omega \in L^q(S^{n-1})$, $h \in \Delta_s$, $q, s \in (1, 2]$. Then

$$\|T(f)\|_{L^p(\mathbb{R}^d)} \le C_p(q-1)^{-1}(s-1)^{-1}\|\Omega\|_{L^q(S^{n-1})}\|h\|_{\Delta_s}\|f\|_{L^p(\mathbb{R}^d)}$$

for all $p \in (1, \infty)$, where the constant C_p is independent of q, s, Ω and h; also, it is independent of the polynomials P_j if they are of fixed degree.

By Theorem 3 and extrapolation we have

Theorem 4. Let $\Omega \in L \log L(S^{n-1})$ and $h \in \mathcal{M}_1$. Then

$$\|T(f)\|_{L^p(\mathbb{R}^d)} \le C_p\|\Omega\|_{L \log L}\|h\|_{\mathcal{M}_1}\|f\|_{L^p(\mathbb{R}^d)}, \quad p \in (1, \infty),$$

where C_p is independent of the polynomials P_j if they are of fixed degree.

Previous result

Recall

$$T(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - P(y))K(y)\, dy,$$

$$K(y) = h(|y|)\Omega(y')|y|^{-n}, \qquad y' = |y|^{-1}y.$$

$\Omega \in L \log L(S^{n-1})$, $\exists s > 1 : h \in \Delta_s$, $P(-y) = -P(y)$
$\implies T : L^p \to L^p$ for all $1 < p < \infty$.
A. Al-Salman and Y. Pan, J. London Math. Soc. (2), 2002.

Theorem 4 improves upon this result by replacing the assumption on h with $h \in \mathcal{M}_1$.

Idea for proofs of Theorems 1, 3.

Framework of the proof comes from

- J. Duoandikoetxea and J. L. Rubio de Francia (D-R), Maximal and singular integral operators via Fourier transform estimates, Invent. Math. 84 (1986).

We apply the methods of D-R with a suitable Littlewood-Paley (L-P) decomposition.

For singular Radon transforms, we also use results for oscillatory integrals in

- D. Fan and Y. Pan, Singular integral operators with rough kernels supported by subvarieties, Amer. J. Math. (1997),
- F. Ricci and E. M. Stein, Harmonic analysis on nilpotent groups and singular integrals, I, J. Func. Anal. 73 (1987).

Littlewood-Paley decomposition.

L-P decomposition adapted to a lacunary sequence $\{\rho^k\}$.

$\{\psi_k\}_{k=-\infty}^{\infty}$: $\psi_k \in C^\infty((0,\infty))$,

$$\text{supp}(\psi_k) \subset [\rho^{-k-1}, \rho^{-k+1}],$$

$$\sum_{k\in\mathbb{Z}} \psi_k(t) = 1,$$

$$|(d/dt)^j \psi_k(t)| \leq c_j/t^j \quad (j = 1, 2, \ldots),$$

where the constants c_j are independent of $\rho \geq 2$.

Define an operator S_k by

$$(S_k(f))\,\hat{}\,(\xi) = \psi_k(|\xi|)\,\hat{f}(\xi).$$

Then

$$\left\| \left(\sum_k |S_k(f)|^2 \right)^{1/2} \right\|_p \le C_p \|f\|_p,$$

$$\|f\|_p \le C'_p \left\| \left(\sum_k |S_k(f)|^2 \right)^{1/2} \right\|_p$$

for $1 < p < \infty$. The constants C_p, C'_p are independent of ρ.

A new element of the proof is to apply L-P decomposition depending on q and s for which $\Omega \in L^q(S^{n-1})$ and $h \in \Delta_s$. More precisely, we apply L-P decomposition adapted to a lacunary sequence with Hadamard gap

$$\boxed{\rho \sim 2^{q's'}}.$$

If we apply L-P decomposition with a fixed Hadamard gap, for example, with $\rho = 2$, and leave the other part of our proof unchanged, then we have

$$\|T\|_{p,p} \lesssim [(q-1)(s-1)]^{-1-\delta(p)} \|\Omega\|_q \|h\|_{\Delta_s},$$

where

$$\delta(p) = |1/p - 1/p'|.$$

This is unfavorable, since $1 + \delta(p) \to 2$ as $p \to 1$ or $p \to \infty$.

§3. Singular integrals on homogeneous groups.

Convolution associated with homogeneous group structure defines singular integrals:

$$T(f)(x) = \mathrm{p.v.} \int f(y)K(y^{-1}x)\,dy$$
$$= \lim_{\epsilon \to 0} \int_{r(y^{-1}x)>\epsilon} f(y)K(y^{-1}x)\,dy,$$

where K is homogeneous of degree $-\gamma$ with respect to dilations A_t, $t > 0$,

$$K(A_t x) = t^{-\gamma} K(x), \quad t > 0, \quad x \neq 0,$$

$$A_t x = (t^{a_1}x_1, t^{a_2}x_2, \ldots, t^{a_n}x_n), \quad x = (x_1, \ldots, x_n),$$
$$0 < a_1 \leq a_2 \leq \cdots \leq a_n, \quad \gamma = a_1 + \cdots + a_n,$$

and $r(x)$ is a norm function associated with $\{A_t\}$.

Also, we consider the maximal singular integral

$$T_*(f)(x) = \sup_{N,\epsilon>0} \left| \int_{\epsilon<r(y^{-1}x)<N} f(y)K(y^{-1}x)\,dy \right| .$$

We prove L^p and weighted L^p boundedness of T and T_* under a sharp condition of the kernel.

We regard $\mathbb{R}^n$ as a homogeneous group. We also write $\mathbb{R}^n = \mathbb{H}$.

- multiplication is given by a polynomial mapping;
- (ux)(vx)= ux+vx, $x \in \mathbb{H}$, $u, v \in \mathbb{R}$;
- the identity is the origin 0, $x^{-1} = -x$;
- $\exists\{A_t\}_{t>0}$: a dilation family on $\mathbb{R}^n$ such that

$$\boxed{A_t x = (t^{a_1}x_1, t^{a_2}x_2, \dots, t^{a_n}x_n),}$$

  $x = (x_1, \dots, x_n)$, $0 < a_1 \le a_2 \le \cdots \le a_n$,
  A_t is an automorphism of the group structure

$$\boxed{A_t(xy) = (A_t x)(A_t y),\ x, y \in \mathbb{H},\ t > 0;}$$

- Lebesgue measure is bi-invariant Haar measure.

$|x|$: the Euclidean norm for $x \in \mathbb{R}^n$,

$r(x)$: a norm function satisfying $r(A_t x) = t r(x)$, $\forall t > 0$, $\forall x \in \mathbb{R}^n$;

(1) r is continuous on $\mathbb{R}^n$ and smooth in $\mathbb{R}^n \setminus \{0\}$;

(2) $r(x + y) \le C_0(r(x) + r(y)), \quad r(xy) \le C_0(r(x) + r(y))$
for some $C_0 \ge 1$;

(3) $r(x^{-1}) = r(x)$;

(4) If $\Sigma = \{x \in \mathbb{R}^n : r(x) = 1\}$, then $\Sigma = S^{n-1}$,
where $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$;

(5) $\exists c_1, c_2, c_3, c_4, \alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ such that

$$c_1|x|^{\alpha_1} \le r(x) \le c_2|x|^{\alpha_2} \quad \text{if } r(x) \ge 1,$$

$$c_3|x|^{\beta_1} \le r(x) \le c_4|x|^{\beta_2} \quad \text{if } r(x) \le 1.$$

Convolution is defined as

$$f * g(x) = \int_{\mathbb{R}^n} f(y) g(y^{-1}x)\, dy.$$

- $$(f * g) * h = f * (g * h).$$

- Let $\gamma = a_1 + \cdots + a_n$. Then, $dx = t^{\gamma-1}\, dS\, dt$, that is,

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_0^\infty \int_\Sigma f(A_t\theta) t^{\gamma-1}\, dS(\theta)\, dt$$

where $dS = \omega\, dS_0$, ω is a strictly positive C^∞ function on Σ and dS_0 is the Lebesgue surface measure on Σ.

Remark Let $\{A_t\}_{t>0}$, $A_t = t^P = \exp((\log t)P)$, be a dilation group on $\mathbb{R}^n$, where P is an $n \times n$ real matrix whose eigenvalues have positive real parts. Then, we have similar results for $\{A_t\}_{t>0}$, with $\gamma = \mathrm{trace}\ P$.

An example. **Heisenberg group $\mathbb{H}_1$.**

$$(x, y, u)(x', y', u') = (x + x', y + y', u + u' + (xy' - yx')/2),$$

$$(x, y, u), (x', y', u') \in \mathbb{R}^3,$$

then $\mathbb{R}^3$ with this group law is the Heisenberg group $\mathbb{H}_1$; a dilation is defined by

$$A_t(x, y, u) = (tx, ty, t^2 u),$$

and a norm function is

$$r(x, y, u) = \frac{1}{\sqrt{2}}\sqrt{\sqrt{(x^2 + y^2)^2 + 4u^2} + x^2 + y^2}.$$

Also, we can adopt

$$A'_t(x, y, u) = (tx, t^2 y, t^3 u).$$

Let

$$K(x) = \Omega(x')r(x)^{-\gamma}, \qquad x' = A_{r(x)^{-1}}x \text{ for } x \neq 0,$$

where $\gamma = a_1 + \cdots + a_n$,

Ω is locally integrable in $\mathbb{R}^n \setminus \{0\}$ and homogeneous of degree 0 with respect to the dilation group $\{A_t\}$, that is,

$$\Omega(A_t x) = \Omega(x) \qquad \text{for } x \neq 0,\ t > 0;$$

also

$$\int_\Sigma \Omega(\theta)\, dS(\theta) = 0.$$

Theorem A. Suppose that $\Omega \in L \log L(\Sigma)$. Then, T is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$.

T. Tao 1999 proved this by interpolation between L^2 estimates and weak $(1,1)$ estimates.

IDEA: $(TT^*)^M$ estimates

Theorem 5. Suppose that $\Omega \in L \log L(\Sigma)$. Then,

$$T_* : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), \qquad \forall p \in (1, \infty).$$

IDEA: Extrapolation

Weighted L^p estimates for T and T_*.

If B is a subset of $\mathbb{H}$ such that

$$B = \{x \in \mathbb{H} : r(a^{-1}x) < s\}$$

for some $a \in \mathbb{H}$ and $s > 0$, then we call B a ball in $\mathbb{H}$ with center a and radius s and write $B = B(a, s)$.

Definition (Muckenhoupt class on $\mathbb{H}$) Let w be a weight function on $\mathbb{R}^n$.

(1) We say $w \in A_p$, $1 < p < \infty$, if

$$\sup_B \left(|B|^{-1} \int_B w(x)\, dx \right) \left(|B|^{-1} \int_B w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{H}$.

(2) The class A_1 is defined to be the set of weight functions w satisfying $Mw \le Cw$ a.e. where M denotes the Hardy-Littlewood maximal operator.

Theorem 6. Suppose that $q > 1$, $\Omega \in L^q(\Sigma)$, $1 < p < \infty$. Then,

(1) T and T_* are bounded on $L^p(w)$ if $q' \leq p < \infty$ and $w \in A_{p/q'}$, $q' = q/(q-1)$;

(2) if $1 < p \leq q$ and $w \in A_{p'/q'}$, T and T_* are bounded on $L^p(w^{1-p})$.

In the Euclidean convolution case, where Fourier transform estimates are available, this was proved independently by

J. Duoandikoetxea, 1993,

D. Watson, 1990.

Idea of proof.

Theory of Duoandikoetxea and Rubio de Francia (1986):

- **Orthogonality arguments for L^2 estimates via Fourier transform estimates and Plancherel's theorem**
- **Littlewood-Paley theory**
- **Interpolation arguments**

Our strategy is:

to employ a version of theory of Duoandikoetxea and Rubio de Francia adapted for analysis on homogeneous groups;

replace use of Fourier transform estimates with $(TT^*)^M$ estimates (basic L^2 estimates for convolution) and apply Cotlar's lemma.

$(TT^*)^M$ method.

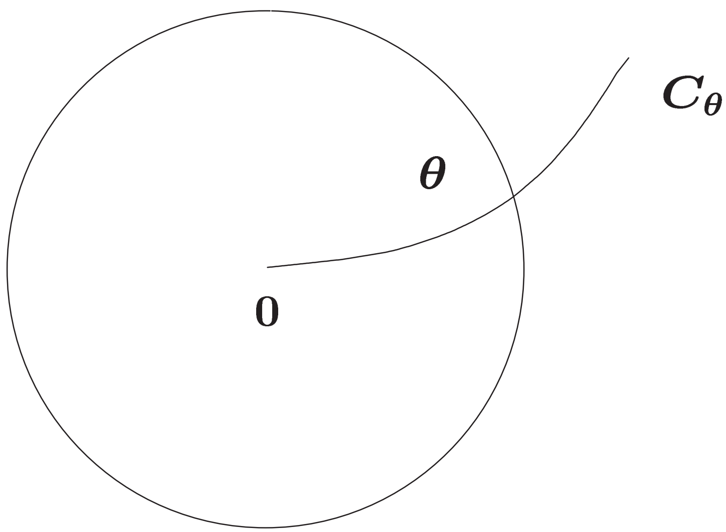
- $\|TT^*\| = \|T\|^2$.

Let Ω be homogeneous of degree 0 on $\mathbb{R}^n \setminus \{0\}$. Define

$$C_\theta = \{A_t\theta : t > 0\}, \qquad \theta \in \Sigma.$$

Then, Ω is smooth on C_θ for every $\theta \in \Sigma$ since

$$\Omega(A_t\theta) = \Omega(\theta).$$



Let $\rho \geq 2$. Let $\psi_j \in C_0^\infty(\mathbb{R})$, $j \in \mathbb{Z}$, be such that

$$\operatorname{supp}(\psi_j) \subset \{t \in \mathbb{R} : \rho^j \leq t \leq \rho^{j+2}\}, \quad \psi_j \geq 0,$$
$$\sum_{j \in \mathbb{Z}} \psi_j(t) = 1 \quad \text{for } t \neq 0,$$
$$\left|(d/dt)^m \psi_j(t)\right| \leq c_m |t|^{-m} \quad \text{for } m = 0, 1, 2, \dots,$$

where c_m is independent of ρ. Let

$$\begin{aligned} D_j K(x) &= (\log 2)^{-1} \int_0^\infty \psi_j(t) \delta_t K_0(x)\, dt/t \\ &= (\log 2)^{-1} \Omega(x') r(x)^{-\gamma} \int_{1/2}^1 \psi_j(tr(x))\, dt/t, \end{aligned}$$

$\delta_t K_0(x) = t^{-\gamma} K_0(A_t^{-1} x), \quad K_0(x) = K(x) \chi_{I_0}(x),$
$I_0 = \{x \in \mathbb{R}^n : 1 \leq r(x) \leq 2\}$.

Then, $\mathrm{supp}(D_j K) \subset \{x : \rho^j \le r(x) \le 2\rho^{j+2}\}$ and

$$\sum_{j\in\mathbb{Z}} D_j K = K, \qquad Tf = \sum_{j\in\mathbb{Z}} f * D_j K.$$

We choose $\boxed{\rho = 2^{s'}}$ if $\Omega \in L^s(\Sigma)$.

Let ϕ be a C^∞ function such that $\mathrm{supp}(\phi) \subset \{1/2 < r(x) < 1\}$, $\int \phi = 1$, $\phi(x) = \phi(x^{-1})$, $\phi(x) \ge 0$.

Define

$$J_k = \delta_{\rho^{k-1}}\phi - \delta_{\rho^k}\phi, \quad k \in \mathbb{Z}, \quad \delta_t\phi(x) = t^{-\gamma}\phi(A_t^{-1}x).$$

Then

$$\boxed{\sum_k J_k = \delta,}$$

where δ is the delta function.

Lemma 2 (basic L^2 estimates). Let $s > 1$, $\Omega \in L^s(\Sigma)$, $\rho = 2^{s'}$. Then,

$$\|f * D_j K * J_{k+j}\|_2 \le C \frac{s}{s-1} 2^{-\epsilon|k|} \|\Omega\|_s \|f\|_2.$$

If $s = \infty$, this was proved by T. Tao 1999 with $(TT^*)^n$ method.

Decompose

$$Tf = \sum_{j\in\mathbb{Z}} f * D_j K = \sum_{k_1,k_2\in\mathbb{Z}} U_{k_1,k_2} f,$$

where

$$U_{k_1,k_2} f = \sum_j f * J_{k_1+j} * D_j K * J_{k_2+j}.$$

Lemma 3. Let $1 < p < \infty$, $1 < s \le 2$, $\rho = 2^{s'}$.

$$\|U_{k_1,k_2} f\|_p \le C(s-1)^{-1} 2^{-\epsilon(|k_1|+|k_2|)} \|\Omega\|_s \|f\|_p$$

for some $\epsilon > 0$.

Lemma 3 implies

$$\|Tf\|_p \le \sum_{k_1,k_2} \|U_{k_1,k_2} f\|_p \le C(s-1)^{-1} \sum_{k_1,k_2} 2^{-\epsilon(|k_1|+|k_2|)} \|\Omega\|_s \|f\|_p$$

$$\le C(s-1)^{-1} \|\Omega\|_s \|f\|_p.$$

§4. Nonisotropic dilations and weak type $(1,1)$ estimates.

Let $\{A_t\}_{t>0}$, $A_t = t^P = \exp((\log t)P)$, be a dilation group on $\mathbb{R}^n$, where P is an $n \times n$ real matrix whose eigenvalues have positive real parts.

Let K be a locally integrable function on $\mathbb{R}^n \setminus \{0\}$ such that

$$K(A_t x) = t^{-\gamma} K(x), \quad \gamma = \mathrm{trace} P.$$

We write

$$K(x) = \Omega(x')r(x)^{-\gamma}, \qquad x' = A_{r(x)^{-1}} x \text{ for } x \neq 0,$$

where Ω is homogeneous of degree 0 with respect to the dilation group $\{A_t\}$. We assume that

$$\int_\Sigma \Omega(\theta)\, dS(\theta) = 0.$$

Let

$$Tf(x) = \text{p.v.} \int f(y)K(x-y)\,dy, \quad K(x) = \Omega(x')r(x)^{-\gamma}.$$

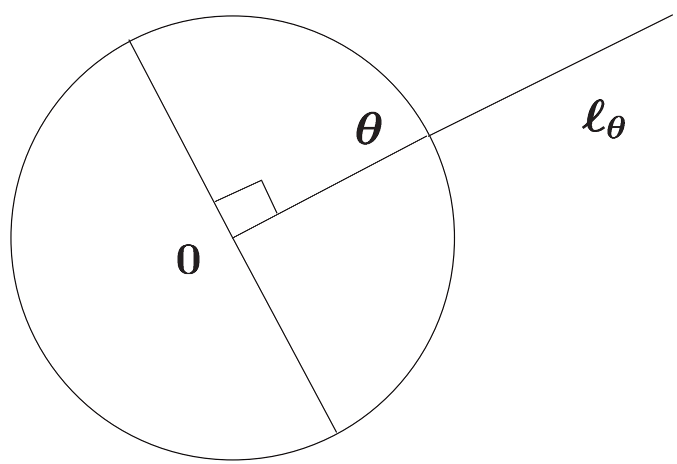
Theorem 7. Suppose that $n = 2$ and $\Omega \in L\log L(\Sigma)$. Then, the operator T is of weak type $(1,1)$, i.e.,

$$|\{|Tf| > \lambda\}| \le \frac{C}{\lambda}\|f\|_1, \quad \lambda > 0.$$

Previous results.

Theorem B (A. Seeger 1996). Suppose that $A_t x = tx$ and $r(x) = |x|$, $x \in \mathbb{R}^n$, $n \geq 2$, $\Omega \in L\log L(\Sigma)$. Then, the operator T is of weak type $(1,1)$.

IDEA: Fourier transform estimates + microlocal analysis; Calderón-Zygmund decomposition.



Theorem C (T. Tao 1999). Let $A_t x = (t^{a_1}x_1, t^{a_2}x_2, \dots, t^{a_n}x_n)$, where $x = (x_1, \dots, x_n)$ and $0 < a_1 \le a_2 \le \cdots \le a_n$. Suppose that $\Omega \in L\log L(\Sigma)$. Then T is of weak type $(1,1)$.

In fact, T. Tao proved the weak type $(1,1)$ boundedness of singular integrals on general homogeneous groups.

IDEA: $(TT^*)^M$ estimates;
Calderón-Zygmund decomposition;
Covering lemma of Vitali type, John-Nirenberg inequality for BMO.

There exists a non-singular real matrix Q such that $Q^{-1}PQ$ is one of the following Jordan canonical forms:

$$P_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad P_2 = \begin{pmatrix} \alpha & 0 \\ 1 & \alpha \end{pmatrix}, \quad P_3 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

where $\alpha, \beta > 0$. Accordingly, we have three kinds of dilations

$$\begin{pmatrix} t^\alpha & 0 \\ 0 & t^\beta \end{pmatrix}, \quad t^\alpha \begin{pmatrix} 1 & 0 \\ \log t & 1 \end{pmatrix}, \quad t^\alpha \begin{pmatrix} \cos(\beta \log t) & \sin(\beta \log t) \\ -\sin(\beta \log t) & \cos(\beta \log t) \end{pmatrix}.$$

The case $P = P_1$ is handled by Theorem C. We have to consider the cases $P = P_2$ and $P = P_3$.

Proof of Theorem 7 follows closely the methods of T. Tao, as the Fourier transform is not readily available in this context.

Also, to handle the case $P = P_3$, we apply a trick that may have difficulty in extending to higher dimensions.

THANK YOU !

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