

**Estimates for singular integrals associated with  
nonisotropic dilations**

**Shuichi Sato**

**Kanazawa University**

We consider two kinds of convolution on  $\mathbb{R}^n$ :

- convolution associated with Euclidean space structure
- convolution associated with homogeneous group structure.

- Convolution associated with homogeneous group structure.

**Singular integrals on a homogeneous group:**

$$\begin{aligned} T(f)(x) &= \text{p.v.} \int f(y)L(y^{-1}x) dy \\ &= \lim_{\epsilon \rightarrow 0} \int_{r(y^{-1}x) > \epsilon} f(y)L(y^{-1}x) dy, \end{aligned}$$

**with rough kernels:**

$$L(x) = h(r(x))K(x),$$

**$K$  is homogeneous of degree  $-\gamma$  with respect to dilations  $A_t$ ,  $t > 0$ ,**

$$K(A_tx) = t^{-\gamma} K(x), \quad t > 0, \quad x \neq 0,$$

$$\begin{aligned} A_tx &= (t^{\alpha_1}x_1, t^{\alpha_2}x_2, \dots, t^{\alpha_n}x_n), \quad x = (x_1, \dots, x_n), \\ 0 < \alpha_1 &\leq \alpha_2 \leq \dots \leq \alpha_n, \quad \gamma = \alpha_1 + \dots + \alpha_n. \end{aligned}$$

Also, we consider a maximal singular integral

$$T_* f(x) = \sup_{N, \epsilon > 0} \left| \int_{\epsilon < r(y^{-1}x) < N} f(y) L(y^{-1}x) dy \right|.$$

We prove  $L^p$  and weighted  $L^p$  boundedness of  $T$  and  $T_*$  under a sharp condition for the kernel

- Convolution associated with Euclidean space structure.

Let  $\{A_t\}_{t>0}$ ,  $A_t = t^P = \exp((\log t)P)$ , be a dilation group on  $\mathbb{R}^n$ , where  $P$  is an  $n \times n$  real matrix whose eigenvalues have positive real parts.

Define

$$T(f)(x) = \text{p.v.} \int f(y) K(x - y) dy,$$

where  $K$  is homogeneous of degree  $-\gamma$ ,  $\gamma = \text{trace } P$ , with respect to dilations  $A_t$ . We prove weak type  $(1, 1)$  estimates for  $T$  on  $\mathbb{R}^2$  under the  $L \log L$  condition on the kernel.

**§1.**  $\mathbb{R}^n$  as a homogeneous group

**§2.** Results for  $L^p$  estimates for  $T$  and  $T_*$

**§3.** Results for weighted  $L^p$  estimates for  $T$  and  $T_*$

**§4.** A basic  $L^2$  estimates

**§5.** Sketch of proof for  $L^p$  estimates of  $T$

**§6.** Weak type  $(1, 1)$  estimates on  $\mathbb{R}^2$ .

## §1. $\mathbb{R}^n$ as a homogeneous group.

$\mathbb{R}^n$ : the  $n$  dimensional Euclidean space,  $n \geq 2$ .

We regard  $\mathbb{R}^n$  as a homogeneous group:

- multiplication is given by a polynomial mapping;
- $\exists \{A_t\}_{t>0}$ : a dilation family on  $\mathbb{R}^n$  such that

$$A_t x = (t^{\alpha_1} x_1, t^{\alpha_2} x_2, \dots, t^{\alpha_n} x_n),$$

$x = (x_1, \dots, x_n)$  and  $0 < \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n$ ,

$A_t$  is an automorphism of the group structure;

- Lebesgue measure is a bi-invariant Haar measure;
- the identity is the origin  $0$ ,  $x^{-1} = -x$ .

**Multiplication  $xy$  satisfies**

(1)  $(ux)(vx) = ux + vx, x \in \mathbb{H}, u, v \in \mathbb{R};$

(2)  $A_t(xy) = (A_tx)(A_ty), x, y \in \mathbb{H}, t > 0;$

(3) if  $z = xy, z = (z_1, \dots, z_n)$ , then  $z_k = P_k(x, y),$

$$P_1(x, y) = x_1 + y_1,$$

$$P_k(x, y) = x_k + y_k + R_k(x, y) \quad \text{for } k \geq 2,$$

where  $R_k(x, y)$  is a polynomial of degree greater than 1 depending only on  $x_1, \dots, x_{k-1}, y_1, \dots, y_{k-1}$ .

We also write  $\mathbb{R}^n = \mathbb{H}$ .

$|x|$ : the Euclidean norm for  $x \in \mathbb{R}^n$ ,

$r(x)$ : a norm function satisfying  $r(A_t x) = t r(x)$ ,  $\forall t > 0$ ,  $\forall x \in \mathbb{R}^n$ ;

- (1)  $r$  is continuous on  $\mathbb{R}^n$  and smooth in  $\mathbb{R}^n \setminus \{0\}$ ;
- (2)  $r(x + y) \leq C_0(r(x) + r(y))$ ,  $r(xy) \leq C_0(r(x) + r(y))$  for some  $C_0 \geq 1$ ;
- (3)  $r(x^{-1}) = r(x)$ ;
- (4)  $\{x \in \mathbb{R}^n : r(x) = 1\} = S^{n-1}$ , where  $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ ;
- (5)  $\exists c_1, c_2, c_3, c_4, \alpha_1, \alpha_2, \beta_1, \beta_2 > 0$  such that

$$c_1|x|^{\alpha_1} \leq r(x) \leq c_2|x|^{\alpha_2} \quad \text{if } r(x) \geq 1,$$

$$c_3|x|^{\beta_1} \leq r(x) \leq c_4|x|^{\beta_2} \quad \text{if } r(x) \leq 1.$$

- The space  $\mathbb{H}$  with a left invariant quasi-metric  $d(x, y) = r(x^{-1}y)$  is a space of homogeneous type.

## Convolution

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(y^{-1}x) dy$$

- $(f * g) * h = f * (g * h)$
- $(f * g)^{\sim} = \tilde{g} * \tilde{f}$  if  $\tilde{f}(x) = f(x^{-1})$ .
- **Euclidean convolution**

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(x - y) dy.$$

**An example.**

**Heisenberg group  $\mathbb{H}_1$ .**

$$(x, y, u)(x', y', u') = (x + x', y + y', u + u' + (xy' - yx')/2),$$

$$(x, y, u), (x', y', u') \in \mathbb{R}^3,$$

then  $\mathbb{R}^3$  with this group law is the Heisenberg group  $\mathbb{H}_1$ ;  
the dilation is defined by

$$A_t(x, y, u) = (tx, ty, t^2u),$$

and the norm function is

$$r(x, y, u) = \frac{1}{\sqrt{2}} \sqrt{\sqrt{(x^2 + y^2)^2 + 4u^2} + x^2 + y^2}.$$

## §2. Results for $L^p$ estimates for $T$ and $T_*$ .

**Definition.**  $F$ : a function on  $\mathbb{R}^n$ .

$F \in L \log L(D_0)$  (Zygmund class),  $D_0 = \{x \in \mathbb{R}^n : 1 \leq r(x) \leq 2\}$ .

$\iff$

$\text{supp}(F) \subset D_0$ ,  $\int_{D_0} |F(x)| \log(2 + |F(x)|) dx < \infty$ .

**Definition** The space  $d_s$ ,  $s \geq 1$ , is defined as

$$d_s = \{h \text{ on } \mathbb{R}_+ : \|h\|_{d_s} < \infty\},$$

$$\|h\|_{d_s} = \sup_{j \in \mathbb{Z}} \left( \int_{2^j}^{2^{j+1}} |h(t)|^s dt / t \right)^{1/s},$$

$\mathbb{Z}$  : the set of integers,  $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$ ;

$$d_\infty = L^\infty(\mathbb{R}_+).$$

•  $s > t \implies d_s \subset d_t$ .

Also, put for  $t \in (0, 1]$ ,

$$\omega(h, t) = \sup_{|s| < tR/2} \int_R^{2R} |h(r - s) - h(r)| dr / r,$$

where the supremum is taken over all  $s$  and  $R$  such that  $|s| < tR/2$ .

$K$ : a locally integrable function on  $\mathbb{R}^n \setminus \{0\}$

- $K(A_t x) = t^{-\gamma} K(x)$ ,  $\gamma = \alpha_1 + \cdots + \alpha_n$ ,

for all  $t > 0$  and  $x \in \mathbb{R}^n \setminus \{0\}$ ;

- $\int_{a < r(x) < b} K(x) dx = 0$  for all  $0 < a < b$ .

Define  $K_0(x) = K(x)\chi_{D_0}(x)$ ,  $D_0 = \{x \in \mathbb{R}^n : 1 \leq r(x) \leq 2\}$ .

Let

$$Tf(x) = \text{p.v.} f * L(x) = \text{p.v.} \int_{\mathbb{R}^n} f(y)L(y^{-1}x) dy,$$

where  $L(x) = h(r(x))K(x)$ ,  $h \in d_1$ .

**Theorem 1.** Suppose that  $h \in d_q$  for some  $q > 1$ ,  $\omega(h, t) \leq Ct^\eta$  for some  $\eta > 0$ ,  $1 < s \leq q$ ,  $K_0 \in L^s(D_0)$ ,  $K_0 = K\chi_{D_0}$ ,  $D_0 = \{1 \leq r(x) \leq 2\}$ . Then,

$$\|Tf\|_p \leq C_p(s/(s-1))\|K_0\|_s\|f\|_p, \quad 1 < p < \infty$$

for all  $s$ ,  $1 < s \leq q$ , and  $K_0 \in L^s(D_0)$ , where the constant  $C_p$  is independent of  $s$  and  $K_0$ .

Extrapolation of Yano using Theorem 1 implies the following result.

**Theorem 2.** Let  $h$  be as in Theorem 1. Suppose  $K_0 \in L \log L(D_0)$ . Then,  $T$  is bounded on  $L^p(\mathbb{R}^n)$  for all  $p \in (1, \infty)$ .

When  $h = 1$ , this is due to T. Tao 1999.

Recall

$$T_* f(x) = \sup_{N, \epsilon > 0} \left| \int_{\epsilon < r(y^{-1}x) < N} f(y) L(y^{-1}x) dy \right|.$$

**Theorem 3.**  $h \in d_q$  for some  $q > 1$ ,  $\omega(h, t) \leq Ct^\eta$  for some  $\eta > 0$ ,  $K_0 \in L^s(D_0)$ ,  $1 < s \leq q$ .

Then, if  $1 < p < \infty$ ,

$$\|T_* f\|_p \leq C_p(s/(s-1)) \|K_0\|_s \|f\|_p$$

for all  $s$ ,  $1 < s \leq q$ , and  $K_0 \in L^s(D_0)$ ; the constant  $C_p$  is independent of  $s$  and  $K_0$ .

Extrapolation of Yano using Theorem 3 implies the following result.

**Theorem 4.** Let  $h$  be as in Theorem 3. Suppose  $K_0 \in L \log L(D_0)$ . Then,

$$T_* : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n) \quad \text{for all } p \in (1, \infty).$$

Perhaps, Theorem 3 is more interesting than Theorem 1, because for the operator  $T$  with  $K_0 \in L \log L(D_0)$ , maybe we are able to prove weak type  $(1, 1)$  estimates. So if we have  $L^2$  estimates for the same  $T$ , then we can apply the Marcinkiewicz interpolation to get  $L^p$  results for  $T$ ; indeed, the case of Theorem 2 with  $h = 1$  was proved in this way by T.Tao. But for  $T_*$  with  $K_0 \in L \log L(D_0)$ , weak type  $(1, 1)$  estimate seems to be not yet available even in the case when  $h = 1$ .

### §3. Results for weighted $L^p$ estimates for $T$ and $T_*$ .

**Definition.**

- $B = B(a, s)$  is called a ball in  $\mathbb{H}$  with center  $a$  and radius  $s$

$$\iff$$

$$B = \{x \in \mathbb{H} : r(a^{-1}x) < s\}$$

for  $a \in \mathbb{H}$  and  $s > 0$ .

- $w \in A_p$ ,  $1 < p < \infty$  (Muckenhoupt class on  $\mathbb{H}$ )

$$\iff$$

$$\sup_B \left( |B|^{-1} \int_B w(x) dx \right) \left( |B|^{-1} \int_B w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty.$$

- $w \in A_1 \iff Mw \leq Cw$  a.e.

where  $M$  denotes the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{x \in B} |B|^{-1} \int_B |f(y)| dy.$$

**Theorem 5.** Suppose that

$K_0 \in L^q(D_0)$ ,  $h \in d_q$  for some  $q > 1$ ,

$\omega(h, t) \leq Ct^\eta$  for some  $\eta > 0$ .

Let  $1 < p < \infty$ . Then,

- (1)  $T$  and  $T_*$  are bounded on  $L^p(w)$  if  $q' \leq p < \infty$  and  $w \in A_{p/q'}$ ,  $q' = q/(q - 1)$ ;
- (2) if  $1 < p \leq q$  and  $w \in A_{p'/q'}$ ,  $T$  and  $T_*$  are bounded on  $L^p(w^{1-p})$ .

In the Euclidean convolution case, where Fourier transform estimates are available, this was proved independently by

J. Duoandikoetxea, 1993,

D. Watson, 1990.

## §4. A basic $L^2$ estimate.

Let  $\psi_j \in C_0^\infty(\mathbb{R})$ ,  $j \in \mathbb{Z}$ , be such that

$$\text{supp}(\psi_j) \subset \{t \in \mathbb{R} : 2^j \leq t \leq 2^{j+2}\}, \quad \psi_j \geq 0,$$

$$\sum_{j \in \mathbb{Z}} \psi_j(t) = 1 \quad \text{for } t \neq 0,$$

$$|(d/dt)^m \psi_j(t)| \leq c_m |t|^{-m} \quad \text{for } m = 0, 1, 2, \dots$$

Let

$$S_j L(x) = (\log 2)^{-1} h(r(x)) \int_0^\infty \psi_j(t) \delta_t K_0(x) dt/t, \quad \delta_t K_0(x) = t^{-\gamma} K_0(A_t^{-1}x).$$

Then

$$\sum_{j \in \mathbb{Z}} S_j L = L, \quad Tf = \sum_{j \in \mathbb{Z}} f * S_j L.$$

**Lemma 1.** Let  $q > 1$ ,  $h \in d_q$ ,  $K_0 \in L^q(D_0)$  and  $\omega(h, t) \leq Ct^\eta$  for some  $\eta > 0$ . Then,

$$\|f * S_j L * \Delta_{k+j}\|_2 \leq C 2^{-\epsilon|k|} \|K_0\|_q \|f\|_2.$$

If  $h = 1$  and  $q = \infty$ , this was proved by T. Tao 1999.

## §5. Sketch of proof for $L^p$ estimates of $T$ .

The case  $s = q$  of

**Theorem 1.**

$$\|Tf\|_p \leq C_p(s/(s-1))\|K_0\|_s\|f\|_p, \quad 1 < s \leq q.$$

Theory of Duoandikoetxea and Rubio de Francia (1986):

- Orthogonality arguments for  $L^2$  estimates via Fourier transform estimates and Plancherel's theorem
- Littlewood-Paley theory
- Interpolation arguments

Our strategy is:

to employ a version of theory of Duoandikoetxea and Rubio de Francia adapted for the present situation;  
replace use of Fourier transform estimates with Lemma 1 and apply Cotlar's lemma.

## Littlewood-Paley inequalities.

$\phi$ : a  $C^\infty$  function,  $\text{supp}(\phi) \subset B(0, 1) \setminus B(0, 1/2)$ ,  $\int \phi = 1$ ,  
 $\phi(x) = \tilde{\phi}(x)$ ,  $\phi(x) \geq 0 \forall x \in \mathbb{R}^n$ , where  $\tilde{\phi}(x) = \phi(x^{-1})$ .

Define

$$\Delta_k = \delta_{2k-1}\phi - \delta_{2k}\phi, \quad k \in \mathbb{Z},$$

where

$$\delta_t\phi(x) = t^{-\gamma}\phi(A_t^{-1}x).$$

Then  $\Delta_k = \tilde{\Delta}_k$ ,

$$\sum_k \Delta_k = \delta$$

where  $\delta$  is the delta function.

**Lemma 2.** Let  $w \in A_p$ ,  $1 < p < \infty$ . Then

$$\left\| \sum_k f_k * \Delta_k \right\|_{L^p(w)} \leq C_{p,w} \left\| \left( \sum_k |f_k|^2 \right)^{1/2} \right\|_{L^p(w)},$$

$$\left\| \left( \sum_k |f * \Delta_k|^2 \right)^{1/2} \right\|_{L^p(w)} \leq C_{p,w} \|f\|_{L^p(w)}.$$

**Decompose**

$$Tf = \sum_{j \in \mathbb{Z}} f * S_j L = \sum_{k_1, k_2 \in \mathbb{Z}} U_{k_1, k_2} f,$$

**where**

$$U_{k_1, k_2} f = \sum_j f * \Delta_{k_1+j} * \nu_j * \Delta_{k_2+j}, \quad \nu_j = S_j L.$$

**Lemma 3.** Let  $1 < p < \infty$ .

$$\|U_{k_1, k_2} f\|_p \leq C_p 2^{-\epsilon(|k_1| + |k_2|)} \|K_0\|_q \|f\|_p$$

for some  $\epsilon > 0$ .

**Lemma 3 implies**

$$\begin{aligned}\|Tf\|_p &\leq \sum_{k_1, k_2} \|U_{k_1, k_2} f\|_p \leq C \sum_{k_1, k_2} 2^{-\epsilon(|k_1| + |k_2|)} \|K_0\|_q \|f\|_p \\ &\leq C \|K_0\|_q \|f\|_p.\end{aligned}$$

To prove Lemma 3 we need

**Lemma 4.**

$$\|U_{k_1, k_2} f\|_2 \leq C 2^{-\epsilon(|k_1| + |k_2|)} \|K_0\|_q \|f\|_2$$

for some  $\epsilon > 0$ .

**Proof of Lemma 4.** Let

$$S_j f = f * \Delta_{k_1+j} * \nu_j * \Delta_{k_2+j}, \quad \nu_j = S_j L.$$

Then

$$U_{k_1, k_2} f = \sum_j S_j f.$$

We prove

$$(1) \quad \|S_j S_{j'}^* f\|_2 \leq C 2^{-2\epsilon(|k_1|+|k_2|)} 2^{-\delta|j-j'|} \|K_0\|_q^2 \|f\|_2$$

and

$$(2) \quad \|S_{j'}^* S_j f\|_2 \leq C 2^{-2\epsilon(|k_1|+|k_2|)} 2^{-\delta|j-j'|} \|K_0\|_q^2 \|f\|_2,$$

where

$$S_{j'}^* f = f * \Delta_{k_2+j'} * \tilde{\nu}_{j'} * \Delta_{k_1+j'}, \quad \tilde{\nu}_{j'}(x) = \nu_{j'}(x^{-1}).$$

Then, Lemma 4 follows from the Cotlar-Knapp-Stein lemma.

**Proof of (2).**

By Lemma 1 and  $\|\Delta_{k_2+j} * \Delta_{k_2+j'}\|_1 \leq C 2^{-\delta|j-j'|}$ ,

$$\begin{aligned}
& \left\| f * (\Delta_{k_1+j} * \nu_j) * (\Delta_{k_2+j} * \Delta_{k_2+j'}) * (\tilde{\nu}_{j'} * \Delta_{k_1+j'}) \right\|_2 \\
& \leq C 2^{-\epsilon|k_1|} \|K_0\|_q \left\| f * (\Delta_{k_1+j} * \nu_j) * (\Delta_{k_2+j} * \Delta_{k_2+j'}) \right\|_2 \\
& \leq C 2^{-\epsilon|k_1|} 2^{-\delta|j-j'|} \|K_0\|_q \|f * (\Delta_{k_1+j} * \nu_j)\|_2, \\
& \leq C 2^{-2\epsilon|k_1|} 2^{-\delta|j-j'|} \|K_0\|_q^2 \|f\|_2.
\end{aligned}$$

Also,

$$\begin{aligned}
& \left\| f * \Delta_{k_1+j} * (\nu_j * \Delta_{k_2+j}) * (\Delta_{k_2+j'} * \tilde{\nu}_{j'}) * \Delta_{k_1+j'} \right\|_2 \\
& \leq C 2^{-\epsilon |k_2|} \|K_0\|_q \|f * \Delta_{k_1+j} * (\nu_j * \Delta_{k_2+j})\|_2 \\
& \leq C 2^{-2\epsilon |k_2|} \|K_0\|_q^2 \|f\|_2
\end{aligned}$$

Taking the geometric mean we have

$$\begin{aligned}
& \left\| f * \Delta_{k_1+j} * \nu_j * \Delta_{k_2+j} * \Delta_{k_2+j'} * \tilde{\nu}_{j'} * \Delta_{k_1+j'} \right\|_2 \\
& \leq C 2^{-\epsilon |k_1|} 2^{-\epsilon |k_2|} 2^{-\delta |j-j'|/2} \|K_0\|_q \|f\|_2.
\end{aligned}$$

Similarly, we can prove (1).

### Proof of Lemma 3.

### Lemma 5.

$$\|U_{k_1, k_2} f\|_r \leq C \|K_0\|_q \|f\|_r, \quad 1 < r < \infty.$$

This follows from the Littlewood-Paley inequality and a certain vector valued inequality.

Interpolating between the  $L^r$  estimate in lemma 5 and the estimate in Lemma 4 ( $\|U_{k_1, k_2} f\|_2 \leq C 2^{-\epsilon(|k_1| + |k_2|)} \|K_0\|_q \|f\|_2$ ), we get Lemma 3.

**Proof of Theorem 1 for  $K_0 \in L^s(D_0)$ ,  $1 < s \leq q$ .**

Let  $\rho \geq 2$ . Let  $\psi_j \in C_0^\infty(\mathbb{R})$ ,  $j \in \mathbb{Z}$ , be such that

$$\text{supp}(\psi_j) \subset \{t \in \mathbb{R} : \rho^j \leq t \leq \rho^{j+2}\}, \quad \psi_j \geq 0,$$

$$\sum_{j \in \mathbb{Z}} \psi_j(t) = 1 \quad \text{for } t \neq 0,$$

$$|(d/dt)^m \psi_j(t)| \leq c_m |t|^{-m} \quad \text{for } m = 0, 1, 2, \dots,$$

where  $c_m$  is independent of  $\rho$ . Let

$$S_j L(x) = (\log 2)^{-1} h(r(x)) \int_0^\infty \psi_j(t) \delta_t K_0(x) dt/t.$$

Then,  $\sum_{j \in \mathbb{Z}} S_j L = L$ ,  $Tf = \sum_{j \in \mathbb{Z}} f * S_j L$ .

We choose  $\rho = 2^{s'}$ . Then, repeat the proof of Theorem 1 for  $s = q$  and check the constants carefully.

## §6. Weak type (1,1) estimates on $\mathbb{R}^2$ .

Let

$$Tf(x) = \text{p.v.} \int f(y)K(x-y) dy.$$

**Theorem A (A. Seeger 1996).**

Suppose that  $A_t x = tx$  and  $r(x) = |x|$ ,  $x \in \mathbb{R}^n$ ,  $n \geq 2$ ,  $K_0 \in L \log L(D_0)$ .

Then, the operator  $T$  is of weak type (1,1), i.e.,

$$|\{|Tf| > \lambda\}| \leq \frac{C}{\lambda} \|f\|_1, \quad \lambda > 0.$$

**Theorem B (T. Tao 1999).**

Let

$$A_t x = (t^{\alpha_1} x_1, t^{\alpha_2} x_2, \dots, t^{\alpha_n} x_n),$$

where  $x = (x_1, \dots, x_n)$  and  $0 < \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n$ .

Suppose that  $K_0 \in L \log L(D_0)$ . Then  $T$  is of weak type  $(1, 1)$ .

In fact, T. Tao proved the weak type  $(1, 1)$  boundedness for singular integrals on general homogeneous groups.

Let  $A_t = t^P = \exp((\log t)P)$ , where  $P$  is an arbitrary  $n \times n$  real matrix whose eigenvalues have positive real parts.

**Theorem 6.**

Suppose that  $n = 2$  and  $K_0 \in L \log L(D_0)$ . Then, the operator  $T$  is of weak type  $(1, 1)$ .

There exists a non-singular real matrix  $Q$  such that  $Q^{-1}PQ$  is one of the following Jordan canonical forms:

$$P_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad P_2 = \begin{pmatrix} \alpha & 0 \\ 1 & \alpha \end{pmatrix}, \quad P_3 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

where  $\alpha, \beta > 0$ . So, we have three kinds of dilations

$$\begin{pmatrix} t^\alpha & 0 \\ 0 & t^\beta \end{pmatrix}, \quad t^\alpha \begin{pmatrix} 1 & 0 \\ \log t & 1 \end{pmatrix}, \quad t^\alpha \begin{pmatrix} \cos(\beta \log t) & \sin(\beta \log t) \\ -\sin(\beta \log t) & \cos(\beta \log t) \end{pmatrix}.$$

**The case where  $P = P_1$  is handled by Theorem B. We have to consider the cases  $P = P_2$  and  $P = P_3$ .**

A proof of Theorem 6 follows closely the methods of T. Tao, as the Fourier transform is not readily available in this context. But we need some new estimates and arguments which do not occur in the work of T. Tao. To handle the case  $P = P_3$ , we apply a trick that may have difficulty in extending to higher dimensions.

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