

# **On certain estimates of singular integrals useful for extrapolation**

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We consider singular integrals of the form:

$$\begin{aligned} T(f)(x) &= \text{p.v.} \int_{\mathbb{R}^n} f(x - y) K(y) dy \\ &= \lim_{\epsilon \rightarrow 0} \int_{|y| > \epsilon} f(x - y) K(y) dy, \end{aligned}$$

with rough kernels:

$$K(y) = h(|y|)\Omega(y')|y|^{-n}, \quad y' = y/|y|.$$

Also, we consider several other classes of singular integrals. We have  $L^p$  estimates ( $1 < p < \infty$ ):

$$\|T\|_{p,p} \lesssim \frac{1}{(q-1)(s-1)} \|\Omega\|_q \|h\|_{\Delta_s}$$

for  $q, s \in (1, 2]$ .

As an application, we can prove  $L^p$  boundedness of the singular integrals under certain sharp size conditions on  $\Omega$  and  $h$  via an extrapolation argument.

**§1.** Singular integrals of Calderón-Zygmund and R. Fefferman

**§2.** Singular Radon transforms

**§3.** Singular integrals along surfaces of revolution

**§4.** Littlewood-Paley functions

**§5.** Idea for proof

**§1.** Let  $\Omega \in L^1(S^{n-1})$  satisfy

$$\int_{S^{n-1}} \Omega(\theta) d\sigma(\theta) = 0$$

where  $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$

$d\sigma$  : the Lebesgue measure on  $S^{n-1}$

$n \geq 2$ .

We consider singular integrals of the form:

$$T(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - y) K(y) dy,$$

$$K(x) = h(|x|) \frac{\Omega(x')}{|x'|^n}, \quad x' = x/|x|.$$

**Definition.** Let  $F \in L^1(S^{n-1})$ .

(1)  $F \in L \log L(S^{n-1})$  (Zygmund class)

$$\iff$$

$$\int_{S^{n-1}} |F(\theta)| \log(2 + |F(\theta)|) d\sigma(\theta) < \infty.$$

(2)  $F \in H^1(S^{n-1})$  (Hardy space)

$$\iff$$

$$\|F\|_{H^1(S^{n-1})} = \|P^+ F\|_{L^1(S^{n-1})} < \infty$$

where

$$P^+ F(\theta) = \sup_{0 \leq r < 1} \left| \int_{S^{n-1}} F(\omega) P_{r\theta}(\omega) d\sigma(\omega) \right|$$

$$P_{r\omega}(\theta) = c_n \frac{1 - r^2}{|r\omega - \theta|^n} \quad (\text{Poisson kernel}),$$

$$0 \leq r < 1, \omega, \theta \in S^{n-1}.$$

- $L \log L(S^{n-1})$  is a proper subspace of  $H^1(S^{n-1})$ .

## The case of homogeneous kernels.

Write  $T = T_\Omega$ . Then,

$$(T_\Omega f)^\wedge(\xi) = m(\xi') \hat{f}(\xi),$$

where

$$m(\xi') = - \int_{S^{n-1}} \Omega(\theta) F(\xi', \theta) d\sigma(\theta),$$

$$F(\xi', \theta) = \left[ i \frac{\pi}{2} \operatorname{sgn}(\langle \xi', \theta \rangle) + \log |\langle \xi', \theta \rangle| \right].$$

This implies

- $\Omega \in L \log L(S^{n-1}) \implies T_\Omega : L^2 \rightarrow L^2.$

The method of rotations of Calderón-Zygmund (1956) implies:

- $\Omega \in L^1(S^{n-1})$  and is odd  
 $\implies T_\Omega : L^p \rightarrow L^p$  for all  $1 < p < \infty;$
- $\Omega \in L \log L(S^{n-1})$   
 $\implies T_\Omega : L^p \rightarrow L^p$  for all  $1 < p < \infty.$

- $\Omega \in H^1(S^{n-1})$

$\implies T_\Omega : L^p \rightarrow L^p$  for all  $1 < p < \infty$ .

This was proved by Coifman-Weiss, Connett, Ricci-Weiss (1977–1979) by applying developed versions of the Calderón-Zygmund method of rotations.

This improves the previous result since  $L \log L(S^{n-1})$  is a proper subspace of  $H^1(S^{n-1})$ .

## The case where $h$ is not constant.

**Definition** The space  $\Delta_s$ ,  $s \geq 1$ , is defined as

$$\Delta_s = \{h \text{ on } \mathbb{R}_+ : \|h\|_{\Delta_s} < \infty\},$$

$$\|h\|_{\Delta_s} = \sup_{j \in \mathbb{Z}} \left( \int_{2^j}^{2^{j+1}} |h(t)|^s dt / t \right)^{1/s},$$

where

$\mathbb{Z}$  : the set of integers,

$\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$ ;

$$\Delta_\infty = L^\infty(\mathbb{R}_+).$$

- $s > t \implies \Delta_s \subset \Delta_t$ .

**(1)**  $h \in L^\infty$ ,  $\Omega \in Lip(S^{n-1})$   
 $\implies T : L^p \rightarrow L^p$  for all  $1 < p < \infty$ .

R. Fefferman, 1979.

**(2)**  $h \in L^\infty$ ,  $\exists q > 1 : \Omega \in L^q(S^{n-1})$   
 $\implies T : L^p \rightarrow L^p$  for all  $1 < p < \infty$ .

J. Namazi, 1986.

**(3)**  $h \in \Delta_2$ ,  $\exists q > 1 : \Omega \in L^q(S^{n-1})$   
 $\implies T : L^p \rightarrow L^p$  for all  $1 < p < \infty$ .

J. Duoandikoetxea and J. L. Rubio de Francia (D-R), 1986.

**(4)**  $\Omega \in H^1(S^{n-1})$ ,  $\exists s > 1 : h \in \Delta_s$   
 $\implies T : L^p \rightarrow L^p$  if  $|1/p - 1/2| < \min(1/2, 1/s')$ ,  $s' = s/(s-1)$ .

D. Fan and Y. Pan, 1997.

**(5)**  $\Omega \in L \log L(S^{n-1})$ ,  $\exists s > 1 : h \in \Delta_s$   
 $\implies T : L^p \rightarrow L^p$  for all  $1 < p < \infty$ .

A. Al-Salman and Y. Pan, 2002.

**Theorem 1.** Suppose that  $\Omega \in L^q(S^{n-1})$ ,  $h \in \Delta_s$ ,  $q, s \in (1, 2]$ . Then

$$\|T(f)\|_{L^p} \leq C_p (q-1)^{-1} (s-1)^{-1} \|\Omega\|_{L^q(S^{n-1})} \|h\|_{\Delta_s} \|f\|_{L^p}$$

for all  $p \in (1, \infty)$ , where the constant  $C_p$  is independent of  $q, s, \Omega$  and  $h$ .

### Definition

For  $h$  on  $\mathbb{R}_+$  and  $a > 0$ , let

$$L_a(h) = \sup_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} |h(r)| (\log(2 + |h(r)|))^a \ dr / r.$$

Define

$$\mathcal{L}_a = \{h : L_a(h) < \infty\}.$$

- $a < b \implies \mathcal{L}_b \subset \mathcal{L}_a$ .

- $\bigcup_{s>1} \Delta_s \not\subseteq \bigcap_{a>0} \mathcal{L}_a$ .

By Theorem 1 and extrapolation of Yano we have

**Theorem 2.** Let  $\Omega \in L \log L(S^{n-1})$  and  $h \in \mathcal{L}_a$  for some  $a > 2$ . Then

$$\|T(f)\|_{L^p} \leq C_p \|f\|_{L^p}$$

for all  $p \in (1, \infty)$ .

Remarks.

- Al-Salman-Pan (2002) proved  $L^p$  boundedness of  $T$  under the condition that  $\Omega \in L \log L(S^{n-1})$  and  $h \in \Delta_s$  for some  $s > 1$ . Theorem 2 improves this result by replacing the assumption on  $h$  with  $h \in \mathcal{L}_a$  for some  $a > 2$ .
- $\Omega \in L \log L(S^{n-1}), h \in \mathcal{L}_1$   
 $\implies T : L^p \rightarrow L^p$  for all  $p \in (1, \infty)$  ?

## §2. Singular Radon transforms.

$$\begin{aligned} T(f)(x) &= \text{p.v.} \int_{\mathbb{R}^n} f(x - P(y)) K(y) dy \\ &= \lim_{\epsilon \rightarrow 0} \int_{|y| > \epsilon} f(x - P(y)) K(y) dy, \end{aligned}$$

where

$$K(y) = h(|y|) \Omega(y') |y|^{-n}, \quad y' = |y|^{-1} y;$$

$n \geq 2$  and  $\Omega \in L^1(S^{n-1})$  satisfies

$$\int_{S^{n-1}} \Omega(\theta) d\sigma(\theta) = 0$$

$f$ : a function on  $\mathbb{R}^d$ ;

$P(y) = (P_1(y), P_2(y), \dots, P_d(y))$ : a polynomial mapping, where each  $P_j$  is a real-valued polynomial on  $\mathbb{R}^n$ ;

We assume that  $P(-y) = -P(y)$ .

**Theorem 3.** Suppose that  $\Omega \in L^q(S^{n-1})$ ,  $h \in \Delta_s$ ,  $q, s \in (1, 2]$ . Then

$$\begin{aligned} \|T(f)\|_{L^p(\mathbb{R}^d)} &\leq C_p(q-1)^{-1}(s-1)^{-1} \\ &\quad \times \|\Omega\|_{L^q(S^{n-1})} \|h\|_{\Delta_s} \|f\|_{L^p(\mathbb{R}^d)} \end{aligned}$$

for all  $p \in (1, \infty)$ , where the constant  $C_p$  is independent of  $q, s, \Omega$  and  $h$ . Also, the constant  $C_p$  is independent of polynomials  $P_j$  if we fix  $\deg(P_j)$  ( $j = 1, 2, \dots, d$ ).

By Theorem 3 and extrapolation we have

**Theorem 4.** Let  $\Omega \in L \log L(S^{n-1})$  and  $h \in \mathcal{L}_a$  for some  $a > 2$ . Then

$$\|T(f)\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}$$

for all  $p \in (1, \infty)$ , where  $C_p$  is independent of polynomials  $P_j$  if the polynomials are of fixed degree.

## Previous results

$$T(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - P(y)) K(y) dy,$$

$$K(y) = h(|y|) \Omega(y') |y|^{-n}, \quad y' = |y|^{-1} y;$$

(1)  $h = 1, \Omega \in C^1(S^{n-1})$

$\implies T : L^p \rightarrow L^p$  for all  $1 < p < \infty$ .

E. M. Stein, Proceedings of International Congress of Mathematicians, Berkeley, 1986.

(2)  $\Omega \in H^1(S^{n-1}), \exists s > 1 : h \in \Delta_s$

$\implies T : L^p \rightarrow L^p$  if  $|1/p - 1/2| < \min(1/2, 1/s')$ .

D. Fan and Y. Pan, Amer. J. Math., 1997.

(3)  $\Omega \in L \log L(S^{n-1}), \exists s > 1 : h \in \Delta_s, P(-y) = -P(y)$

$\implies T : L^p \rightarrow L^p$  for all  $1 < p < \infty$ .

A. Al-Salman and Y. Pan, J. London Math. Soc. (2), 2002.

Theorem 4 improves result (3) by replacing the assumption on  $h$  with  $h \in \mathcal{L}_a$  for some  $a > 2$ .

### §3. Singular integrals along surfaces of revolution.

Let

$$\Gamma : [0, \infty) \rightarrow \mathbb{R}^m$$

be a continuous mapping satisfying  $\Gamma(0) = 0$ . We define a singular integral operator along the surface  $(y, \Gamma(|y|))$  by

$$Tf(x, z) = \text{p.v.} \int_{\mathbb{R}^n} f(x - y, z - \Gamma(|y|)) K(y) dy$$

where  $K(y) = h(|y|)\Omega(y')|y|^{-n}$ .

Let

$$M_\Gamma g(z) = \sup_{R>0} R^{-1} \int_0^R |g(z - \Gamma(t))| dt.$$

We assume that

$$M_\Gamma : L^p(\mathbb{R}^m) \rightarrow L^p(\mathbb{R}^m) \quad \text{for all } p > 1.$$

**Theorem 5.** Suppose that  $\Omega \in L^q(S^{n-1})$  for some  $q \in (1, 2]$  and  $h \in \Delta_s$  for some  $s > 1$ . Then,

$$\|Tf\|_{L^p(\mathbb{R}^{n+m})} \leq C_p(q-1)^{-1}\|\Omega\|_q\|h\|_{\Delta_s}\|f\|_{L^p(\mathbb{R}^{n+m})}$$

if  $|1/p - 1/2| < \min(1/s', 1/2)$ , where the constant  $C_p$  is independent of  $q$  and  $\Omega$ .

**Theorem 6.** Suppose  $\Omega \in L \log L(S^{n-1})$  and  $h \in \Delta_s$  for some  $s > 1$ . Then,  $T$  is bounded on  $L^p(\mathbb{R}^{n+m})$  if  $|1/p - 1/2| < \min(1/s', 1/2)$ .

When  $m = 1$  and  $\Gamma$  is a  $C^2$ , convex, increasing function, Theorem 6 was proved by Al-Salman and Pan 2002.

## §4. Littlewood-Paley functions.

We consider the Littlewood-Paley function on  $\mathbb{R}^n$  defined by

$$S_\psi(f)(x) = \left( \int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where  $\psi$  is in  $L^1(\mathbb{R}^n)$ ,

$$\psi_t(x) = t^{-n} \psi(t^{-1}x).$$

We assume that

$$\int_{\mathbb{R}^n} \psi(x) dx = 0.$$

## Marcinkiewicz integral of Stein 1958.

Let

$$\psi(x) = |x|^{-n+1} \Omega(x') \chi_{(0,1]}(|x|), \quad x' = x/|x|,$$

where  $\Omega \in L^1(S^{n-1})$ ,  $\int_{S^{n-1}} \Omega d\sigma = 0$ ,  
 $\chi_E$  is the characteristic function of  $E$ .

Define

$$\mu_\Omega(f) = S_\psi(f).$$

### Theorem A (T.Walsh 1972).

If  $\Omega \in L(\log L)^{1/2}(S^{n-1})$ , then

$$\mu_\Omega : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n).$$

Here  $\Omega \in L(\log L)^{1/2}(S^{n-1})$  means

$$\int_{S^{n-1}} |\Omega(\theta)| [\log(2 + |\Omega(\theta)|)]^{1/2} d\sigma(\theta) < \infty.$$

Al-Salman, Al-Qassem, Cheng and Pan 2002 extended Theorem A to  $L^p$ ,  $1 < p < \infty$ .

**Theorem B.** If  $\Omega \in L(\log L)^{1/2}(S^{n-1})$ , then

$$\mu_\Omega : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n) \quad \text{for all } p \in (1, \infty).$$

We can give a different proof of Theorem B by extrapolation and the following:

**Theorem 7.** If  $\Omega \in L^q(S^{n-1})$  for some  $q \in (1, 2]$ , we have

$$\|\mu_\Omega(f)\|_p \leq C_p(q-1)^{-1/2} \|\Omega\|_q \|f\|_p$$

for  $p \in (1, \infty)$ , where the constant  $C_p$  is independent of  $q$  and  $\Omega$ .

## §5. Idea for proof of Theorem 1.

**Theorem 1.** Suppose that  $\Omega \in L^q(S^{n-1})$ ,  $h \in \Delta_s$ ,  $q, s \in (1, 2]$ . Then

$$\|T(f)\|_{L^p} \leq C_p (q-1)^{-1} (s-1)^{-1} \|\Omega\|_{L^q(S^{n-1})} \|h\|_{\Delta_s} \|f\|_{L^p}$$

for all  $p \in (1, \infty)$ , where the constant  $C_p$  is independent of  $q, s, \Omega$  and  $h$ .

$$T(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y) K(y) dy,$$

$$K(x) = h(|x|) \frac{\Omega(x')}{|x|^n}, \quad x' = x/|x|.$$

Framework of the proof comes from

- J. Duoandikoetxea and J. L. Rubio de Francia (D-R), Maximal and singular integral operators via Fourier transform estimates, *Invent. Math.* 84 (1986).

We apply the method of D-R involving the Littlewood-Paley (L-P) theory.

For singular Radon transforms, we also use results for oscillatory integrals in

- D. Fan and Y. Pan, Singular integral operators with rough kernels supported by subvarieties, *Amer. J. Math.* (1997),
- F. Ricci and E. M. Stein, Harmonic analysis on nilpotent groups and singular integrals, I, *J. Func. Anal.* 73 (1987).

Let  $\rho \geq 2$  and

$$E_k = \{x \in \mathbb{R}^n : \rho^k < |x| \leq \rho^{k+1}\}.$$

Then

$$T(f)(x) = \sum_{-\infty}^{\infty} \sigma_k * f(x),$$

where

$$\begin{aligned} \sigma_k(x) &= K(x)\chi_{E_k}(x), \\ \sigma_k * f(x) &= \int_{E_k} f(x-y)K(y) dy. \end{aligned}$$

Note that

$$(\sigma_k * f)^{\hat{}}(\xi) = \hat{f}(\xi) \int_{E_k} e^{-2\pi i \langle y, \xi \rangle} K(y) dy.$$

## Littlewood-Paley decomposition.

L-P decomposition adapted to a lacunary sequence  $\{\rho^k\}$ .

$$\{\psi_k\}_{k=-\infty}^{\infty}: \psi_k \in C^\infty((0, \infty)),$$

$$\text{supp}(\psi_k) \subset [\rho^{-k-1}, \rho^{-k+1}],$$

$$\sum_{k \in \mathbb{Z}} \psi_k(t) = 1,$$

$$|(d/dt)^j \psi_k(t)| \leq c_j / t^j \quad (j = 1, 2, \dots),$$

where the constants  $c_j$  are independent of  $\rho \geq 2$ .

Define an operator  $S_k$  by

$$(S_k(f))^{\hat{}}(\xi) = \psi_k(|\xi|) \hat{f}(\xi).$$

Then

$$\left\| \left( \sum_k |S_k(f)|^2 \right)^{1/2} \right\|_p \leq C_p \|f\|_p,$$

$$\|f\|_p \leq C'_p \left\| \left( \sum_k |S_k(f)|^2 \right)^{1/2} \right\|_p$$

for  $1 < p < \infty$ .

The constants  $C_p, C'_p$  are independent of  $\rho$ .

Let  $p \in (1, \infty)$ ,  $\delta(p) = |1/p - 1/p'|$ ,  
 $\theta \in (0, 1)$ .

**Lemma 1.** For  $q, s \in (1, 2]$  and  $\rho \geq 2$ , let

$$A = (\log \rho) \|\Omega\|_{L^q(S^{n-1})} \|h\|_{\Delta_s},$$

$$B = \left(1 - \rho^{-\theta/(2q's')}\right)^{-1}.$$

Then

$$\|T(f)\|_p \leq CAB^{1+\delta(p)} \|f\|_p$$

for  $p \in (1 + \theta, (1 + \theta)/\theta)$ .

The constants  $C$  is independent of  $q, s \in (1, 2]$ ,  $\Omega \in L^q(S^{n-1})$ ,  
 $h \in \Delta_s$ ,  $\rho$ .

**Lemma 1  $\implies$  Theorem 1.**

By Lemma 1 we have

$$\|T(f)\|_p \leq CAB^{1+\delta(p)}\|f\|_p,$$

where

$$A = (\log \rho) \|\Omega\|_{L^q(S^{n-1})} \|h\|_{\Delta_s},$$

$$B = \left(1 - \rho^{-\theta/(2q's')}\right)^{-1}.$$

$$p \in (1 + \theta, (1 + \theta)/\theta), \quad \theta \in (0, 1).$$

Take

$$\rho = 2^{q's'}.$$

Then

$$\|T(f)\|_p \leq C_p(q-1)^{-1}(s-1)^{-1}\|\Omega\|_q\|h\|_{\Delta_s}\|f\|_p$$

for all  $p \in (1 + \theta, (1 + \theta)/\theta)$ . This completes the proof of Theorem 1, since

$$(1 + \theta, (1 + \theta)/\theta) \rightarrow (1, \infty) \quad \text{as } \theta \rightarrow 0.$$

A new element of the proof is to apply L-P decomposition depending on  $q$  and  $s$  for which  $\Omega \in L^q(S^{n-1})$  and  $h \in \Delta_s$ . We apply L-P decomposition adapted to a lacunary sequence with Hadamard gap  $\rho$ ,

$$\rho \sim 2^{q's'}.$$

If we apply L-P decomposition with a fixed Hadamard gap, for example, with  $\rho = 2$ , and if we do not change the other part of our proof, then we have

$$\|T\|_{p,p} \lesssim [(q-1)(s-1)]^{-1-\delta(p)} \|\Omega\|_q \|h\|_{\Delta_s},$$

where

$$\delta(p) = |1/p - 1/p'|.$$

This is not favorable, since  $1 + \delta(p) \rightarrow 2$  as  $p \rightarrow 1$  or  $p \rightarrow \infty$ .

The method of appropriately choosing a lacunary sequence has been already used in a different way from ours by A. Al-Salman and Y. Pan, J. London Math. Soc. (2), 2002.

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